Renormalization of non-singlet quark OMEs for DIS

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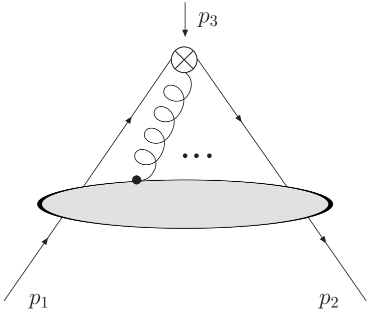
Introduction

- In QCD: Factorization between hard scales (e.g. hard partonic cross-sections, perturbative) and soft scales (e.g. PDFs, non-perturbative)
- Important non-perturbative information from operator matrix elements (OMEs)

$$\langle\psi(p_1)|\,\mathcal{O}(p_3)\,|\bar{\psi}(p_2)\rangle$$



Introduction

OMEs describe different physical quantities depending on momentum flow, e.g.

- Forward kinematics, $\langle \psi(p) | \mathcal{O}(0) \left| \bar{\psi}(-p) \right\rangle$: PDFs
- Off-forward kinematics, $\langle \psi(p_1) | \mathcal{O}(p_3) \left| \bar{\psi}(p_2) \right\rangle$: GPDs and pion form factor

All these non-perturbative quantities can be extracted from experiment (e.g. DIS and DVCS) or lattice QCD (e.g. [1, 2] and for recent progress [3, 4, 5, 6, 7, 8])

The scale dependence of these distributions can be calculated perturbatively (RGE)

$$\frac{d\mathcal{O}}{d\log\mu^2} = -\gamma\mathcal{O}, \;\; \gamma \equiv a_s\gamma^{(0)} + a_s^2\gamma^{(1)} + ...$$

During this talk: Off-forward kinematics

Operator definitions

We are interested in spin-N non-singlet quark operators of the form

$$O^{NS}_{\mu_1 \dots \mu_N} = \mathcal{S}\bar{\psi}\lambda^{\alpha}\gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N}\psi, \;\; D_{\mu} = \partial_{\mu} - ig_s A_{\mu},$$

which e.g. show up in the OPE analysis of DIS. In practice we consider

$$\Delta^{\mu_1} \dots \Delta^{\mu_N} \mathcal{O}^{NS}_{\mu_1 \dots \mu_N} \equiv \mathcal{O}_N, \;\; \Delta^2 = 0$$

Because of the non-zero momentum flow through the operator-vertex, these will mix with total-derivative operators

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} = \mathcal{S}\, \partial^{\mu_1} \dots \partial^{\mu_p} \left((D^{\nu_1} \dots D^{\nu_q} \overline{\psi}) \, \lambda^{\alpha} \gamma_{\mu} \, (D^{\sigma_1} \dots D^{\sigma_r} \psi) \right),$$
$$p + q + r = N - 1$$

Choice of basis for total-derivative operators is not unique!
This one: (Total) derivative basis $\mathcal{D}$ (see e.g. [9, 10])

Operator renormalization

In the chosen basis the operators renormalize as

$$\mathcal{O}_{N+1} = Z_\psi \left(Z_{N,N}[\mathcal{O}_{N+1}] + Z_{N,N-1}[\partial \mathcal{O}_N] + \cdots + Z_{N,0}[\partial^N \mathcal{O}_1] \right)$$

For the full spin-(N+1) sector:

$$\begin{pmatrix} \mathcal{O}_{N+1} \\ \partial \mathcal{O}_N \\ \vdots \\ \partial^N \mathcal{O}_1 \end{pmatrix} = Z_\psi \begin{pmatrix} Z_{N,N} & Z_{N,N-1} & \dots & Z_{N,0} \\ 0 & Z_{N-1,N-1} & \dots & Z_{N-1,0} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & Z_{0,0} \end{pmatrix} \begin{pmatrix} [\mathcal{O}_{N+1}] \\ [\partial \mathcal{O}_N] \\ \vdots \\ [\partial^N \mathcal{O}_1] \end{pmatrix}$$

Anomalous dimension matrix

We are interested in the corresponding anomalous dimension matrix (ADM)

$$\gamma^{\mathcal{D}}_{N,k} = -\left(\frac{d}{d \ln \mu^2} Z_{N,j} \right) Z^{-1}_{j,k}$$

and

$$\begin{pmatrix} \gamma_{N,N} & \gamma^{\mathcal{D}}_{N,N-1} & \cdots & \gamma^{\mathcal{D}}_{N,0} \\ 0 & \gamma_{N-1,N-1} & \cdots & \gamma^{\mathcal{D}}_{N-1,0} \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \gamma_{0,0} \end{pmatrix}$$

The diagonal elements correspond to the forward anomalous dimensions. The off-diagonal elements are a priori unknown (only low-N results known in literature up to 3-loops in $\mathcal{D}$-basis [11, 12])

Is it possible to determine the full mixing matrix for arbitrary N?

Diagonal (forward) anomalous dimensions

$$\begin{pmatrix} \gamma_{N,N} & \gamma^{\mathcal{D}}_{N,N-1} & \cdots & \gamma^{\mathcal{D}}_{N,0} \\ 0 & \gamma_{N-1,N-1} & \cdots & \gamma^{\mathcal{D}}_{N-1,0} \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \gamma_{0,0} \end{pmatrix}$$

The scale dependence of the PDFs is determined by the DGLAP equation

$$\frac{\mathrm{d} f_a(x,\mu^2)}{\mathrm{d}\log\mu^2} = [P_{ab} \otimes f_b](x),$$

and the splitting functions are related to the forward anomalous dimensions

$$\gamma_{NS}(N) \equiv \gamma_{N-1,N-1} = -\int_0^1 \mathrm{d}x\, x^{N-1} P_{NS}(x)$$

Known completely up to the 3-loop level, and in certain limits up to the 5-loop level [13, 14, 15, 16, 17, 18, 19, 20, 21]

Constraints on the anomalous dimensions

Working in the chiral limit, left- and right-derivative operators[1] renormalize with the same renormalization constants

$$\mathcal{O}^{\mathcal{D}}_{p,0,r} = \sum_{j=0}^{r} Z_{r,r-j}\,[\mathcal{O}^{\mathcal{D}}_{p+j,0,r-j}]\,,$$

$$\mathcal{O}^{\mathcal{D}}_{p,q,0} = \sum_{j=0}^{q} Z_{q,q-j}\,[\mathcal{O}^{\mathcal{D}}_{p+j,q-j,0}]$$

and total derivatives act as

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} \;=\; \mathcal{O}^{\mathcal{D}}_{p-1,q+1,r} + \mathcal{O}^{\mathcal{D}}_{p-1,q,r+1},$$

e.g.

$$\mathcal{O}^{\mathcal{D}}_{0,N,0} - (-1)^N \sum_{j=0}^{N} (-1)^j \binom{N}{j} \mathcal{O}^{\mathcal{D}}_{j,0,N-j} \;=\; 0\,.$$

[1] $\mathcal{O}^{\mathcal{D}}_{p,q,r} \;=\; \mathcal{S}\,\partial^{\mu_1}\ldots\partial^{\mu_p}\,((D^{\nu_1}\ldots D^{\nu_q}\overline{\psi})\,\lambda^{\alpha}\gamma_{\mu}\,(D^{\sigma_1}\ldots D^{\sigma_r}\psi)$

Constraints on the anomalous dimensions

These lead to a relation between sums of the elements of the mixing matrix

$$\forall k: \sum_{j=k}^{N} \left\{ (-1)^k \binom{j}{k} \gamma^{\mathcal{D}}_{N,j} - (-1)^j \binom{N}{j} \gamma^{\mathcal{D}}_{j,k} \right\} = 0$$

$$\begin{pmatrix} \gamma_{N,N} & \gamma^{\mathcal{D}}_{N,N-1} & \gamma^{\mathcal{D}}_{N,N-2} & \cdots & \gamma^{\mathcal{D}}_{N,k} & \cdots & \gamma^{\mathcal{D}}_{N,0} \\ 0 & \gamma_{N-1,N-1} & \gamma^{\mathcal{D}}_{N-1,N-2} & \cdots & \gamma^{\mathcal{D}}_{N-1,k} & \cdots & \gamma^{\mathcal{D}}_{N-1,0} \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_{k,k} & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & \gamma_{0,0} \end{pmatrix}$$

Valid to all orders in a_s

Constraints on the anomalous dimensions

Special cases:

- $k = N - 1$ [2]

$$\gamma^{\mathcal{D}}_{N,N-1} = \frac{N}{2}\left(\gamma_{N-1,N-1} - \gamma_{N,N}\right)$$

$$\begin{pmatrix} \gamma_{N,N} & \gamma^{\mathcal{D}}_{N,N-1} & \gamma^{\mathcal{D}}_{N,N-2} & \cdots & \gamma^{\mathcal{D}}_{N,k} & \cdots & \gamma^{\mathcal{D}}_{N,0} \\ 0 & \gamma_{N-1,N-1} & \gamma^{\mathcal{D}}_{N-1,N-2} & \cdots & \gamma^{\mathcal{D}}_{N-1,k} & \cdots & \gamma^{\mathcal{D}}_{N-1,0} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma_{k,k} & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \ldots & \gamma_{0,0} \end{pmatrix}$$

[2] The low-N version of this relation was pointed out to us by J. Gracey in a private communication.

Constraints on the anomalous dimensions

- $k = 0$

$$\gamma^{\mathcal{D}}_{N,0} = (-)^N \left[\sum_{i=0}^{N} \gamma^{\mathcal{D}}_{N,i} - \sum_{j=1}^{N-1} (-)^j \binom{N}{j} \gamma^{\mathcal{D}}_{j,0} \right]$$

$$\begin{pmatrix} \gamma_{N,N} & \gamma^{\mathcal{D}}_{N,N-1} & \gamma^{\mathcal{D}}_{N,N-2} & \cdots & \gamma^{\mathcal{D}}_{N,k} & \cdots & \gamma^{\mathcal{D}}_{N,0} \\ 0 & \gamma_{N-1,N-1} & \gamma^{\mathcal{D}}_{N-1,N-2} & \cdots & \gamma^{\mathcal{D}}_{N-1,k} & \cdots & \gamma^{\mathcal{D}}_{N-1,0} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma_{k,k} & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \ldots & \gamma_{0,0} \end{pmatrix}$$

Constraints on the anomalous dimensions

General case:

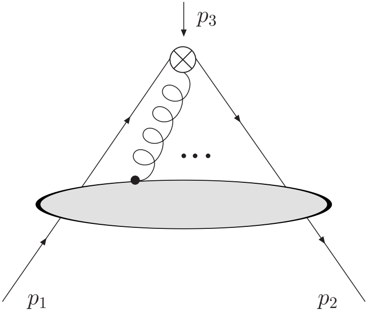
$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,j+k} + \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,j}$$

Can be used to construct the complete ADM from the knowledge of the forward anomalous dimensions $\gamma_{N,N}$ and the last column $\gamma^{\mathcal{D}}_{N,0}$!

4-step algorithm for constructing the ADM

1. Determine all-N expressions for $\gamma^{\mathcal{D}}_{N,N-1}$ and $\gamma^{\mathcal{D}}_{N,0}$
2. Calculate
$$\binom{N}{k}\sum_{j=0}^{N-k}(-1)^j\binom{N-k}{j}\gamma_{j+k,j+k}$$
and construct an Ansatz for the off-diagonal piece
3. Calculate
$$\sum_{j=k}^{N}(-1)^k\binom{j}{k}\sum_{l=j+1}^{N}(-1)^l\binom{N}{l}\gamma^{\mathcal{D}}_{l,j}$$
4. Substitute into the consistency relation $\Rightarrow$ System of equations + boundary condition for $k=0$

Feynman diagrams and $\gamma_{N,0}$



The diagrams are generated with QGRAF [22] and calculated using FORCER [23], which efficiently deals with massless propagator-type diagrams in dimensional regularization

$\rightarrow$ Nullify one of the external momenta: $\langle\psi(p)|\,\mathcal{O}(-p)\,|\bar{\psi}(0)\rangle$

Feynman diagrams and $\gamma_{N,0}$

Remember the renormalization pattern

$$\mathcal{O}_{N+1} = Z_\psi \left(Z_{N,N}[\mathcal{O}_{N+1}] + Z_{N,N-1}[\partial \mathcal{O}_N] + \cdots + Z_{N,0}[\partial^N \mathcal{O}_1] \right)$$

$$\Rightarrow \mathcal{B}(N+1) \equiv \mathcal{O}_{N+1}/\epsilon \sim \sum_{i=0}^{N} \gamma_{N,i}^{\mathcal{D}}$$

and the relation for the last column

$$\gamma_{N,0}^{\mathcal{D}} = (-)^N \left[\sum_{i=0}^{N} \gamma_{N,i}^{\mathcal{D}} - \sum_{j=1}^{N-1} (-)^j \binom{N}{j} \gamma_{j,0}^{\mathcal{D}} \right]$$

$$\Rightarrow \gamma_{N,0}^{\mathcal{D}} = \sum_{i=0}^{N} (-1)^i \binom{N}{i} \mathcal{B}(i+1)$$

Example new result: Leading-n_f part of 4-loop ADM

We present here the leading-n_f part of the 4-loop off-diagonal elements of the ADM. The Riemann-Zeta function

$$\zeta_3 \equiv \sum_{i=1}^{\infty} \frac{1}{i^3}$$

will appear as well as harmonic sums, which are recursively defined as [24, 25]

$$S_{\pm m}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^m} ,$$

$$S_{\pm m_1, m_2, \ldots, m_d}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} \; S_{m_2, \ldots, m_d}(i) ,$$

Example new result: Leading-n_f part of 4-loop ADM

$$
\begin{aligned}
\gamma_{N,k}^{\mathcal{D},(3)} = & \frac{8}{27} n_f^3 C_F \Big\{ \frac{1}{3}\Big(S_1(N) - S_1(k)\Big)^3 \Big(\frac{1}{N+2} - \frac{1}{N-k} \Big) \\
& + \Big(S_1(N) - S_1(k)\Big)^2 \Big(\frac{5}{3}\frac{1}{N-k} + \frac{2}{N+1} - \frac{11}{3}\frac{1}{N+2} + \frac{1}{(N+2)^2} \Big) \\
& + \Big(S_1(N) - S_1(k)\Big)\Big(S_2(N) - S_2(k)\Big)\Big(\frac{1}{N+2} - \frac{1}{N-k} \Big) \\
& + 2\Big(S_1(N) - S_1(k)\Big)\Big(\frac{1}{3}\frac{1}{N-k} - \frac{13}{3}\frac{1}{N+1} + \frac{2}{(N+1)^2} + \frac{4}{N+2} \\
& - \frac{11}{3}\frac{1}{(N+2)^2} + \frac{1}{(N+2)^3} \Big) + \Big(S_2(N) - S_2(k)\Big)\Big(\frac{5}{3}\frac{1}{N-k} \\
& + \frac{2}{N+1} - \frac{11}{3}\frac{1}{N+2} + \frac{1}{(N+2)^2} \Big) + \frac{2}{3}\Big(S_3(N) - S_3(k)\Big)\Big(\frac{1}{N+2} \\
& - \frac{1}{N-k} \Big) + \frac{2}{3}\frac{1}{N-k} + \frac{2}{N+1} - \frac{26}{3}\frac{1}{(N+1)^2} + \frac{4}{(N+1)^3} \\
& - \frac{8}{3}\frac{1}{N+2} + \frac{8}{(N+2)^2} - \frac{22}{3}\frac{1}{(N+2)^3} + \frac{2}{(N+2)^4} + 4\zeta_3\Big(\frac{1}{N+2} - \frac{1}{N-k} \Big) \Big\}
\end{aligned}
$$

Comparison with calculations in the Gegenbauer basis

In conformal calculations, the ADM has been calculated completely up to 3-loop order in another basis for the total derivative operators (see e.g. [26, 27] for details). It is possible to relate the anomalous dimensions in both bases (new)

$$\sum_{j=0}^{N} (-1)^j \frac{(j+2)!}{j!} \gamma^{\mathcal{G}}_{N,j} = \frac{1}{N!} \sum_{j=0}^{N} (-1)^j \binom{N}{j} \frac{(N+j+2)!}{(j+1)!} \sum_{l=0}^{j} \gamma^{\mathcal{D}}_{j,l}$$

Using this relation and our 4-loop result, we can also derive the 4-loop leading-n_f part of the ADM in the Gegenbauer basis

Leading-n_f part of 4-loop Gegenbauer ADM (new)

$$
\begin{aligned}
\gamma_{N,k}^{\mathcal{G},(3)} =& \frac{64}{27}\frac{n_f^3 C_F}{a(N,k)}\vartheta_{N,k}\Bigg\{ -\frac{2}{3}\Big(S_1(N)-S_1(k)\Big)^3(2k+3) + \Big(S_1(N)-S_1(k)\Big)^2 \\
& \times (2k+3)\Big(\frac{5}{3}-\frac{1}{N+1}-\frac{1}{N+2}\Big) + \Big(S_1(N)-S_1(k)\Big)^2\Big(4+\frac{1}{k+1} \\
& -\frac{1}{k+2}\Big) + \Big(S_1(N)-S_1(k)\Big)(2k+3)\Big(\frac{1}{3}-\frac{1}{3}\frac{1}{N+1}-\frac{1}{(N+1)^2} \\
& +\frac{11}{3}\frac{1}{N+2}-\frac{1}{(N+2)^2}\Big) + \Big(S_1(N)-S_1(k)\Big)\Big(-\frac{20}{3}+\frac{4}{N+1}+\frac{2}{N+1}\frac{1}{k+1} \\
& -\frac{5}{3}\frac{1}{k+1}-\frac{1}{(k+1)^2}+\frac{4}{N+2}-\frac{2}{N+2}\frac{1}{k+2}+\frac{5}{3}\frac{1}{k+2}+\frac{1}{(k+2)^2}\Big) \\
& +(2k+3)\Big(\frac{7}{3}\frac{1}{N+1}-\frac{1}{6}\frac{1}{(N+1)^2}-\frac{1}{2}\frac{1}{(N+1)^3}-\frac{2}{N+2}+\frac{11}{6}\frac{1}{(N+2)^2} \\
& -\frac{1}{2}\frac{1}{(N+2)^3}-\frac{1}{3}\Big(S_3(N)-S_3(k)\Big)\Big) - \frac{2}{3}+\frac{2}{3}\frac{1}{N+1}+\frac{2}{(N+1)^2} \\
& +\frac{1}{(N+1)^2}\frac{1}{k+1}-\frac{5}{3}\frac{1}{N+1}\frac{1}{k+1}-\frac{1}{N+1}\frac{1}{(k+1)^2}-\frac{8}{3}\frac{1}{k+1}+\frac{5}{6}\frac{1}{(k+1)^2} \\
& +\frac{1}{2}\frac{1}{(k+1)^3}-\frac{22}{3}\frac{1}{N+2}+\frac{2}{(N+2)^2}-\frac{1}{(N+2)^2}\frac{1}{k+2}+\frac{5}{3}\frac{1}{N+2}\frac{1}{k+2} \\
& +\frac{1}{N+2}\frac{1}{(k+2)^2}+\frac{5}{3}\frac{1}{k+2}-\frac{5}{6}\frac{1}{(k+2)^2}-\frac{1}{2}\frac{1}{(k+2)^3}\Bigg\}
\end{aligned}
$$

Conclusions and outlook

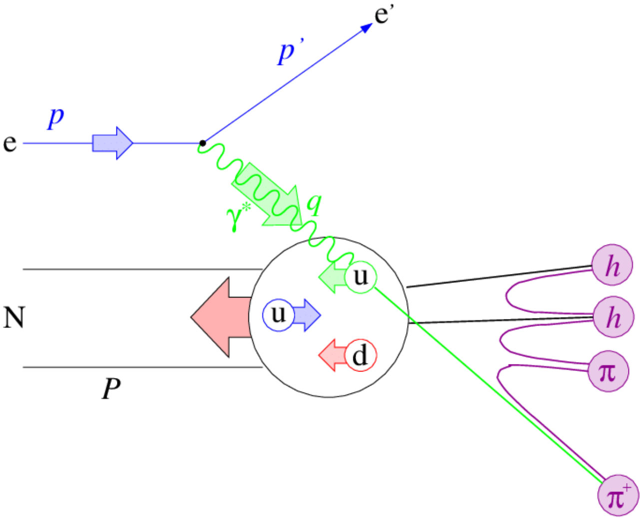
- New method for calculating off-diagonal elements of ADM, based on renormalization structure in chiral limit
- Checks and extends previous calculations, both in the derivative and in Gegenbauer bases
- Nice advantage of our method: Well-suited for automation with computer algebra programs
- Possible to go beyond the leading-n_f limit, see [hep-ph/2107.02470] for more details
- Generalize method to different operators (e.g. flavor singlet operators) in QCD and different models (e.g. gradient operators in scalar field theories) $\rightarrow$ needs to be studied

Thank you for your attention!

Backup and references

Deep inelastic scattering (DIS) [3]

Particularly important process: DIS (can e.g. probe proton structure)



[3]See e.g. M. Schwarz, *Quantum Field Theory and the Standard Model*

Deep inelastic scattering

Physical cross section is proportional to

$$\frac{1}{q^4} L_{\mu\nu} H^{\mu\nu}$$

- $L_{\mu\nu}$: Leptonic tensor (electron information, trivial)
- $H^{\mu\nu}$: Hadronic tensor (quantum corrections, hard)

Relate $H_{\mu\nu}$ to electromagnetic currents

$$H_{\mu\nu} = \int d^4x \, e^{iq\cdot x} \langle P| \, J_\mu(x) J_\nu(0) \, |P\rangle$$

where

$$J_\mu(x) = \bar{q}(x)\gamma_\mu q(x) = (c\rho, \vec{j})$$

- $\bar{q}/q$: (Anti-) quark
- γ_μ: Dirac gamma matrix

Operator product expansion (OPE)

Main idea: Write a product of operators at the same point as a sum, which is easier to handle

$$\lim_{x \to y} O(x)O(y) = \sum_n C_n(x-y)O_n(x)$$

More practical in momentum space

$$\int d^4x \, e^{iq\cdot x} O(x)O(0) = \sum_n C_n(q)O_n(0)$$

$C_n(q)$: Wilson coefficients (universal numbers)

Operator product expansions and DIS

Now apply the OPE to DIS

$$J^{\mu}(x)J^{\nu}(y)=\sum_{n}C_{n}(x-y)O_{n}^{\mu\nu}(x)$$

The type of operators that will appear on the RHS are of the following form

$$O_{\mu_1\ldots\mu_N}=\mathcal{S}\;\bar{q}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}q$$

where

- $D_{\mu}=\partial_{\mu}-ig_{s}A_{\mu}$; A_{μ} gluon
- $\mathcal{S}$: Symmetrization

Compare this with the standard electromagnetic current

$$J_{\mu}(x)=\bar{q}(x)\gamma_{\mu}q(x)$$

ADM in the Gegenbauer basis

We start by introducing the renormalized non-local light-ray operators $[\mathcal{O}]$, which act as generating functions for local operators, see e.g. [27], as

$$[\mathcal{O}](x; z_1, z_2) = \sum_{m,k} \frac{z_1^m z_2^k}{m!\ k!} \left[\overline{\psi}(x)(\overleftarrow{D} \cdot n)^m \not{n}(n \cdot \overrightarrow{D})^k \psi(x)\right] .$$

The renormalization group equation for these light-ray operators can be written as

$$\Big(\mu^2 \partial_{\mu^2} + \beta(a_s)\partial_{a_s} + \mathcal{H}(a_s)\Big)[\mathcal{O}](z_1, z_2) = 0$$

The evolution operator $\mathcal{H}(a_s)$ is an integral operator and acts on the light-cone coordinates of the fields [28]

$$\mathcal{H}(a_s)[\mathcal{O}](z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, h(\alpha, \beta)[\mathcal{O}](z_{12}^\alpha, z_{21}^\beta)$$

with $z_{12}^\alpha \equiv z_1(1-\alpha) + z_2\alpha$ and the evolution kernel $h(\alpha, \beta)$. The moments of the evolution kernel correspond to the anomalous dimensions of the local operators

ADM in the Gegenbauer basis

The light-ray operators admit an expansion in a basis of local operators in terms of Gegenbauer polynomials, see e.g. [26],

$$\mathcal{O}^{\mathcal{G}}_{N,k} = (\partial_{z_1} + \partial_{z_2})^k C_N^{3/2}\left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}}\right)\mathcal{O}(z_1, z_2)\bigg|_{z_1=z_2=0},$$

where $k \geq N$ is the total number of derivatives. The Gegenbauer polynomials can be written as [29]

$$C_N^{\nu}(z) = \frac{\Gamma(\nu + 1/2)}{\Gamma(2\nu)} \sum_{l=0}^{N} (-1)^l \frac{\Gamma(2\nu + N + l)}{l!\,(N-l)!\,\Gamma(\nu + 1/2 + l)} \left(\frac{1}{2} - \frac{z}{2}\right)^l.$$

The renormalized operators $[\mathcal{O}^{\mathcal{G}}_{N,k}]$ obey the evolution equation

$$\left(\mu^2\partial_{\mu^2} + \beta(a_s)\partial_{a_s}\right)[\mathcal{O}^{\mathcal{G}}_{N,k}] = \sum_{j=0}^{N} \gamma^{\mathcal{G}}_{N,j}[\mathcal{O}^{\mathcal{G}}_{j,k}].$$

Calculating sums

Using our constructive algorithm, we have to evaluate single and double sums involving harmonic sums and denominators. This can be done by using the principles of symbolic summation, see e.g. [30, 31] for extensive overviews. Particularly helpful for these purposes is the MATHEMATICA package SIGMA [32], which uses the creative telescoping algorithm.

Calculating sums

Telescoping: Suppose we have a summation $\sum_{k=a}^{N} f(k)$
$\rightarrow$ Find function $g(N)$ such that the summand can be written as

$$f(k) = \Delta g(k) \equiv g(k+1) - g(k)$$

$$\Rightarrow \sum_{k=a}^{N} f(k) = \sum_{k=a}^{N} g(k+1) - \sum_{k=a}^{N} g(k)$$
$$= g(N+1) - g(a)$$

$\sim$ discrete version of symbolic integration:

$$f(x) = Dg(x) \equiv \frac{d}{dx} g(x) \Rightarrow \int_a^b f(x) dx = g(b) - g(a)$$

Calculating sums

Creative telescoping [33]: Suppose we have the summation $\sum_{k=a}^{b} f(n,k) \equiv S(n)$
$\Rightarrow$ Find functions $c_0(n), ..., c_d(n)$ and $g(n,k)$ such that

$$g(n,k+1) - g(n,k) = c_0(n)f(n,k) + ... + c_d(n)f(n+d,k)$$

Now apply summation on both sides of the equation

$$\Rightarrow g(n,b+1) - g(n,a) = c_0(n)\sum_{k=a}^{b} f(n,k) + ... + c_d(n)\sum_{k=a}^{b} f(n+d,k)$$

$\Rightarrow$ Inhomogeneous recurrence for original sum

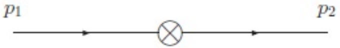
$$q(n) = c_0(n)S(n) + ... + c_d(n)S(n+d)$$

Calculating sums

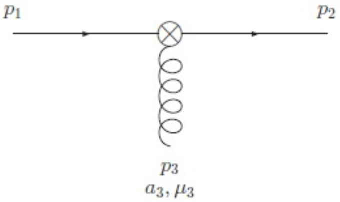
- In this way, SIGMA generates and solves recurrence for given summation problem
- Solution consists of solution set for homogeneous recurrence + particular solution
- For final closed expression of summation: Determine linear combination of solutions that has same initial values as the given sum

Feynman rules for operator insertions

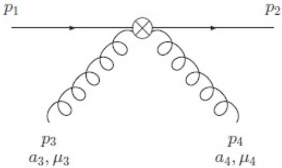
Contract OMEs with $\Delta_{\mu_1}...\Delta_{\mu_N}$, $\Delta^2 = 0$, see e.g. [19]



$$\not{\Delta}\,(\Delta\cdot p_2)^{N-1}$$



$$-g t^{a_3} \not{\Delta} \Delta^{\mu_3} \sum_{j_1=0}^{N-2} (p_2\cdot\Delta)^{N-2-j_1} (q\cdot\Delta - p_1\cdot\Delta)^{j_1}$$



$$g^2 \not{\Delta} \Delta^{\mu_3} \Delta^{\mu_4} \sum_{j_1=0}^{N-3} \sum_{j_2=0}^{j_1} (p_2\cdot\Delta)^{N-3-j_1} (q\cdot\Delta - p_1\cdot\Delta)^{j_2}$$
$$\Big(t^{a_3} t^{a_4} (q\cdot\Delta - p_1\cdot\Delta - p_3\cdot\Delta)^{j_1-j_2}$$
$$+ t^{a_4} t^{a_3} (q\cdot\Delta - p_1\cdot\Delta - p_4\cdot\Delta)^{j_1-j_2} \Big)$$

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