Lecture on Generalised Partons Distributions of Pseudo-Goldstone bosons and the nucleon

Cédric Mezrag

CEA Saclay, Irfu DPhN

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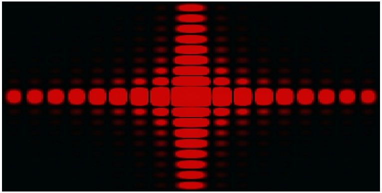
The 2021 School of Physics of Baryons

Introduction : probing the internal structure of matter

- Fraunhofer diffraction



Simulation of Fraunhofer diffraction due to a rectangle slit.

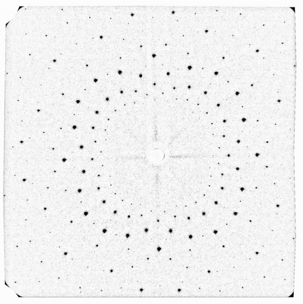
source : Wikimedia Commons

- Far field diffraction
- Diffraction
  $\rightarrow$ Fourier transform of transmission coefficient

Scattering experiments

A key tool to understand the structure of matter

- Fraunhofer diffraction
- X-ray scattering



Silicium crystal diffractive pattern

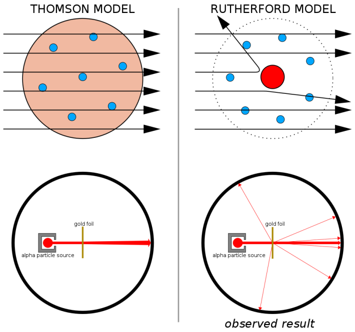
source : UK's national synchrotron

- X-ray wavelength
  $\rightarrow \lambda \simeq$ typical size
- Bragg Law
- Diffraction pattern
  $\rightarrow$ Fourier transform of electronic density
- Provide information on the cristal structure

Scattering experiments

A key tool to understand the structure of matter

- Fraunhofer diffraction
- X-ray scattering
- Rutherford experiment



- α particles scattering on a gold foil
- Some of which are scattered at large angles
- Invalidate the Thomson Model (Plum Pudding)
- Allows to develop the Rutherford planetary model

source : Wikimedia Commons

A pattern a study matter

- Scattering without breaking
- Fourier transform relation between matter structure and diffraction figure
- Repeat itself for different orders of magnitude
- Can we extend that to hadron structure?

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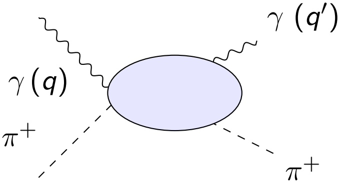

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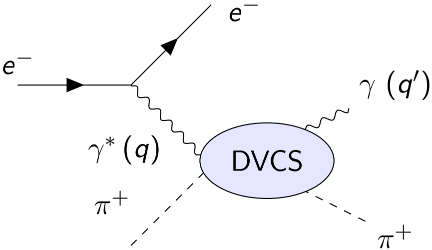
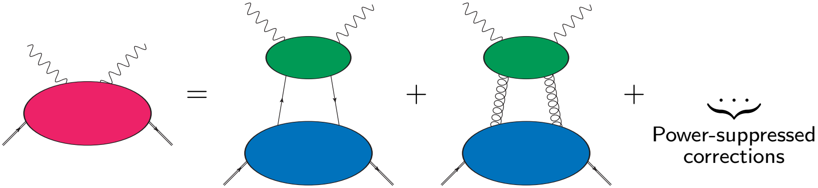


Large virtuality and factorisation

- When the photon is strongly virtual : $Q^2 = -q^2 >> M^2, t$



- Decomposition of DVCS between perturbative (green) and non-perturbative (blue) subparts.
- Perturbative part $\rightarrow$ description of the interaction between the probe and a parton inside hadron
- Non-perturbative part : description of a parton hadron amplitude called Generalised Partons Distributions (GPDs)
- GPDs is where the information on the hadrons structure lies.

Generalised Parton Distributions

References

- General review on GPDs:
  M. Diehl, Phys.Rept., 2003, 388, 41-277
  A. Belitsky and A. Radyushkin, Phys.Rept., 2005, 418, 1-387
- Modern phenomenological applications
  K. Kumericki *et al.*, Eur. Phys. J., 2016, A52, 157
- Future experimental opportunities
  EIC Yellow Report, arXiv:2103.05419

Definitions and some properties

Formal Definition for the pion

$$H_\pi^q(x,\xi,t) = \frac{1}{2}\int \frac{e^{ixP^+z^-}}{2\pi}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}$$

$$H_\pi^g(x,\xi,t) = \frac{1}{2}\int \frac{e^{ixP^+z^-}}{2\pi}\langle P+\frac{\Delta}{2}|G^{+\mu}(-\frac{z}{2})G_\mu^+(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}$$

D. Müller *et al.*, Fortsch. Phy. 42 101 (1994)
X. Ji, Phys. Rev. Lett. **78**, 610 (1997)
A. Radyushkin, Phys. Lett. **B380**, 417 (1996)

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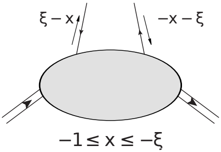
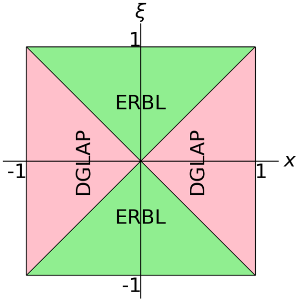
- Caveat ! In gauges other than the lightcone one, a Wilson line is necessary to make the GPDs gauge invariant

Kinematical Range

Different values of (x, ξ) yields different lightfront interpretations:





- Modifies our understanding of what is probed
- Different type of contributions
- It determines two big regions
- Relevant for evolution equations
- $|\xi| > 1$ region of Generalised Distribution Amplitudes (GDA)

Connection with the PDF

Coming back to the definition:

$$H_\pi^q(x,\xi,t) = \frac{1}{2}\int \frac{e^{ixP^+z^-}}{2\pi}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}$$

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$$H_\pi^q(x,0,0) = q(x)\Theta(x) - \bar{q}(-x)\Theta(-x)$$
$$H_\pi^g(x,0,0) = xg(x)\Theta(x) - xg(-x)\Theta(-x)$$

In the limit $(\xi, t) \to (0,0)$, one recover the PDFs.

Connection with the form factor

Looking at the quark definition:

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we would recover the Form Factor if we could make the operator "local". Simple way to do that $\rightarrow$ integrate on Fourier conjugate variable:

$$\begin{aligned}\int \mathrm{d}x\, H_\pi^q(x,\xi,t) &= \frac{1}{2}\int \delta(P^+z^-)\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}\\ &= \frac{1}{2P^+}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(0)\gamma^+\psi^q(0)|P-\frac{\Delta}{2}\rangle\end{aligned}$$

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We recover the pion electromagnetique Form Factor

The soft pion theorem

- It relates the pion GPD to the pion Distribution Amplitude (DA)
- The standard proof is more technical, and involve Generalised Distribution Amplitudes (GDAs):
  - First relate GDA (two pions DA) to the standard DA in the low energy limit (PCAC)
  - Then use crossing symmetry to connect the GDA to the GPD

M. Polyakov, Nucl.Phys.B 555 (1999) 231

- An alternative proof based on the Bethe-Salpeter formalism is available (no crossing symmetry but a simplified description of the pion)

C. Mezrag *et al.*, Phys.Lett.B 741 (2015) 190-196

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$$H_\pi^q(x,1,0) + H_\pi^q(-x,1,0) = \varphi\left(\frac{1+x}{2}\right)$$

$$H_\pi^q(x,1,0) - H_\pi^q(-x,1,0) = 0$$

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A probabilistic interpretation can be obtained only for $\xi = 0$

$$\rho(x,\tilde{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} e^{i\Delta_\perp \tilde{b}_\perp} H(x,0,-\Delta_\perp^2)$$

M. Burkardt, PRD 62 (2000) 071503, PRD 66 (2002) 119903 (erratum)

Computations

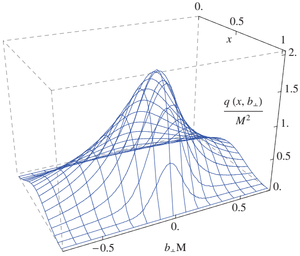


fig. from C. Mezrag *et al.*, PLB 741 (2015) 190-196

Extractions

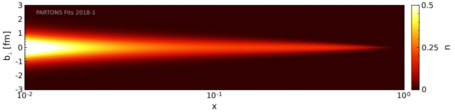


fig. from H. Moutarde *et al.*, EPJ C 78 (2018) 11, 890

Extractions require extrapolations and are model dependent.

Place of GPDs in the Hadron physics context

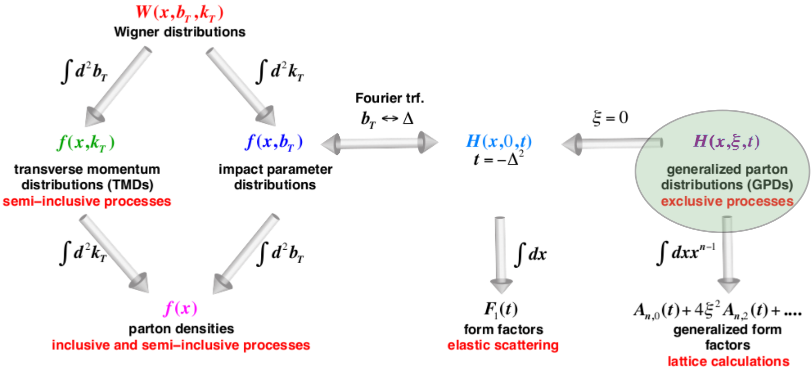
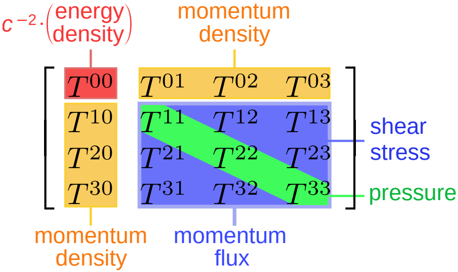


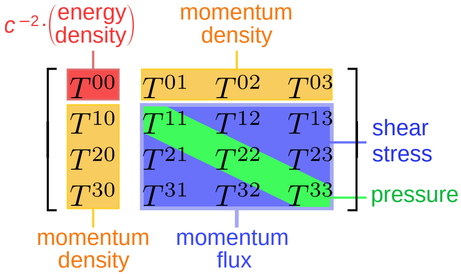
figure from A. Accardi *et al.*, Eur.Phys.J.A 52 (2016) 9, 268



How energy, momentum, pressure are shared between quarks and gluons

Caveat: renormalization scheme and scale dependence

C. Lorcé *et al.*, PLB 776 (2018) 38-47,
M. Polyakov and P. Schweitzer,
IJMPA 33 (2018) 26, 1830025
C. Lorcé *et al.*, Eur.Phys.J.C 79 (2019) 1, 89



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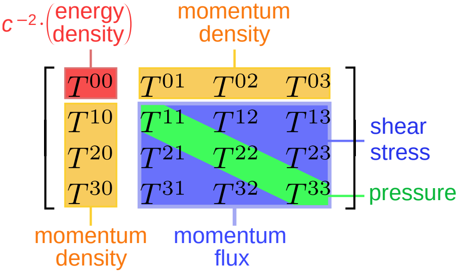
$$\langle p' | T^{\mu\nu}_{q,g} | p \rangle = 2P^\mu P^\nu A_{q,g}(t;\mu) + \frac{1}{2}\left(\Delta^\mu \Delta^\nu - g^{\mu\nu}\Delta^2\right) C_{q,g}(t;\mu) + 2M^2 g^{\mu\nu} \bar{C}_{q,g}(t;\mu)$$

Interpretation of GPDs II

Connection to the Energy-Momentum Tensor



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$$\int_{-1}^{1} \mathrm{d}x\, x\, H_q(x,\xi,t;\mu) = A_q(t;\mu) + \xi^2 C_q(t;\mu)$$

- Ji sum rule (nucleon)
- Fluid mechanics analogy

X. Ji, PRL 78, 610-613 (1997)
M.V. Polyakov PLB 555, 57-62 (2003)

Questions ?

Polynomiality and its consequences

We can generalise what we obtained on the EFF for higher moments:

$$\int \mathrm{d}x\, x^m H(x,\xi,t) = \frac{1}{2}\int \mathrm{d}x\, x^m \int \frac{e^{ixP^+z^-}}{2\pi} \langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}$$

$$= \int \frac{\mathrm{d}x}{2(iP^+)^m}\frac{\mathrm{d}^m}{(\mathrm{d}z^-)^m}\left[\frac{e^{ixP^+z^-}}{2\pi}\right]\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|^{\mathbf{z}=0}_{z^+=0}$$

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$$= \frac{i^m}{2(P^+)^{m+1}}\langle P+\frac{\Delta}{2}|\frac{\mathrm{d}}{\mathrm{d}z^-}\left[\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})\right]|P-\frac{\Delta}{2}\rangle|_{z=0}$$

$$= \frac{1}{2(P^+)^{m+1}}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(0)\gamma^+\left(i\overleftrightarrow{\partial}^+\right)^m\psi^q(0)|P-\frac{\Delta}{2}\rangle$$

- we recover local operators as in DIS $\mathcal{O}^{\mu\mu_1\cdots\mu_m} = \mathbf{S}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}^{\mu_1}\ldots\overleftrightarrow{\partial}^{\mu_m}\psi$
- ... but evaluated between off-diagonal states

$$\int \mathrm{d}x\, x^m H(x,\xi,t) = \frac{1}{2(P^+)^{m+1}} \langle P + \frac{\Delta}{2} | \bar{\psi}^q(0) \gamma^+ \left(i \overleftrightarrow{\partial}^+ \right)^m \psi^q(0) | P - \frac{\Delta}{2} \rangle$$

$$= \frac{1}{(P^+)^m} \sum_{\substack{i=0 \\ \text{even}}}^{m} A_{i,m}(t) \Delta^{\mu_1} \ldots \Delta^{\mu_i} P^{\mu_{i+1}} \ldots P^{\mu_m} n_{\mu_1} \ldots n_{\mu_m} + \mathrm{mod}(m,2) (\frac{\Delta^+}{P^+})^{m+1} C_{m+1}(t)$$

$$= \sum_{\substack{i=0 \\ \text{even}}}^{m} A_{i,m}(t) (-2\xi)^i + \mathrm{mod}(m,2) (-2\xi)^{m+1} C_{m+1}(t)$$

Mellin Moments of GPDs

Polynomiality property

$$\int \mathrm{d}x\, x^m H(x,\xi,t) = \frac{1}{2(P^+)^{m+1}} \langle P + \frac{\Delta}{2} | \bar{\psi}^q(0)\gamma^+ \left(i\overleftrightarrow{\partial}^+\right)^m \psi^q(0) | P - \frac{\Delta}{2}\rangle$$

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$$= \sum_{\substack{i=0\\ \text{even}}}^{m} A_{i,m}(t)(-2\xi)^i + \mathrm{mod}(m,2)(-2\xi)^{m+1} C_{m+1}(t)$$

Mellin Moments of GPDs are even polynomials in ξ of a given degree !

- $A_{0,m}(0)$ are the moments of the PDF
- $A_{0,0}(t)$ is proportional to the form factor
- $C_{m+1}(t)$ are the Mellin moment of a new object: the D-term

Introducing the D-term

- We want to define a function D so that for odd m:

$$\int_{-1}^{1} \mathrm{d}y\, y^m D(y,t) = (-2)^{m+1} C_{m+1}(t)$$

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$$\sum_{\substack{i=0\\ \text{even}}}^{m} A_{i,m}(t)(-2\xi)^i = \int_{-1}^{1} \mathrm{d}x\, x^m H(x,\xi,t) - \xi^{m+1} \int_{-1}^{1} \mathrm{d}y\, y^m D(y,t)$$

$$= \int_{-1}^{1} \mathrm{d}x\, x^m \left[H(x,\xi,t) - \Theta(-\xi \leq x \leq \xi) D\left(\frac{x}{\xi}, t\right)\right]$$

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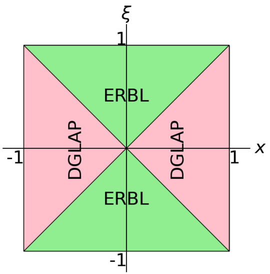

- D-term is a function of 2 variables only ! (like the PDF)
- It lives *only* in the so-called ERBL region

Consequence of Polynomiality

$$\sum_{\substack{i=0\\ \text{even}}}^{m} A_{i,m}(t)(-2\xi)^i = \int_{-1}^{1} \mathrm{d}x\, x^m \left[H(x,\xi,t) - \Theta(-\xi \le x \le \xi) D\left(\frac{x}{\xi}, t\right)\right]$$

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O. Teryaev, PLB510 125-132 (2001)
N. Chouika *et al.*, EPJC 77 906 (2017)

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- It implies that $H - D$ is the Radon transform of a third function, called a Double Distribution F.

Radon transform and Double Distributions

- The connection between GPDs and DDs is given through:

$$H(x,\xi,t) - \Theta(-\xi \le x \le \xi) D\left(\frac{x}{\xi}, t\right) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi) F(\beta,\alpha,t)$$

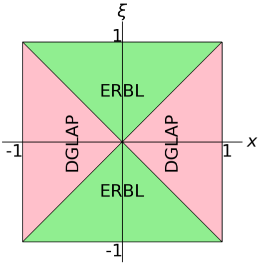
A. Radysuhkin, PRD 56 (1997) 5524-5557
D. Müller *et al.*, Fortsch. Phy. 42 101 (1994)

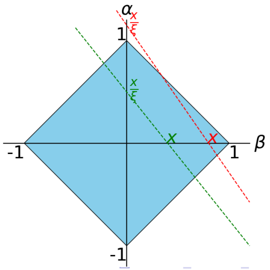
- The D-term can be reabsorbed as:

$$H(x,\xi,t) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi) \left[F(\beta,\alpha,t) + \xi\delta(\beta) D(\alpha,t)\right]$$

M. Polyakov and C. Weiss, PRD60 114017 (1999)

- The properties of the DD guarantee the one of the GPD





Polynomiality revisited with DD

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$$\int \mathrm{d}x\, x^m H(x,\xi,t) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\, (\beta+\alpha\xi)^m F(\beta,\alpha,t) + \xi^{m+1}\int_{-1}^{1} \mathrm{d}\alpha \alpha^m D(\alpha,t)$$
$$= \sum_i^m \xi^i \underbrace{\binom{m}{i} \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\, \alpha^i \beta^{m-i} F(\beta,\alpha,t)}_{=(-2)^i A_{i,m}(t)} + + \xi^{m+1} \underbrace{\int_{-1}^{1} \mathrm{d}\alpha \alpha^m D(\alpha,t)}_{=(-2)^{m+1} C_{m+1}(t)}$$

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$$\begin{aligned}\int \mathrm{d}x\, x^m H(x,\xi,t) &= \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\, (\beta+\alpha\xi)^m F(\beta,\alpha,t) + \xi^{m+1}\int_{-1}^{1} \mathrm{d}\alpha \alpha^m D(\alpha,t) \\ &= \sum_i^m \xi^i \underbrace{\binom{m}{i} \int_\Omega \mathrm{d}\beta\mathrm{d}\alpha\, \alpha^i \beta^{m-i} F(\beta,\alpha,t)}_{=(-2)^i A_{i,m}(t)} + + \xi^{m+1} \underbrace{\int_{-1}^{1} \mathrm{d}\alpha\alpha^m D(\alpha,t)}_{=(-2)^{m+1} C_{m+1}(t)}\end{aligned}$$

- A direct consequence is the link between the DD and the PDF:

$$q(x) = \int_{-1+|x|}^{1-|x|} \mathrm{d}\alpha F(x,\alpha,0)$$

Model of Double Distributions

- Many GPDs models rely on DD in order to fulfil the polynomiality condition.
- The most common way is to use the Radyushkin DD Ansatz:

$$F(\beta,\alpha,t) = q(\beta,t) \times \pi_N(\beta,\alpha)$$

$$\pi_N(\beta,\alpha) = \frac{\Gamma\left(N+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma(N+1)} \frac{((1-|\beta|)^2-\alpha^2)^N}{(1-|\beta|)^{2N+1}}$$

$$1 = \int_{-1+|\beta|}^{1-|\beta|} \mathrm{d}\alpha\, \pi_N(\beta,\alpha)$$

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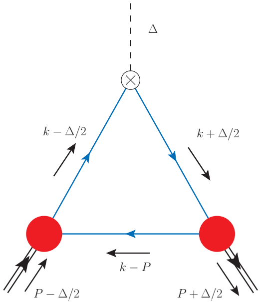
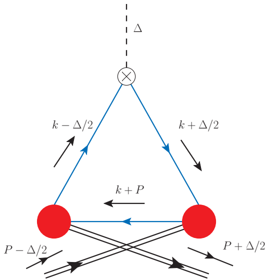
  - Simple to implement
  - Gives results driven by the PDF (much better known)
  - It allows to fulfil easily the GPDs sum rules (connection to EFF)
- However, this functional form has been shown not to be a very flexible fitting parametrisation

C. Mezrag *et al.*,PRD 88 (2013) 1, 014001

- DDs naturally appear in explicitly covariant computations





- Inserting local operators, one recovers polynomials in ξ and therefore DDs.

B.C. Tiburzi and G. A. Miller, PRD 67 (2003) 113004
C. Mezrag *et al.*, arXiv:1406.7425 and FBS 57 (2016) 9, 729-772

Covariant computations and DD

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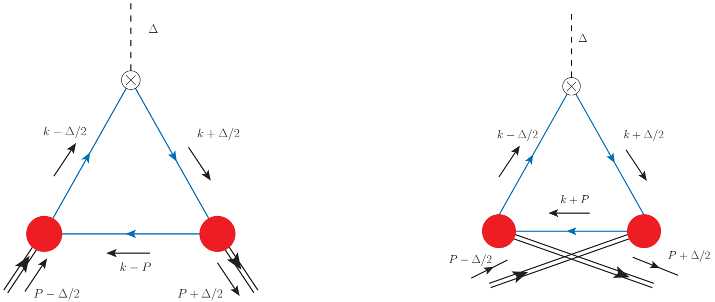

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- However these computations suffer from other issue, for instance regarding the so-called positivity property.

The lightfront wave functions (LFWFs) formalism

- Lightfront quantization allows to expand hadrons on a Fock basis:

$$|P,\pi\rangle \propto \sum_\beta \Phi_\beta^{q\bar{q}}|q\bar{q}\rangle + \sum_\beta \Phi_\beta^{q\bar{q},q\bar{q}}|q\bar{q},q\bar{q}\rangle + \dots$$

$$|P,N\rangle \propto \sum_\beta \Phi_\beta^{qqq}|qqq\rangle + \sum_\beta \Phi_\beta^{qqq,q\bar{q}}|qqq,q\bar{q}\rangle + \dots$$

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Hadrons seen as Fock States

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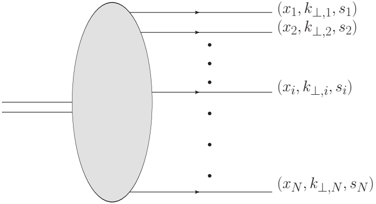
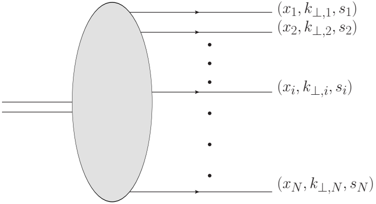
- Non-perturbative physics is contained in the N-particles Lightfront-Wave Functions (LFWF) Φ^N

see for instance S. Brodsky *et al.*, Phys.Rept.S 301 (1998) 299-486



- Momentum information for each parton:
  - Momentum fraction along the lightcone x_i carried by each partons such that $\sum_i^N x_i = 1$ with $0 \leq x_i \leq 1$.
  - Momentum in the transverse plane $k_{\perp,i}$ for each parton
- other quantum number such as parton spin projection



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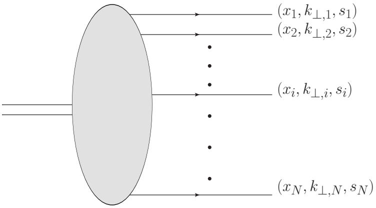
Example: pion

The pion has two independent two-body LFWFs:

$$|\pi, P\rangle = \int [\mathrm{d}x_i \mathrm{d}^2 k_{\perp,i}] \left[\phi^{\uparrow\downarrow}_{q_1 q_2}(x_i, k_{\perp,i}) |q_1(\uparrow) q_2(\downarrow)\rangle + \phi^{\uparrow\uparrow}_{q_1 q_2}(x_i, k_{\perp,i}) |q_1(\uparrow) q_2(\uparrow)\rangle\right] + \dots$$



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- Starting from the matrix element:

$$\langle \pi, P + \frac{\Delta}{2} | \bar{\psi}\left(-\frac{z}{2}\right) \gamma^+ \psi\left(\frac{z}{2}\right) | \pi, P - \frac{\Delta}{2} \rangle$$

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- These matrix elements can be computed, leaving us with an overlap of LFWFs of the type:

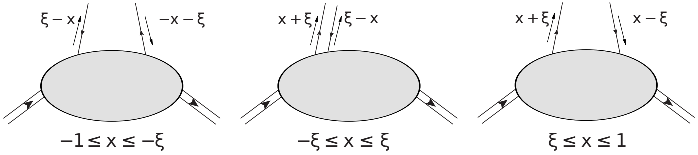
$$\langle \pi, P + \frac{\Delta}{2}|\bar{\psi}\left(-\frac{z}{2}\right)\gamma^+\psi\left(\frac{z}{2}\right)|\pi, P - \frac{\Delta}{2}\rangle \propto \int [\mathrm{d}x_i \mathrm{d}^2 k_{\perp,i}][\mathrm{d}x'_i \mathrm{d}^2 k'_{\perp,i}]\delta(\dots)(\phi^{\uparrow\downarrow}_{q_1 q_2})^* \phi^{\uparrow\downarrow}_{q_1 q_2} + \dots$$

where $\delta(\dots)$ guarantees the momentum conservation.

M. Diehl *et al.*, Nucl.Phys. B596 (2001) 33-65

GPD partonic interpretation

- Two different partonic interpretations:



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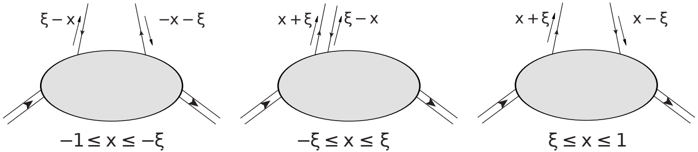
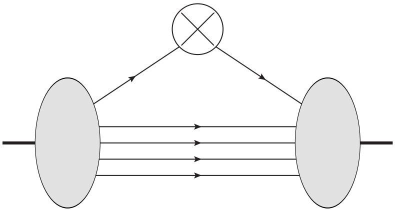


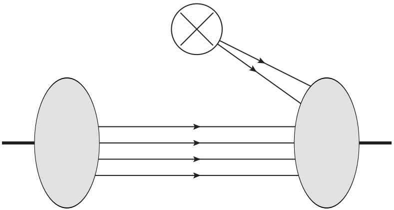
- This has a impact on the way the LFWFs overlap:

DGLAP: $|x| > |\xi|$



- Same N LFWFs
- No ambiguity

ERBL: $|x| < |\xi|$



- N and $N + 2$ partons LFWFs
- Ambiguity

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- In the forward limit $\Delta \to 0$
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- Note that we recover formally a expression of a norm:

$$\langle \pi, P|\bar{\psi}\left(-\frac{z}{2}\right)\gamma^+\psi\left(\frac{z}{2}\right)|\pi, P\rangle \sim \sum_N^{\infty} |\phi^N|^2$$

- Beyond the forward limit, in the DGLAP region, the overlap of LFWFs keeps an interesting structure:

$$\langle \pi, P + \frac{\Delta}{2} | \bar{\psi}\left(-\frac{z}{2}\right) \gamma^+ \psi\left(\frac{z}{2}\right) | \pi, P - \frac{\Delta}{2} \rangle \sim \sum_N^{\infty} (\phi_{out}^N)^* \times \phi_{in}^N$$

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$$|H(x, \xi, t)_{x \geq \xi \geq 0}| \leq \sqrt{q\left(\frac{x-\xi}{1-\xi}\right) q\left(\frac{x+\xi}{1+\xi}\right)}$$

A. Radysuhkin, Phys. Rev. **D59**, 014030 (1999)
B. Pire *et al.*, Eur. Phys. J. **C8**, 103 (1999)
M. Diehl *et al.*, Nucl. Phys. **B596**, 33 (2001)
P.V. Pobilitsa, Phys. Rev. **D65**, 114015 (2002)

- Same type of inequality for gluon GPDs.

Polynomiality vs. Positivity

Polynomiality

- Properties of Mellin moments (local operators)
- Comes from Lorentz Covariance and discrete symmetries
- Delicate cancellations between DGLAP and ERBL region
- Equivalent to the existence of underlying Double Distributions

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Is there a way to fulfil both?

Pragmatic solution: DD-based fit

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"Try and test" way to fulfil positivity in DD space

Pragmatic solution: DD-based fit

- For fitting strategies in the DD space :
  - Specific form better than others

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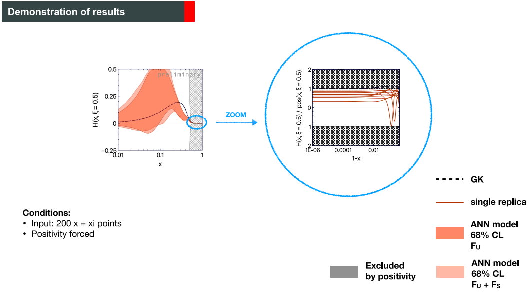
"Try and test" way to fulfil positivity in DD space

- It has been tested on pseudo-data and it really helps constraining GPDs



slide from P. Sznajder *et al.*,

SPIN 2021

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Systematic Way: The covariant extension

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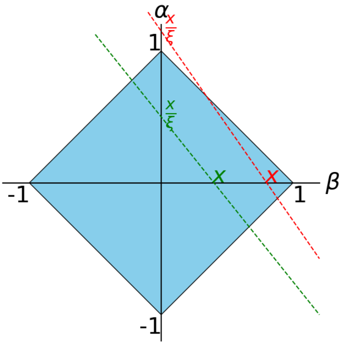

- two types of lines: DGLAP and ERBL lines
- All point of the support are crossed by infinitely many DGLAP lines
- But the line $\beta = 0$!
- when getting close to $\beta = 0$ the slope of DGLAP lines $\to \infty$

- Mathematical answer: yes! We can uniquely extract the DD but not the D-term.

N. Chouika *et al.*, EPJC78, 478 (2018)

Numerical Solution

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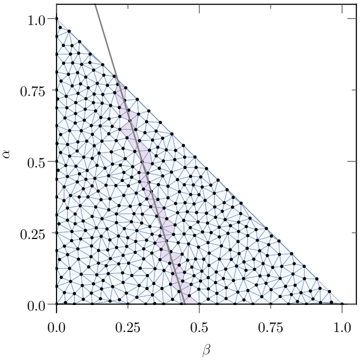

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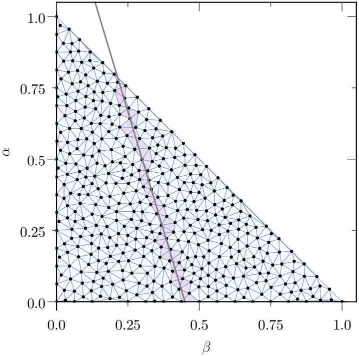

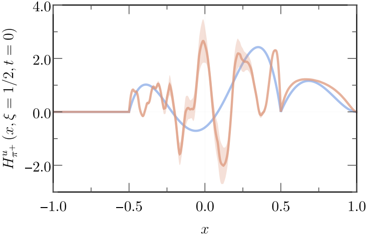

Figures from J.M. Morgado Chavez *et al.*, arXiv:2110.06052

Summary so far

- GPDs are related to EFF and PDFs
- They have to obey multiples properties
- Modelling them so that they fulfil these properties is *difficult*

Next steps

- Scale evolution properties
- Connection to experimental processes

questions?

Evolution properties of GPDs

UV singularities of operators

- Coming back to the operator definition of GPDs:

$$\langle \pi, P + \frac{\Delta}{2} | \bar{\psi}\left(-\frac{z}{2}\right)\gamma^+\psi\left(\frac{z}{2}\right)|\pi, P - \frac{\Delta}{2}\rangle$$

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Two approaches

- Renormalisation of local operators
- Renormalisation using "in partons" matrix elements

- The idea is to "Taylor expand" an operator:

$$\bar{\psi}\left(-\frac{z}{2}\right)\gamma^+\psi\left(\frac{z}{2}\right) = \sum_N^{\infty} c_N(z)\mathcal{O}^N(0)$$

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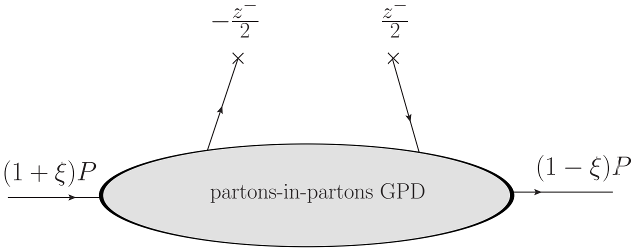
- But it requires to "resum" the renormalised local operators afterward: we saw already when talking about polynomiality that these operators are given by Mellin moment of GPDs $\rightarrow$ solve the inverse moment problem
- Caveat: operator mixing !

- Instead of moments, one can consider partons-in-partons GPDs



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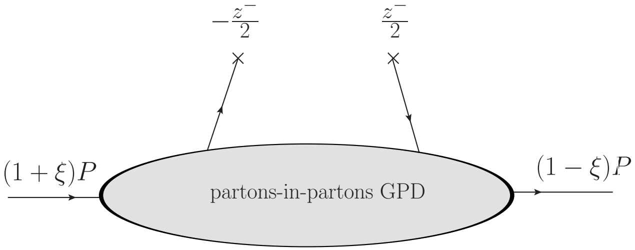

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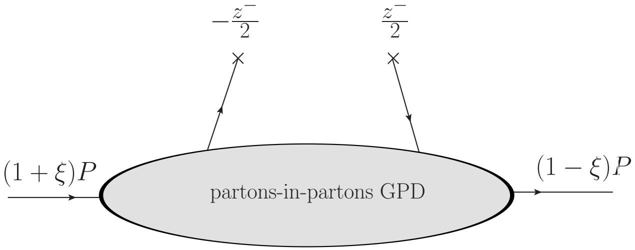

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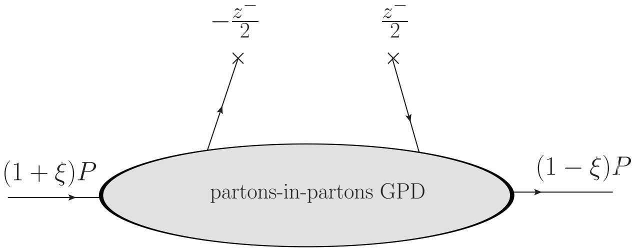

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For that purpose, $\overline{\mathrm{MS}}$ is well suited
GPDs (3D structure, pressure) become *scheme dependent* !

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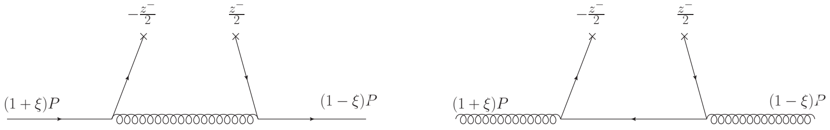

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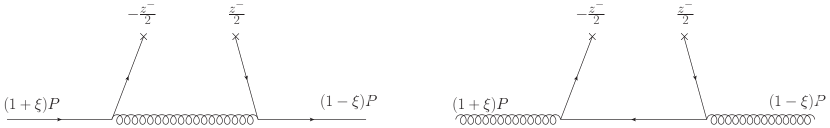

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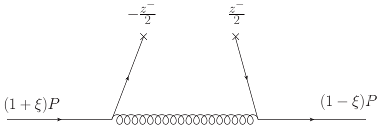

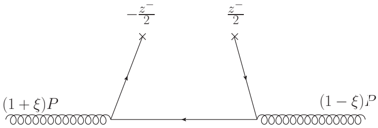

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Final result

$$H^i(x,\xi,t,\mu) = \int_{-1}^{1} \frac{\mathrm{d}y}{|y|} Z_{i,j}\left(\frac{x}{y}, \frac{\xi}{x}, \alpha_s(\mu), \epsilon\right) H^j_{reg}(y,\xi,t,\epsilon)$$

Renormalisation group

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Renormalisation Group

- Knowing the GPD at a scale μ we want to know how it behaves at $\mu + \mathrm{d}\mu$
- we describe perturbatively the impact of this $\mathrm{d}\mu$ leap

$$H(x, \xi, t, \mu + \mathrm{d}\mu) - H(x, \xi, t, \mu)$$

- we obtain like this a first-order integro-differential equation
- α_S becomes “exponentiated”

Non-Singlet Case

$$\frac{\mathrm{d}H^q_{NS}(x,\xi,t,\mu)}{\mathrm{d}\ln(\mu)} = \frac{\alpha_s(\mu)}{4\pi}\int_0^1 \frac{\mathrm{d}y}{y}\mathcal{P}^0_{q\leftarrow q}\left(\frac{x}{y},\frac{\xi}{x}\right) H^q_{NS}(y,\xi,t,\mu)$$

Singlet Case

$$\begin{pmatrix} \frac{\mathrm{d}H^q_S(x,\xi,t,\mu)}{\mathrm{d}\ln(\mu)} \\ \frac{\mathrm{d}H^g(x,\xi,t,\mu)}{\mathrm{d}\ln(\mu)} \end{pmatrix} = \frac{\alpha_s(\mu)}{4\pi}\int_0^1 \frac{\mathrm{d}y}{y} \begin{pmatrix} \mathcal{P}^0_{q\leftarrow q}\left(\frac{x}{y},\frac{\xi}{x}\right) & \mathcal{P}^0_{q\leftarrow g}\left(\frac{x}{y},\frac{\xi}{x}\right) \\ \mathcal{P}^0_{g\leftarrow q}\left(\frac{x}{y},\frac{\xi}{x}\right) & \mathcal{P}^0_{g\leftarrow g}\left(\frac{x}{y},\frac{\xi}{x}\right) \end{pmatrix} \begin{pmatrix} H^q_S(y,\xi,t,\mu) \\ H^g(y,\xi,t,\mu) \end{pmatrix}$$

Evolution equations for GPDs

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The $\mathcal{P}$ distributions can in principle be computed in pQCD

- Splitting function have been computed at:
  - LO (α_s)

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X. Ji PRD55, 7114–7125, 1997
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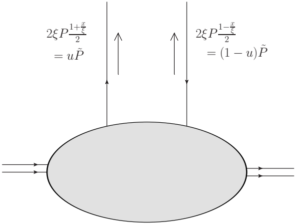
ERBL connection



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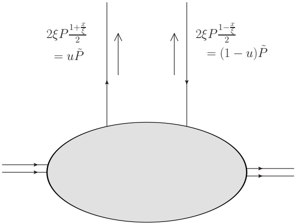

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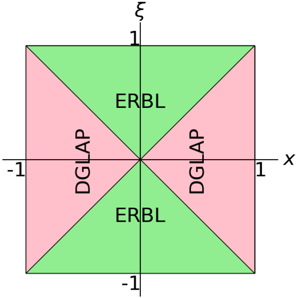
- For $|\xi| = 1$, this interpretation holds for the entire x-range
- We recover there, the so-called ERBL evolution equations

$$\lim_{\xi \to 1} \mathcal{P}\left(\frac{x}{y}, \frac{\xi}{x}\right) = P_{\mathrm{ERBL}}(x, y)$$



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- Yet, evolution equations are written for *matrix elements*, not only operators.
  $\rightarrow$ therefore evolution equations *are* different !

- The ERBL kernal is diagonalised by the 3/2-Gegenbauer polynomials:

$$\int \mathrm{d}u V_{NS}(v,u) C_n^{\frac{3}{2}}(2u-1) \propto \gamma_n C_n(2v-1)$$

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Moments analysis II

Conformal Moments

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GPD Conformal moments $\int \xi^n C_n^{3/2}(\frac{x}{\xi}) H(x,\xi)$ do not mix under evolution !

Other properties

- Charge conservation: $\gamma_0 = 0$
- Energy-Momentum Conservation: $\int \mathrm{d}x x(q(x) + g(x))$ is independent of μ
- Continuity at the crossover lines $|x| = |\xi|$

Evolution in conformal space

- Conformal moments do not mix → easy evolution

$$\xi^n \int_{-1}^{1} \mathrm{d}x C_n^{3/2}\left(\frac{x}{\xi}\right) H(x,\xi,\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{\gamma_n}{\beta_0}} \xi^n \int_{-1}^{1} \mathrm{d}x C_n^{3/2}\left(\frac{x}{\xi}\right) H(x,\xi,\mu_0)$$

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- Inverse moment problem must be solved
  $\rightarrow$ requires analytic continuation in the complex plane
  $\rightarrow$ solution is not unique

D. Mueller and A. Schafer, Nucl.Phys.B739 1-59, 2006

Solving evolution equations

Evolution in conformal space

- Conformal moments do not mix $\rightarrow$ easy evolution

$$\xi^n \int_{-1}^{1} \mathrm{d}x C_n^{3/2}\left(\frac{x}{\xi}\right) H(x,\xi,\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{\gamma_n}{\beta_0}} \xi^n \int_{-1}^{1} \mathrm{d}x C_n^{3/2}\left(\frac{x}{\xi}\right) H(x,\xi,\mu_0)$$

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  $\rightarrow$ requires analytic continuation in the complex plane
  $\rightarrow$ solution is not unique

D. Mueller and A. Schafer, Nucl.Phys.B739 1-59, 2006

Evolution in x-space

- Numerical solution of integro-differential equations
- Dedicated routines do it
- Splitting functions not easily available above one loop

Questions ?

The Nucleon

Main difference: spin-1/2 $\rightarrow$ more tensorial structures!

$$\frac{1}{2}\int \frac{e^{ixP^+z^-}}{2\pi}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}$$
$$= \frac{1}{2P^+}\left[H^q(x,\xi,t)\bar{u}\gamma^+u + E^q(x,\xi,t)\bar{u}\frac{i\sigma^{+\alpha}\Delta_\alpha}{2M}u\right].$$

$$\frac{1}{2}\int \frac{e^{ixP^+z^-}}{2\pi}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\gamma_5\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}$$
$$= \frac{1}{2P^+}\left[\tilde{H}^q(x,\xi,t)\bar{u}\gamma^+\gamma_5 u + \tilde{E}^q(x,\xi,t)\bar{u}\frac{\gamma_5\Delta^+}{2M}u\right].$$

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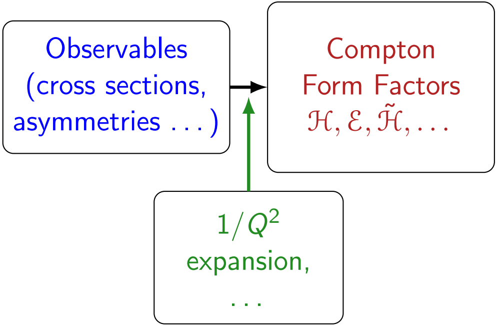
The nucleon has 4 chiral-even and 4 chiral-odd quark GPDs.
All previous properties apply, except the soft pion theorem.

Probing GPDs through exclusive processes

Observables
(cross sections,
asymmetries . . .)

Experimental connection to GPDs

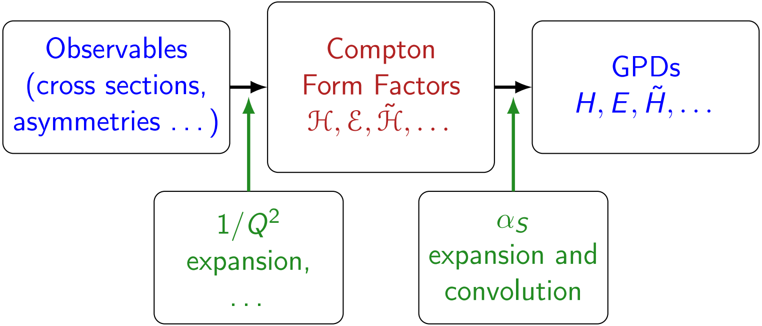


Experimental connection to GPDs

Observables (cross sections, asymmetries . . .) → Compton Form Factors $\mathcal{H}, \mathcal{E}, \tilde{\mathcal{H}}, \dots$ → GPDs $H, E, \tilde{H}, \dots$

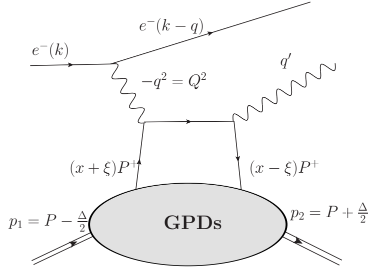
$1/Q^2$ expansion, . . .

α_S expansion and convolution



- CFFs play today a central role in our understanding of GPDs
- Extraction generally focused on CFFs

Deep Virtual Compton Scattering



- Best studied experimental process connected to GPDs
  $\rightarrow$ Data taken at Hermes, Compass, JLab 6, JLab 12

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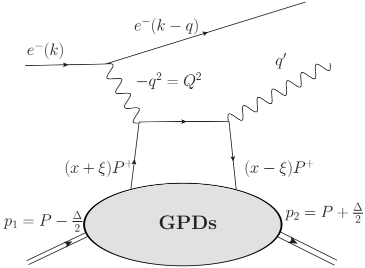

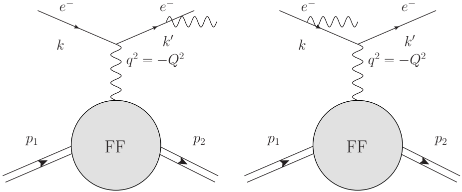

- Best studied experimental process connected to GPDs
  → Data taken at Hermes, Compass, JLab 6, JLab 12
- Interferes with the Bethe-Heitler (BH) process
  - Blessing: Interference term boosted w.r.t. pure DVCS one
  - Curse: access to the angular modulation of the pure DVCS part difficult

M. Defurne *et al.*, Nature Commun. 8 (2017) 1, 1408

QCD corrections to DVCS

- At LO, the DVCS coefficient function is a QED one

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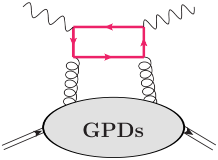
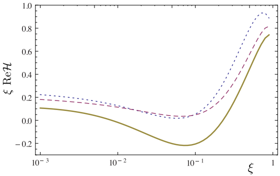
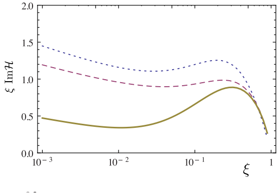
- At NLO, gluon GPDs play a significant role in DVCS







H. Moutarde *et al.*, PRD 87 (2013) 5, 054029

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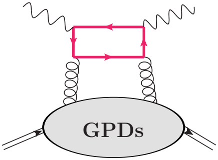
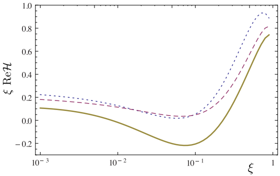
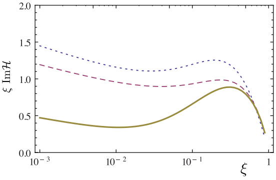




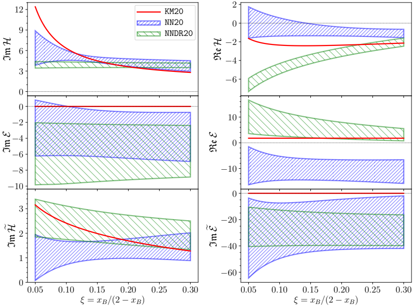


H. Moutarde *et al.*, PRD 87 (2013) 5, 054029

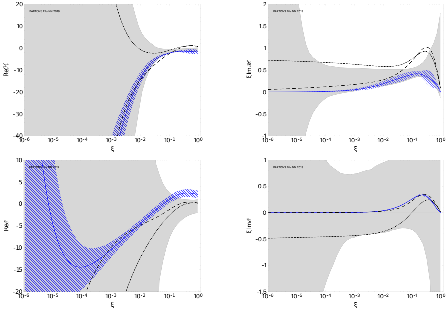
- Recent N2LO studies, impact needs to be assessed

V. Braun *et al.*, JHEP 09 (2020) 117

Recent CFF extractions



M. Cuiċ *et al.*, PRL 125, (2020), 232005



H. Moutarde *et al.*, EPJC 79, (2019), 614

- Recent effort on bias reduction in CFF extraction (ANN)

  additional ongoing studies, J. Grigsby *et al.*, PRD 104 (2021) 016001

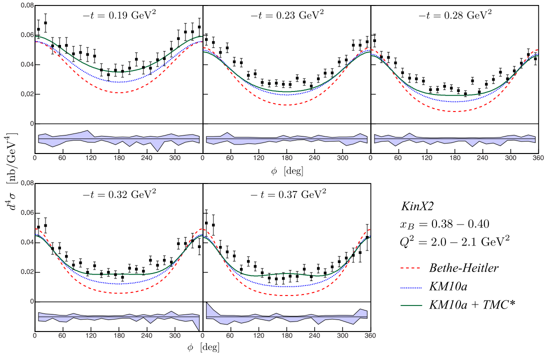
- Studies of ANN architecture to fulfil GPDs properties (dispersion relation,polynomiality,. . .)
- Recent efforts on propagation of uncertainties (allowing impact studies for JLAB12, EIC and EicC)

see *e.g.* H. Dutrieux *et al.*, EPJA 57 8 250 (2021)

Finite t corrections

Kinematical corrections in t/Q^2 and M^2/Q^2

V. Braun *et al.*, PRL 109 (2012), 242001



M. Defurne *et al.* PRC 92 (2015) 55202

- Sizeable even for $t/Q^2 \sim 0.1$
- Not currently included in global fits.

Dispersion relation and the D-term

- At all orders in α_S, dispersion relations relate the real and imaginary parts of the CFF.

I. Anikin and O. Teryaev, PRD 76 056007
M. Diehl and D. Ivanov, EPJC 52 (2007) 919-932

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- For instance at LO:

$$Re(\mathcal{H}(\xi,t)) = \frac{1}{\pi}\int_{-1}^{1}\mathrm{d}x\ Im(\mathcal{H}(x,t))\left[\frac{1}{\xi - x} - \frac{1}{\xi + x}\right] + \underbrace{2\int_{-1}^{1}\mathrm{d}\alpha\frac{D(\alpha,t)}{1-\alpha}}_{\text{Independent of } \xi}$$

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$$\underbrace{Re(\mathcal{H}(\xi,t))}_{\text{Extracted from data}} = \frac{1}{\pi}\int_{-1}^{1} \mathrm{d}x \underbrace{Im(\mathcal{H}(x,t))}_{\text{Extracted from data}} \left[\frac{1}{\xi - x} - \frac{1}{\xi + x}\right] + 2\int_{-1}^{1} \mathrm{d}\alpha \frac{D(\alpha,t)}{1-\alpha}$$

- $D(\alpha,t)$ is related to the EMT (pressure and shear forces)

M.V. Polyakov PLB 555, 57-62 (2003)

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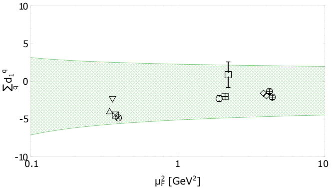


figure from H. Dutrieux *et al.*, Eur.Phys.J.C 81 (2021) 4

- First attempt from JLab 6 *GeV* data

Burkert *et al.*, Nature 557 (2018) 7705, 396-399

- Tensions with other studies
  → uncontroled model-dependence

K. Kumericki, Nature 570 (2019) 7759, E1-E2
H. Moutarde *et al.*, Eur.Phys.J.C 79 (2019) 7, 614
H. Dutrieux *et al.*, Eur.Phys.J.C 81 (2021) 4

- Scheme/scale dependence

The DVCS deconvolution problem I

From CFF to GPDs

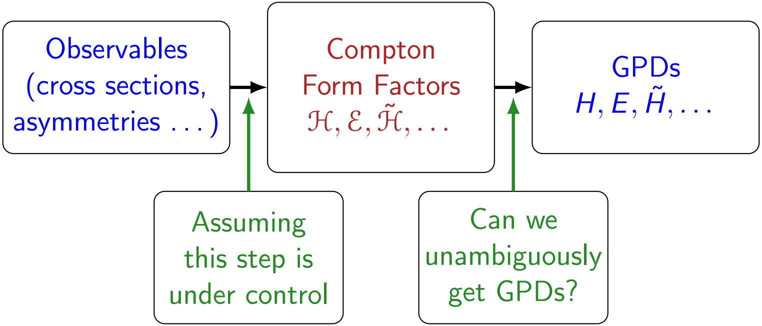
Observables (cross sections, asymmetries ...) → Compton Form Factors $\mathcal{H}, \mathcal{E}, \tilde{\mathcal{H}}, \ldots$ → GPDs $H, E, \tilde{H}, \ldots$

Assuming this step is under control

Can we unambiguously get GPDs?

The DVCS deconvolution problem I

From CFF to GPDs



- It has been known for a long time that this is not the case at LO
  Due to dispersion relations, any GPD vanishing on $x = \pm\xi$ would not contribute to DVCS at LO (neglecting D-term contributions).

The DVCS deconvolution problem I

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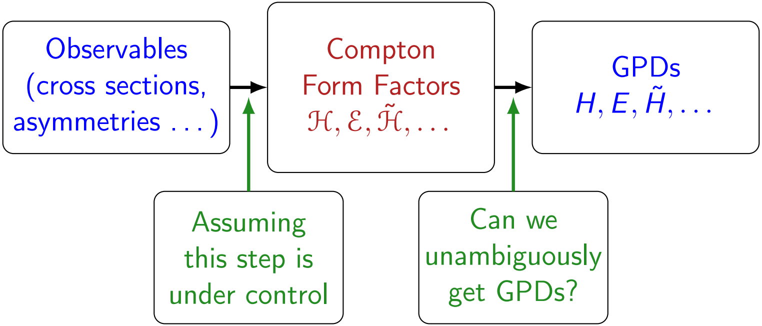

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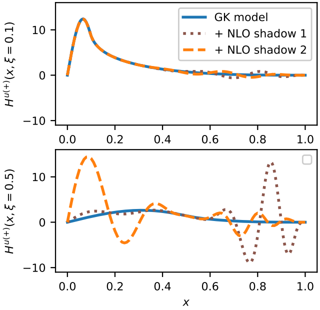
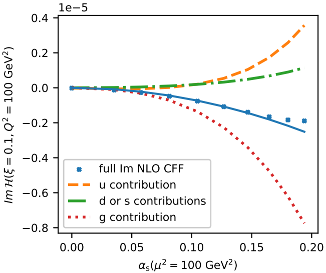
  Due to dispersion relations, any GPD vanishing on $x = \pm\xi$ would not contribute to DVCS at LO (neglecting D-term contributions).
- Are QCD corrections improving the situation?

The DVCS deconvolution problem II

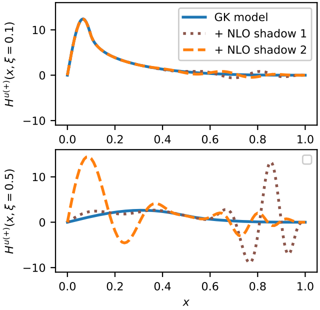


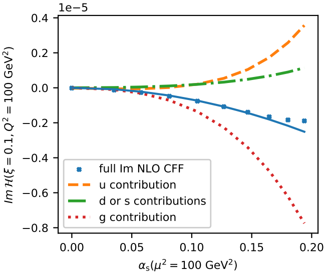


- NLO analysis of shadow GPDs:
  - Cancelling the line $x=\xi$ is necessary but **no longer** sufficient
  - Additional conditions brought by NLO corrections reduce the size of the “shadow space”...
  - ... but do not reduce it to 0
    $\rightarrow$ NLO shadow GPDs

H. Dutrieux *et al.*, PRD 103 114019 (2021)

The DVCS deconvolution problem II





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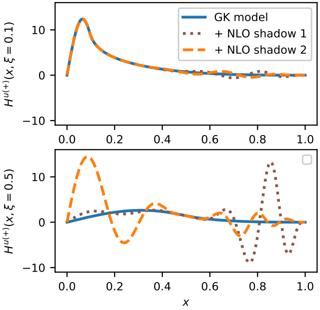
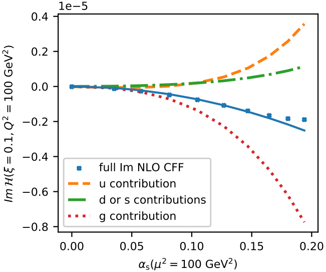
- Evolution
  - it was argued that evolution would solve this issue

    A. Freund PLB 472, 412 (2000)
  - but in practice it is not the case

    H. Dutrieux *et al.*, PRD 103 114019 (2021)

The DVCS deconvolution problem II





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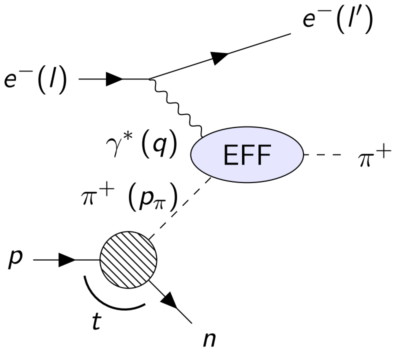
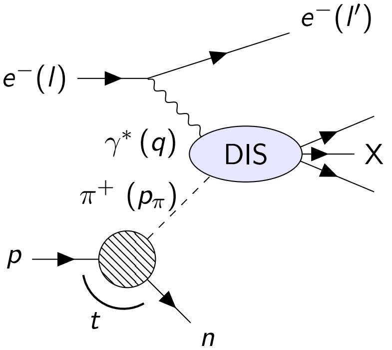
Multichannel Analysis required to fully determine GPDs

Sullivan processes





- Tested at JLab 6
  Huber *et al.*,PRC78, 045203
- Planned for JLab 12
  Aguilar *et al.*, EPJA 55 10, 190
- Envisioned at EIC and EicC
  see EIC Yellow Report and EicC white paper

- Not done at JLab 6
- Planned for JLab 12
  Aguilar *et al.*, EPJA 55 10, 190
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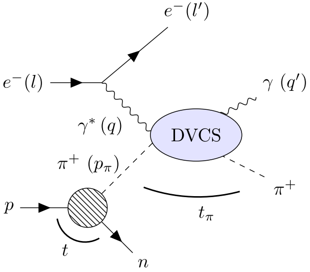
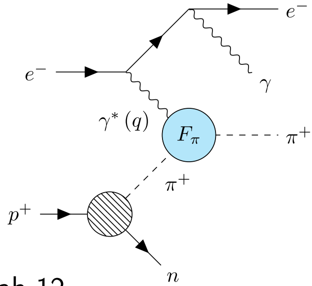
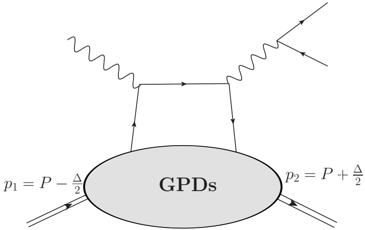
DVCS on virtual Pion Target



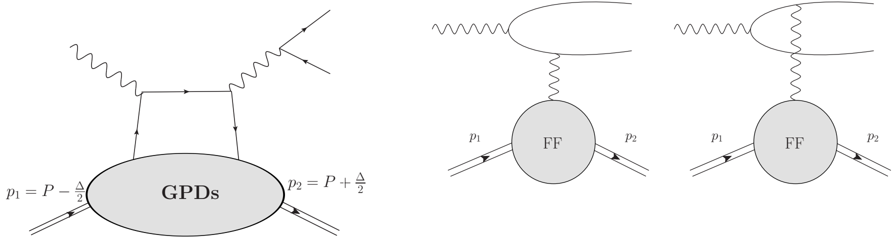


- Question already raised in 2008 for JLab 12.
  Amrath *et al.*, EPJC 58, 179-192
- Would such processes be measurable at the future EIC and EicC? Answering the question of measurability of DVCS requires:
  - A pion GPD model
  - An evolution code
  - A phenomenological code able to compute amplitudes from GPDs
  - An event generator simulating how many events could be detected

Timelike Compton Scattering



- Amplitude related to the DVCS one ($Q^2 \to -Q^2, \ldots$)
  $\to$ theoretical development for DVCS can be extended to TCS
  E. Berger *et al.*, EPJC 23 (2002) 675
- Excellent test of GPD universality but not the best option to solve the deconvolution problem



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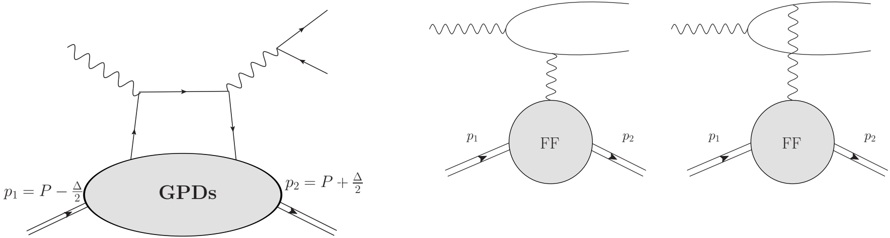

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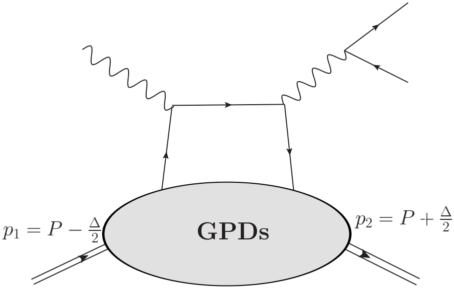
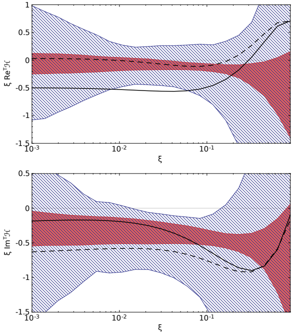
- Interferes with the Bethe-Heitler (BH) process
- Same type of final states as exclusive quarkonium production





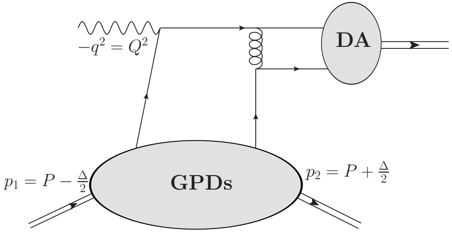
O. Grocholski *et al.*, EPJC 80, (2020) 61

- DVCS Data-driven prediction for TCS at LO and NLO
- First experimental measurement at JLab through forward-backward asymmetry (interference term)

P. Chatagnon *et al.*,arXiv:2108.11746

- Measurable at the LHC in UPC ?

Deep Virtual Meson Production



- Factorization proven for γ_L^*
  J. Collins *et al.*, PRD 56 (1997) 2982-3006
- Same GPDs than previously
- Depends on the meson DA
- Formalism available at NLO
  D. Müller *et al.*, Nucl.Phys.B 884 (2014) 438-546

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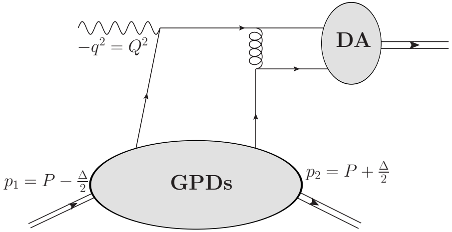

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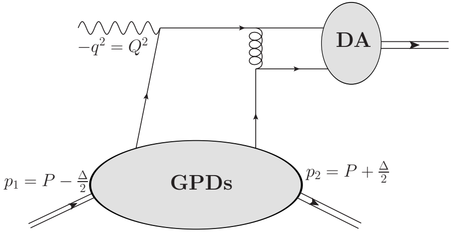
- Mesons can act as filters:
  - Select singlet (V_L), non-singlet (pseudo-scalar mesons) contributions or chiral-odd distributions (V_T)
  - Help flavour separation
  - Leading-order access to gluon GPDs

Deep Virtual Meson Production



- Factorization proven for γ_L^*
  J. Collins *et al.*, PRD 56 (1997) 2982-3006
- Same GPDs than previously
- Depends on the meson DA
- Formalism available at NLO
  D. Müller *et al.*, Nucl.Phys.B 884 (2014) 438-546

- Mesons can act as filters:
  - Select singlet (V_L), non-singlet (pseudo-scalar mesons) contributions or chiral-odd distributions (V_T)
  - Help flavour separation
  - Leading-order access to gluon GPDs
- Factorisation proven $\neq$ factorisation visible at achievable Q^2
  - Leading-twist dominance at a given Q^2 is process-dependent
    $\rightarrow$ for DVMP it can change between mesons.
  - At JLab kinematics, higher-twist contributions are very strong
    $\rightarrow$ hide factorisation of σ_L

Status of DVMP

- π^0 electroproduction
  - $\sigma_T > \sigma_L$ at JLab 6 and likely at JLab 12 kinematics ($Q^2 = 8.3 GeV^2$)
    M. Dlamini *et al.*, arXiv:2011.11125
  - No extraction of σ_L at JLab 12 yet
  - Model-dependent treatment of σ_T using higher-twist contributions
    S. V. Goloskokov and P. Kroll, EPJC 65, 137 (2010)
    G. Goldstein *et al.*, PRD 91 (2015) 11, 114013

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- ρ^0 electroproduction
  - $\sigma_T = \sigma_L$ for $Q^2 \simeq 1.5 GeV^2$ and $\frac{\sigma_L}{\sigma_T}$ increases with Q^2
    see *e.g.* L. Favart, EPJA 52 (2016) 6, 158
  - $\sigma_T \neq 0$ though $\rho_{0;T}$ production vanishes at leading twist
    $\rightarrow$ No LT access to chiral-odd GPDs.
    M. Diehl *et al.*, PRD 59 (1999) 034023
  - Sizeable higher-twist effects need to be understood
    I. Anikin *et al.*, PRD 84 (2011) 054004

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DVMP is as interesting as challenging
Additional data would be more than welcome

PARTONS and Gepard

PARTONS

partons.cea.fr

B. Berthou *et al.*, EPJC 78 (2018) 478

Gepard

calculon.phy.hr/gpd/server/index.html

K. Kumericki, EPJ Web Conf. 112 (2016) 01012

- Similarities : NLO computations, BM formalism, ANN, ...
- Differences : models, evolution, ...

Physics impact

These integrated softwares are the mandatory path toward reliable multichannel analyses.

Conclusion

Summary

- Introduction to GPDs and their place in hadron structure studies
- Focus on two important properties: polynomiality and positivity
- Evolution of GPD
- Connection to experimental processes

Conclusion

- GPD field is as complicated as interesting
- Many theoretical and phenomenological works remain required
- Forthcoming facilities will likely shed new light on them
- Progresses in ab-initio computations (continuum and lattice) expected to be significant in the forthcoming years

Thank you for your attention !

Some final questions ?