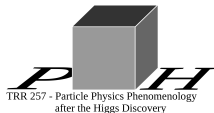


# The pion-photon transition form factor at two loops in QCD

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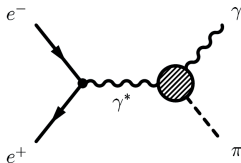
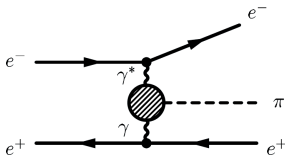
- Pion-photon transition form factor: theoretically (one of) the simplest hadronic matrix elements

$$\langle \pi(p) | j_{\mu}^{\text{em}} | \gamma(p') \rangle = g_{\text{em}}^2 \epsilon_{\mu\nu\alpha\beta} q^{\alpha} p^{\beta} \epsilon^{\nu}(p') F_{\gamma^* \gamma \rightarrow \pi^0}(Q^2)$$

- Ideally suited for
  - precision studies of the partonic landscape of composite hadrons
  - investigating the factorization properties of hard exclusive QCD reactions
- First studies date back half a century (before QCD!)  
[Cornwall'66; Gross, Treiman'71; Brodsky, Kinoshita, Terazawa'71]
  - Later one-loop and subleading power-corrections  
[del Aguila, Chase'81; Braaten'83; Kadantseva, Mikhailov, Radyushkin'86; Shen, Wang'17]
- Also with connections to
  - HLbL contribution to  $(g-2)_{\mu}$  in the dispersive framework (double-virtual FF)
  - exclusive B-meson decays such as  $B \rightarrow \pi \ell \nu$  or  $B \rightarrow \pi \pi$  (pion LCDA)

# Intro / motivation

- Experimental measurements in  $e^+e^-$  collisions



- Status of experimental measurements

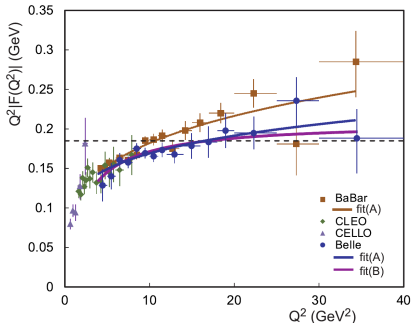
[figures from Wang'18]

- Asymptotic limit (dashed line)

$$\lim_{Q^2 \rightarrow \infty} Q^2 F_{\gamma^* \gamma \rightarrow \pi^0}(Q^2) = \sqrt{2} f_\pi$$

[Brodsky, Lepage'80]

- Scaling violation?



# The pion-photon transition form factor

- Pion-photon transition form factor  $F_{\gamma^* \gamma \rightarrow \pi^0}$
- extracted from matrix element of e.m. current  $j_\mu^{\text{em}} = \sum_q g_{\text{em}} Q_q \bar{q} \gamma_\mu q$  between on-shell photon of momentum  $p'$  and pion of momentum  $p$

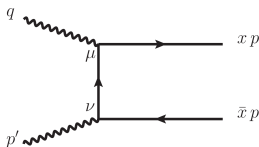
$$\langle \pi(p) | j_\mu^{\text{em}} | \gamma(p') \rangle = g_{\text{em}}^2 \epsilon_{\mu\nu\alpha\beta} q^\alpha p^\beta \epsilon^\nu(p') F_{\gamma^* \gamma \rightarrow \pi^0}(Q^2)$$

- Have  $q = p - p'$  and  $Q^2 = -q^2$
- Kinematics at leading power

$$p'_\mu = (n p') \frac{\bar{n}_\mu}{2} \quad p_\mu = (\bar{n} p) \frac{n_\mu}{2} \quad (n p') \sim (\bar{n} p) \sim \mathcal{O}(\sqrt{Q^2})$$

- Goal: Establish QCD factorization formula for  $F_{\gamma^* \gamma \rightarrow \pi^0}(Q^2)$  at leading power

# The pion-photon transition form factor



- Consider four-point QCD matrix element ( $x$ : momentum fraction,  $\bar{x} \equiv 1 - x$ )

$$\Pi_\mu = \langle q(xp) \bar{q}(\bar{x}p) | j_\mu^{\text{em}} | \gamma(p') \rangle$$

- Allows to derive factorization formula for  $F_{\gamma^* \gamma \rightarrow \pi^0}$  at leading power

$$F_{\gamma^* \gamma \rightarrow \pi^0}^{\text{LP}}(Q^2) = \frac{(Q_u^2 - Q_d^2) f_\pi}{\sqrt{2} Q^2} \int_0^1 dx T_2(x) \phi_\pi(x, \mu)$$

- $T_2(x)$ : hard function, computable in perturbation theory
- $\phi_\pi(x, \mu)$ : Leading twist pion light-cone distribution amplitude (LCDA)

$$\langle \pi(p) | \bar{\xi}(y) W_c(y, 0) \gamma_\mu \gamma_5 \xi(0) | 0 \rangle = -i f_\pi p_\mu \int_0^1 du e^{i u p \cdot y} \phi_\pi(u, \mu) + \mathcal{O}(y^2)$$

# Matching onto SCET

- Goal: Extract hard function  $T_2^{(L)}$  at given loop-order  $L$ .
- Introduce SCET operator basis:  $\{O_1^{\mu\nu}(x), O_2^{\mu\nu}(x), O_E^{\mu\nu}(x)\}$

$$O_j^{\mu\nu}(x) = \frac{\bar{n} \cdot p}{2\pi} \int d\tau e^{i x \tau \bar{n} \cdot p} \bar{\xi}(\tau \bar{n}) W_c(\tau \bar{n}, 0) \Gamma_j^{\mu\nu} \xi(0),$$

- Dirac structures

$$\Gamma_{1, \mu\nu} = g_{\mu\nu}^\perp \not{\bar{n}}, \quad \Gamma_{2, \mu\nu} = i \epsilon_{\mu\nu}^\perp \not{\bar{n}} \gamma_5,$$

$$\Gamma_{E, \mu\nu} = \not{\bar{n}} \left( \frac{1}{2} [\gamma_{\mu, \perp}, \gamma_{\nu, \perp}] - i \epsilon_{\mu\nu}^\perp \gamma_5 \right).$$

- For our calculation  $O_2^{\mu\nu}(x)$  is the relevant operator
  - $O_E^{\mu\nu}(x)$  is evanescent
  - $O_1^{\mu\nu}(x)$  cannot couple to a collinear pion state (parity). Relevant for DVCS.

# Matching onto SCET

- Start with

$$\langle j^{\text{em}} \rangle = \sum_{a=1,2,E} T_a \langle O_a \rangle \quad (1)$$

- Expand both sides in  $\tilde{\alpha}_s = \alpha_s/(4\pi)$ , superscripts give loop order

$$T_a = T_a^{(0)} + \tilde{\alpha}_s T_a^{(1)} + \tilde{\alpha}_s^2 T_a^{(2)} + \dots$$

$$\langle j^{\text{em}} \rangle = \sum_{a=1,2,E} \left\{ A_a^{(0)} + \tilde{\alpha}_s A_a^{(1)} + \tilde{\alpha}_s^2 [A_a^{(2)} + Z_\alpha^{(1)} A_a^{(1)}] + \dots \right\} \langle O_a \rangle^{(0)}$$

- On-shell matrix elements of  $\langle O_a \rangle$  simplify due to scaleless integrals

$$\langle O_a \rangle = \sum_{b=1,2,E} \left\{ \delta_{ab} + \tilde{\alpha}_s Z_{ab}^{(1)} + \tilde{\alpha}_s^2 Z_{ab}^{(2)} + \dots \right\} \langle O_b \rangle^{(0)}$$

- Compare coefficients of  $\langle O_i \rangle^{(0)}$  on both sides of (1) at given order in  $\tilde{\alpha}_s$ .

# Matching onto SCET: Master formulas

- Tree level  $T_a^{(0)} = A_a^{(0)}$

- One loop

$$T_2^{(1)} = A_2^{(1)} - T_2^{(0)} * Z_{22}^{(1)} - T_E^{(0)} * Z_{E2}^{(1)}$$

$$T_E^{(1)} = A_2^{(1)} - T_2^{(0)} * Z_{22}^{(1)} = T_2^{(1)} + \underbrace{T_E^{(0)} * Z_{E2}^{(1)}}_{\text{finite shift}}$$

- Asterisk denotes convolution, due to non-locality of SCET operators  $O_a$ .
- Have used that

$$A_2^{(L)} = A_E^{(L)}$$

$$Z_{1E}^{(1)} = Z_{2E}^{(1)} = 0$$

$$Z_{12}^{(1)} = 0 \text{ (parity)}$$

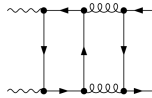
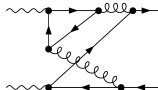
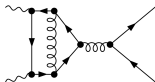
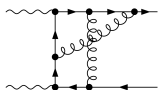
$$Z_{EE}^{(1)} = Z_{22}^{(1)}$$

- Two loops

$$T_2^{(2)} = A_2^{(2)} + Z_\alpha^{(1)} A_2^{(1)} - \sum_{\alpha=2,E} \left[ T_\alpha^{(1)} * Z_{\alpha 2}^{(1)} + T_\alpha^{(0)} * Z_{\alpha 2}^{(2)} \right]$$



# Two-loop calculation



- Generate diagrams with FeynArts and in-house routine

[Hahn'00+]

- Many diagrams vanish

- Zero color factor
- Furry's theorem
- projection onto the  $\pi^0$  wave-function  $\propto |u\bar{u}\rangle - |d\bar{d}\rangle$

- 42 diagrams (plus  $\gamma \leftrightarrow \gamma^*$ ) remain

- Use dimensional regularization with  $D = 4 - 2\epsilon$ , NDR for  $\gamma_5$

- Perform Dirac reduction to  $\Gamma_{1,2,E}^{\mu\nu}$

- In individual diagrams two additional Dirac structures appear,

$$n^\mu n^\nu \bar{\not{n}} \text{ and } \bar{n}^\mu n^\nu \bar{\not{n}}$$

# Two-loop calculation

- Reduction of scalar integrals: IBP relations, Laporta algorithm in FIRE

[Tkachov'81; Chetyrkin,Tkachov'81] [Laporta'01; Smirnov'08]

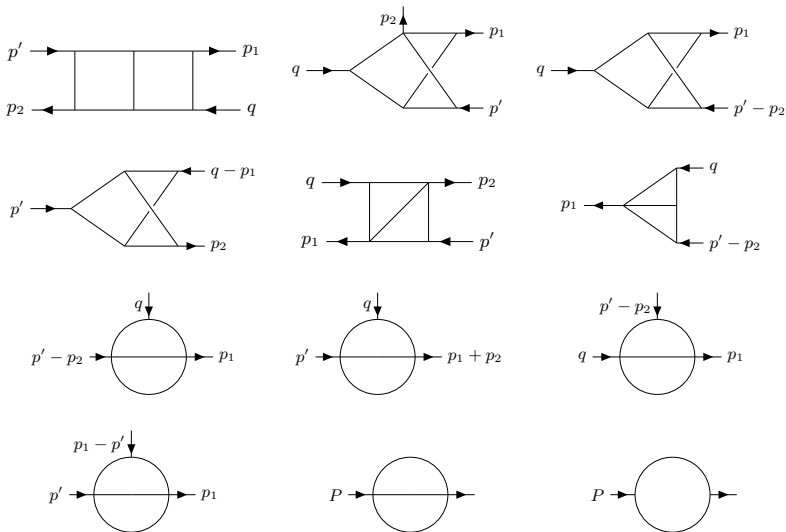
- Reduction of most complicated diagram takes  $\sim 1$  day
- In addition, exploit relations based on momentum conservation
  - Required since  $p_1 = x p$  and  $p_2 = \bar{x} p$  are parallel

$$\bar{x}k_1^2 - \bar{x}(k_1 + p_1)^2 + xk_2^2 + x(k_2 - k_1 - p_2)^2 - x(k_1 - k_2)^2 - x(k_2 - p_2)^2 = 0$$

$$k_2^2 - x(k_2 - p_2)^2 - \bar{x}(k_2 + p_1)^2 = 0$$

- Reduction yields (only) 12 master integrals
- After IBP reduction, Dirac structures  $n^\mu n^\nu \bar{\eta}$  and  $\bar{n}^\mu n^\nu \bar{\eta}$  drop out identically
  - Manifestation of QED Ward identity

# Master integrals



# Evaluating Master integrals

- Closed forms, valid to all orders in  $\epsilon$ . Gives  $\Gamma$ - and hypergeometric functions.
  - Expand with `HypExp` [Maître, TH'05'07]
- Method of differential equations [Kotikov'90'91; Remiddi'97]
  - Partially in canonical form [Henn'13]
- Mellin Barnes representations [Czakon'05; Gluza, Riemann et al.'07+; Kosower'09]
  - compute boundary conditions for DEs as  $x \rightarrow 0$  or  $x \rightarrow 1$  [Czakon'06]
  - derive full  $x$ -dependence of analytic functions
- Numerical checks with FIESTA [Tentyukov, Smirnov'08+]
- Obtain analytic  $\epsilon$ -expansion of all master integrals
  - Harmonic polylogarithms (HPLs) of weights 0, 1 [Remiddi, Vermaseren'99]
  - HPLs of at most weight four appear in the amplitude.

- Recall master formula

$$T_2^{(2)} = A_2^{(2)} + Z_\alpha^{(1)} A_2^{(1)} - \sum_{a=2,E} \left[ T_a^{(1)} * Z_{a2}^{(1)} + T_a^{(0)} * Z_{a2}^{(2)} \right]$$

- $Z_{22}^{(1)}$  is the one-loop ERBL kernel

[Efremov,Radyushkin'80; Brodsky,Lepage'80]

$$Z_{22}^{(1)}(x, x') = -\frac{2C_F}{\epsilon} \left[ \frac{\bar{x}}{\bar{x}'} \left( 1 + \frac{1}{x-x'} \right) \theta(x-x') + (x \leftrightarrow \bar{x}, x' \leftrightarrow \bar{x}') \right]_+$$

- $Z_{22}^{(2)}$  is the two-loop ERBL kernel

$$Z_{22}^{(2)}(x, x') = \frac{1}{2\epsilon^2} \left\{ Z_{22}^{(1)}(x, x'') * Z_{22}^{(1)}(x'', x') - \beta_0 Z_{22}^{(1)}(x, x') \right\} \\ + \frac{1}{2\epsilon} \left\{ 2n_f C_F V_F + 2C_F C_A V_G + C_F^2 V_F \right\}_+$$

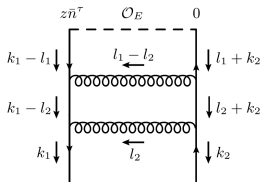
- Need lower-loop quantities beyond  $\mathcal{O}(\epsilon^0)$ .

# Infrared subtraction

- Most delicate point:

Evanescent  $\rightarrow$  physical mixing

at two loops,  $Z_{E2}^{(2)}$



- Insertion of non-local evanescent operator, regularize IR divergences

$$I_{(2),A}^{\mu\nu} = C_F^2 (\bar{n} \cdot k) \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-itzk \cdot \bar{n}} \int \frac{d^d l_1 d^d l_2}{(2\pi)^{2d}} \bar{u}(k_1) (ig\gamma^\rho) \frac{i(k_1' - l_2')}{(k_1 - l_2)^2} (ig\gamma^\alpha) \frac{i(k_1' - l_1')}{(k_1 - l_1)^2} \\ \times O_E^{\mu\nu} \frac{-i(l_1' + k_2')}{(l_1 + k_2)^2} (ig\gamma^\beta) \frac{-i(l_2' + k_2')}{(l_2 + k_2)^2} (ig\gamma^\sigma) v(k_2) \frac{-ig_{\rho\sigma}}{l_2^2} \frac{-ig_{\alpha\beta}}{(l_1 - l_2)^2} e^{iz(k_1 - l_1) \cdot \bar{n}}$$

- Take coefficient of  $O_2^{\mu\nu}$ , Fourier transform

$$I_{(2),A}^{\mu\nu} = 8a_s^2 C_F^2 \left\{ \frac{\bar{t}}{\bar{x}} [3 + 2L_m] + \frac{t \ln t}{\bar{x}} [2(1 + L_m) + \ln t] + \frac{\bar{t}}{x} \ln \bar{x} [2(1 + L_m + \ln \bar{t}) - \ln \bar{x}] \right\} \theta(t - x) + (t \rightarrow \bar{t}, x \rightarrow \bar{x})$$

- IR regulator drops in sum of all diagrams

- Sum everything up, poles in  $\epsilon$  cancel, extract  $T_2^{(2)}$

$$T_2^{(2)} = \beta_0 C_F \left( \mathcal{K}_\beta^{(2)}(x)/x + \mathcal{K}_\beta^{(2)}(\bar{x})/\bar{x} \right) + C_F^2 \left( \mathcal{K}_F^{(2)}(x)/x + \mathcal{K}_F^{(2)}(\bar{x})/\bar{x} \right) + C_F/N_c \left( \mathcal{K}_N^{(2)}(x)/x + \mathcal{K}_N^{(2)}(\bar{x})/\bar{x} \right)$$

[agrees with Braun,Manashov,Moch,Schoenleber'21]

where

$$\begin{aligned} \mathcal{K}_\beta^{(2)}(x) = & -L^2 \left( H_0(x) + \frac{3}{2} \right) + L \left( -\frac{10}{3}H_0(x) - H_1(x) + 2H_{0,0}(x) - 2H_{1,0}(x) - 2\zeta_2 - \frac{19}{2} \right) - \zeta_2 H_1(x) - \frac{19}{9}H_0(x) \\ & - \frac{1}{2}H_1(x) + \frac{10}{3}H_{0,0}(x) - \frac{14}{3}H_{1,0}(x) - H_{1,1}(x) - 2H_{0,0,0}(x) + 2H_{1,0,0}(x) - H_{1,1,0}(x) - \frac{14}{3}\zeta_2 - \zeta_3 - \frac{457}{24} \end{aligned}$$

$$\begin{aligned} \mathcal{K}_F^{(2)}(x) = & L^2 \left( 6H_0(x) - 2H_1(x) + 4H_{0,0}(x) + 2H_{1,0}(x) + \frac{9}{2} \right) + L \left( 8\zeta_2 H_0(x) + \frac{38}{3}H_0(x) + 4\zeta_2 H_1(x) - 17H_1(x) \right. \\ & \left. - 6H_{0,0}(x) + 8H_{1,0}(x) - 2H_{1,1}(x) - 12H_{0,0,0}(x) + 4H_{1,1,0}(x) - 4\zeta_3 + 6\zeta_2 + \frac{47}{2} \right) + 6\zeta_2 H_0(x) + 4\zeta_2 H_1(x) \\ & - 2\zeta_2 H_2(x) - 8\zeta_2 H_{0,0}(x) - 2\zeta_2 H_{1,0}(x) + 2\zeta_2 H_{1,1}(x) + 32\zeta_3 H_0(x) - 4\zeta_3 H_1(x) - \frac{64}{9}H_0(x) - \frac{71}{2}H_1(x) \\ & - \frac{38}{3}H_{0,0}(x) + \frac{34}{3}H_{1,0}(x) - 11H_{1,1}(x) - 2H_{1,2}(x) - 8H_{1,0,0}(x) + 4H_{1,1,0}(x) - 2H_{1,1,1}(x) \\ & - 2H_{1,2,0}(x) - 4H_{2,0,0}(x) - 2H_{2,1,0}(x) + 12H_{0,0,0,0}(x) - 2H_{1,0,0,0}(x) - 2H_{1,1,0,0}(x) \\ & + 2H_{1,1,1,0}(x) + 3\zeta_2^2 + \frac{34}{3}\zeta_2 + 39\zeta_3 + \frac{701}{24} \end{aligned}$$

- Sum everything up, poles in  $\epsilon$  cancel, extract  $T_2^{(2)}$

$$T_2^{(2)} = \beta_0 C_F \left( \mathcal{K}_\beta^{(2)}(x)/x + \mathcal{K}_\beta^{(2)}(\bar{x})/\bar{x} \right) + C_F^2 \left( \mathcal{K}_F^{(2)}(x)/x + \mathcal{K}_F^{(2)}(\bar{x})/\bar{x} \right) + C_F/N_c \left( \mathcal{K}_N^{(2)}(x)/x + \mathcal{K}_N^{(2)}(\bar{x})/\bar{x} \right)$$

[agrees with Braun,Manashov,Moch,Schoenleber'21]

where

$$\begin{aligned} \mathcal{K}_N^{(2)}(x) = & L \left( 4\zeta_2 H_0(x) - \frac{8}{3} H_0(x) - 4H_1(x) - 4H_3(x) + 4H_{2,0}(x) + 12\zeta_3 - 1 \right) + 12x (\zeta_2 H_0(x) - H_3(x) + H_{2,0}(x)) \\ & - 6\zeta_2 H_1(x) - 4\zeta_2 H_2(x) - 4\zeta_2 H_{0,0}(x) + 4\zeta_2 H_{1,0}(x) + 2\zeta_2 H_{1,1}(x) + 14\zeta_3 H_0(x) - \frac{32}{9} H_0(x) + 11H_1(x) \\ & + 4H_2(x) + 8H_4(x) + \frac{8}{3} H_{0,0}(x) + \frac{2}{3} H_{1,0}(x) + 2H_{1,1}(x) + 6H_{1,2}(x) - 2H_{1,3}(x) + 6H_{2,2}(x) - 4H_{3,0}(x) \\ & - 4H_{3,1}(x) - 6H_{1,1,0}(x) - 2H_{1,1,2}(x) + 4H_{1,2,0}(x) - 6H_{2,0,0}(x) - 4H_{2,1,0}(x) - 2H_{1,1,0,0}(x) \\ & + 2H_{1,1,1,0}(x) + \frac{1}{5} \zeta_2^2 - \frac{22}{3} \zeta_2 + 54\zeta_3 - \frac{73}{12} \end{aligned}$$

- The terms proportional to  $x$  are reproducible directly from the coefficient function in DIS.

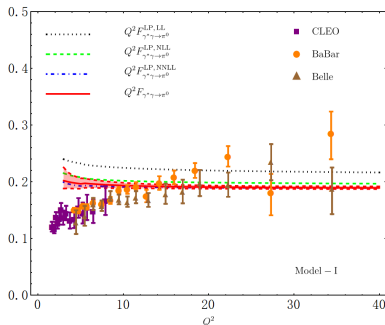
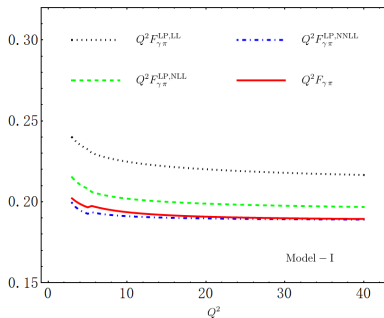


# Numerical results

- Need to model the pion LCDA, choose five models
- Use three-loop evolution of pion LCDA, expand to first 12 Gegenbauer moments

- Model I: 
$$\phi_\pi(x, \mu_0) = \frac{\Gamma(2 + 2\alpha_\pi(\mu_0))}{\Gamma^2(1 + \alpha_\pi(\mu_0))} (x \bar{x})^{\alpha_\pi(\mu_0)}, \quad \alpha_\pi(1 \text{ GeV}) = 0.422^{+0.076}_{-0.067}$$

[Khodjamirian, Melic, Wang, Wei'20]



- Red line includes subleading power corrections (twist 4, hadronic photon effect)

[Shen, Wang'17]

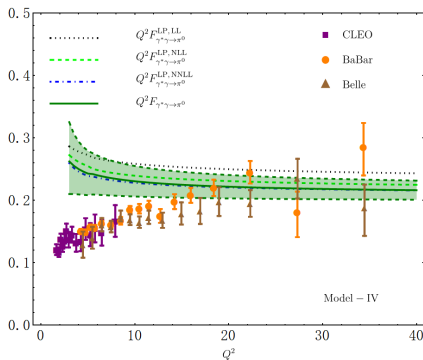
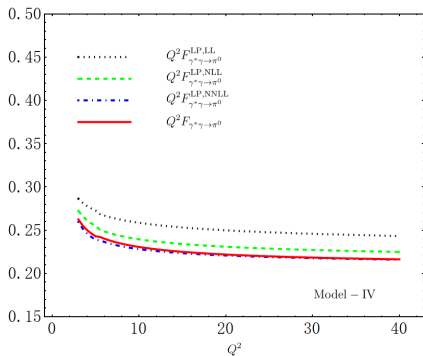
- Only perturbative uncertainties are shown

# Numerical results

● Model IV:  $a_2(1 \text{ GeV}) = 0.269 \pm 0.047$   
 $a_6(1 \text{ GeV}) = 0.141 \pm 0.096$

$a_4(1 \text{ GeV}) = 0.185 \pm 0.062$   
 $a_8(1 \text{ GeV}) = 0.049 \pm 0.116$

[Cheng, Khodjamirian, Rusov'20]

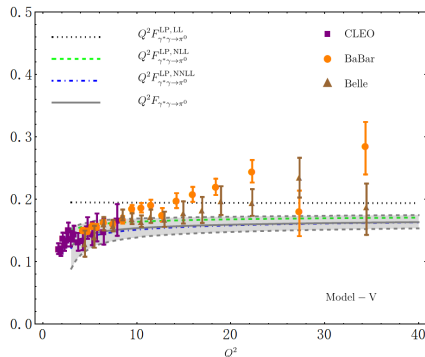
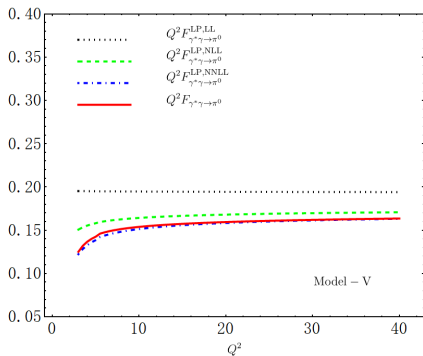


# Numerical results

● Model V:  $a_2(1 \text{ GeV}) = 0.203_{-0.057}^{+0.069}$

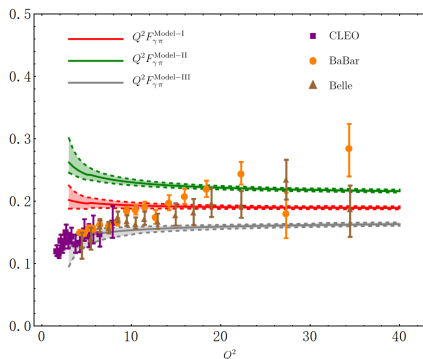
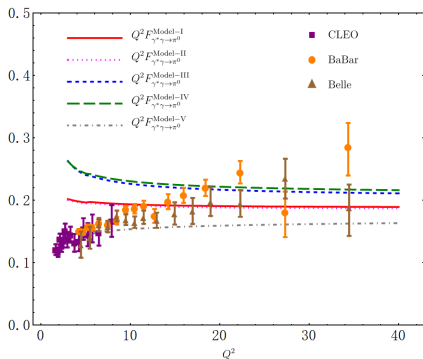
$a_4(1 \text{ GeV}) = -0.143_{-0.087}^{+0.094}$

[Bakulev,Mikhailov,Stefanis'01,Stefanis'20]



# Numerical results

- Comparison between models
- Only perturbative uncertainties are shown



- Belle II data will allow to distinguish between LCDA models

# Conclusion and Outlook

- We computed the pion-photon transition form factor to two loops in QCD
  - Prime example of QCD factorization in hard exclusive processes
  - Two-loop computation of bare amplitude uses standard multi-loop methods
  - IR subtraction non-trivial due to evanescent-physical mixing at two loops
  - Studied models of pion LCDA, comparison to data
- Future plans, e.g.
  - Include massive quarks in fermion loops
  - Take second photon off-shell