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LABORATOIRE DE PHYSIQUE
THEORIQUE ET HAUTES ENERGIES

Baltic School of High Energy Physics
and Accelerator Technologies
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QCD, Higher Orders and Jets

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Lecture 1: Basics of QCD

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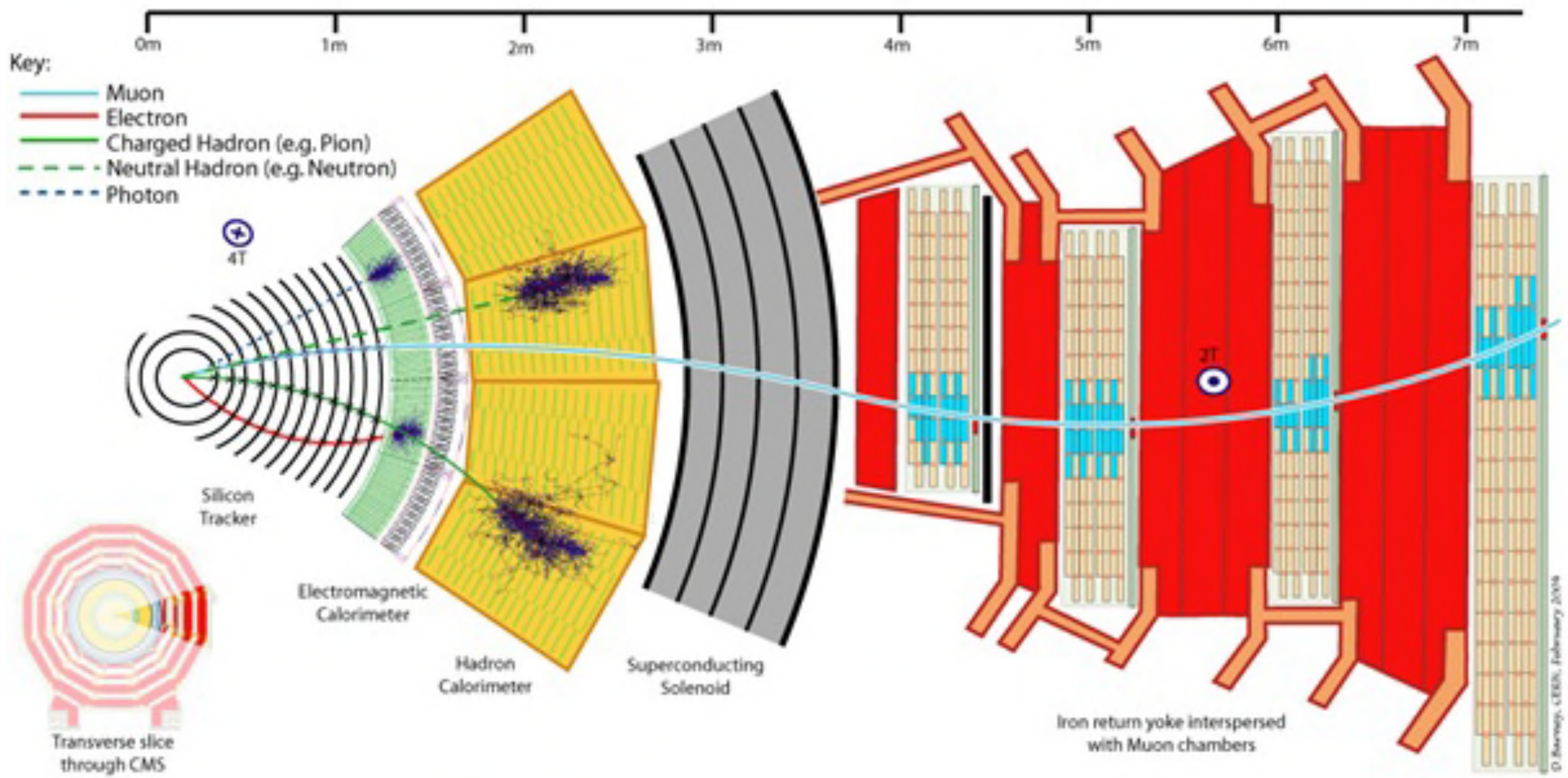
We use colliders to discover particles Beyond the Standard Model particles. This is what a typical LHC detector looks like:



What can we actually directly observe?

Not much. Only particles that travel this far (i.e. at least a few centimetres) without decaying.

The relevant relation is distance = cτ. Distance = 1m needs $\tau = 0.3\ 10^{-8}$ s

What makes it this far? / What can we detect?

- Particles that are absolutely stable:
  - protons, electrons/positrons, photons, neutrinos

- Particles with $c\tau > 1$m:
  - muons, neutrons, pions, kaons

- Particles that decay very quickly ($c\tau \sim 0.01$ mm to 1m) but that we can easily infer either via reconstructed invariant mass of their decays, or displaced vertices (especially if boosted), or other characteristics of their decay products:
  - strange, charm, beauty hadrons

Collider Physics

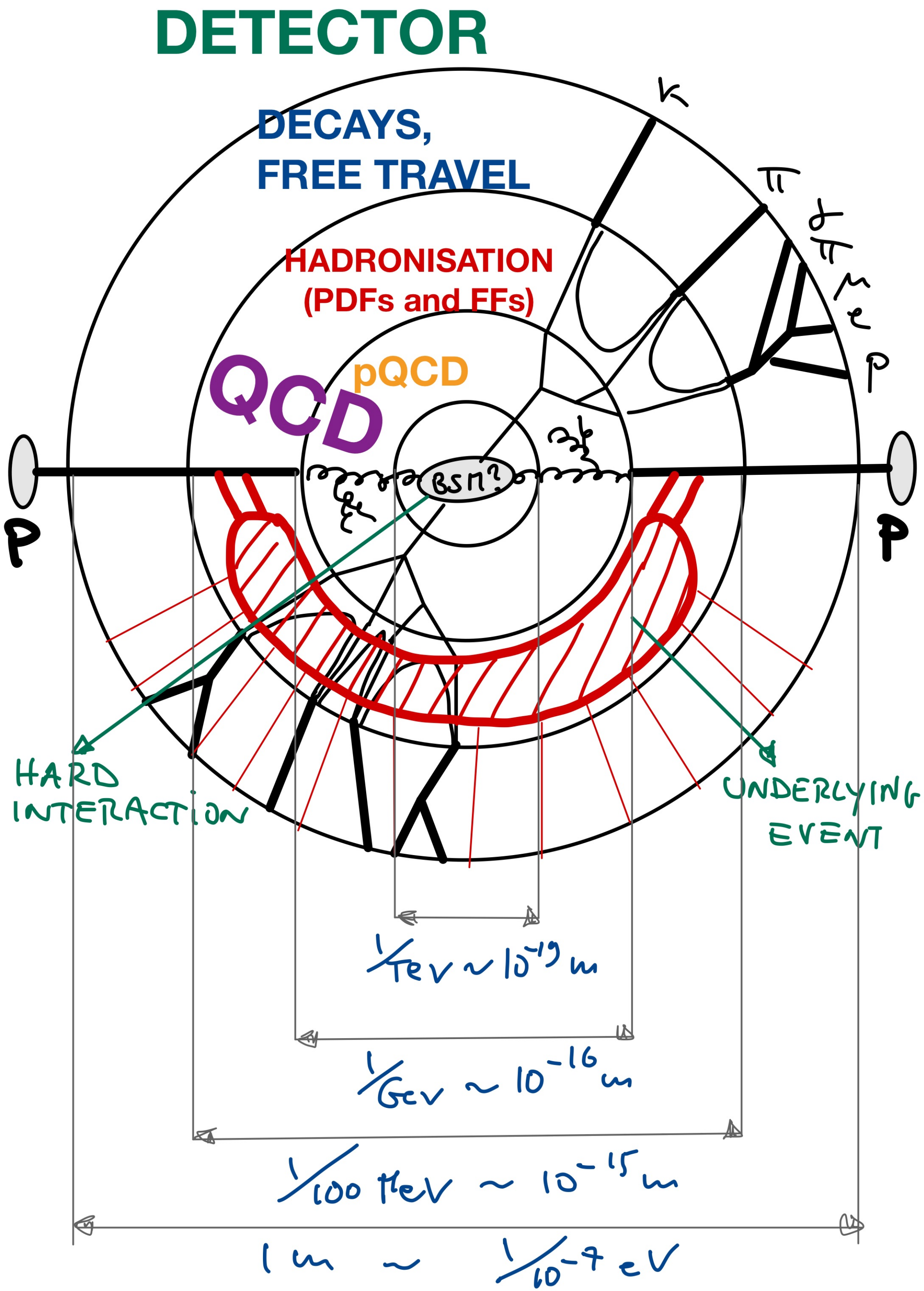
This is about it. All other particles (W, Z, top, Higgs, BSM physics…) must be deduced from measurements of

- electron/positron candidates
- muon/antimuon candidates
- charged hadron
- neutral hadron (no tracks, calo only)
- missing transverse momentum

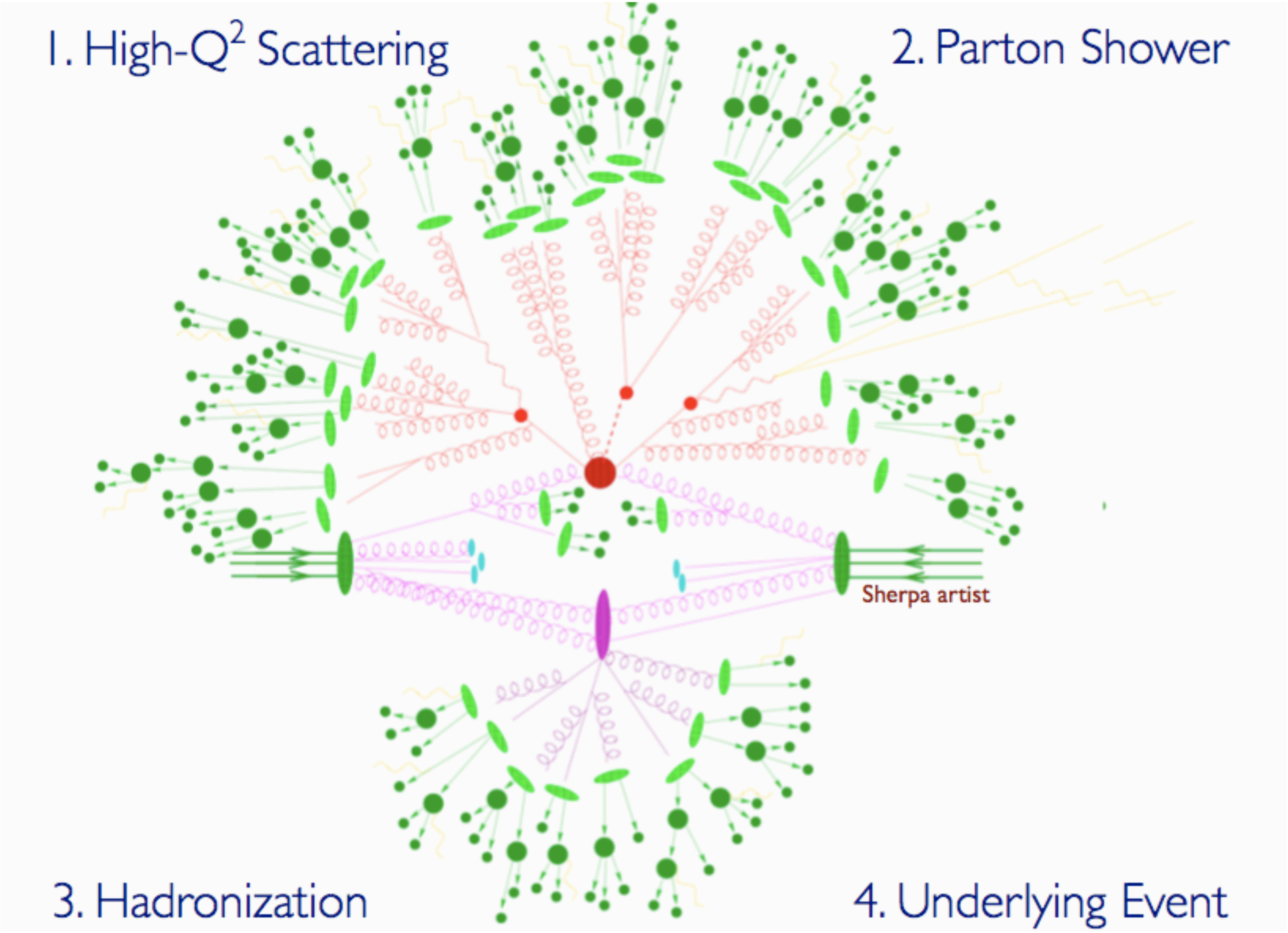
The challenge is to calculate predictions at the "fundamental physics" scale (<< proton size) and connect it to what we observe at macroscopic scales (detector size)

A hadron collider event

[NB. NOT to scale!]



Strong interactions are complicated



Predictions

“We are driven to the conclusion that the Hamiltonian method for strong interactions is dead and must be buried, although of course with deserved honor”

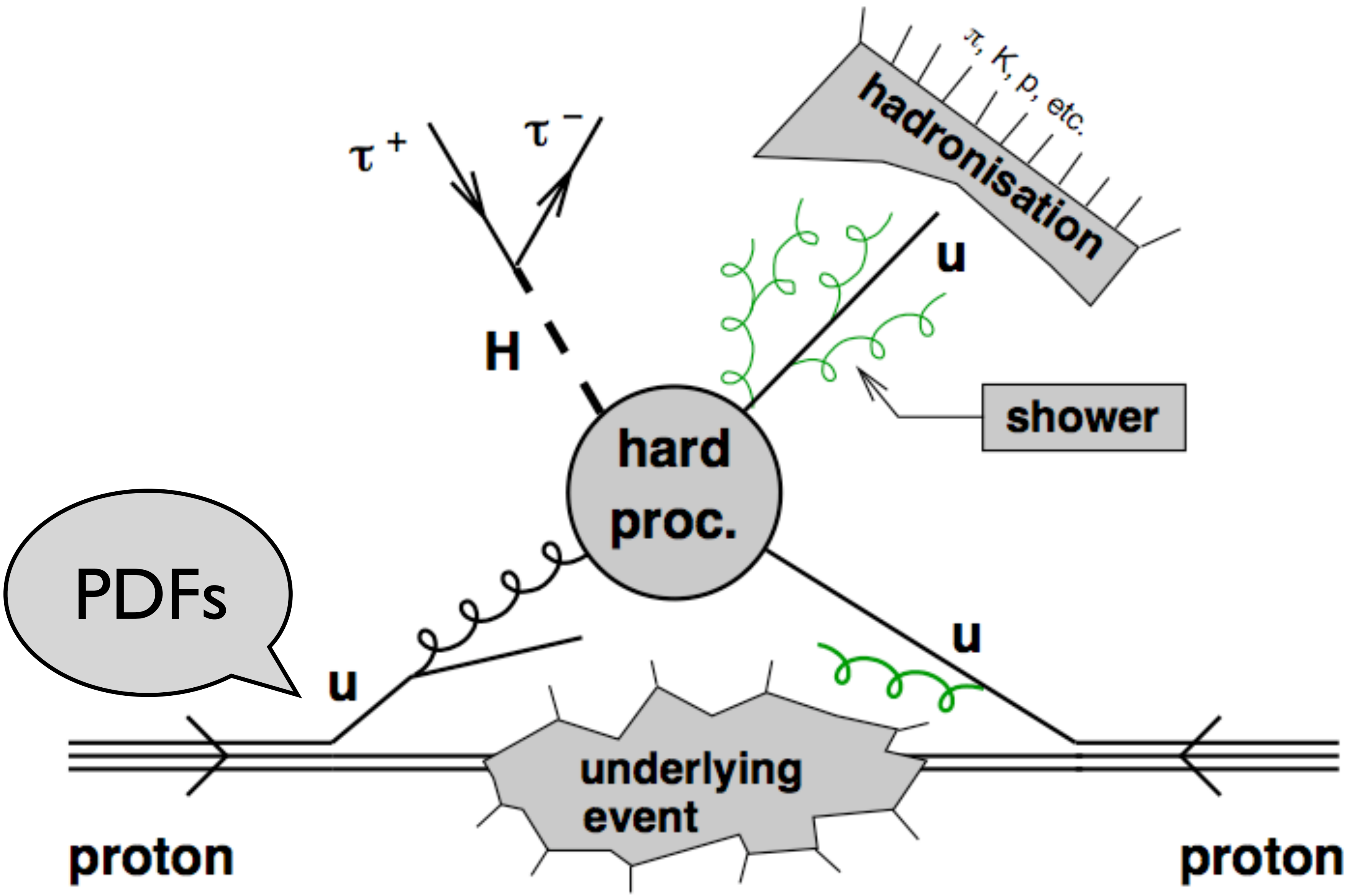
Lev Landau

“The correct theory [of strong interactions] will not be found in the next hundred years”

Freeman Dyson

We have come a long way towards disproving these predictions

A hadronic process



Final state

Hard process

Initial state

Drawing: G. Salam

Books and "classics"...

- T. Muta, *Foundations of Quantum Chromodynamics*, World Scientific (1987)
- R.D. Field, *Applications of perturbative QCD*, Addison Wesley (1989)
  Great for specific examples of detailed calculations
- R.K. Ellis, W.J. Stirling and B.R. Webber, *QCD and Collider Physics*, Cambridge University Press (1996)
  Phenomenology-oriented
- G. Sterman, *An Introduction to Quantum Field Theory*, Cambridge University Press (1993)
  A QFT book, but applications tilted towards QCD
- Dokshitzer, Khoze, Muller, Troyan, *Basics of perturbative QCD*, http://www.lpthe.jussieu.fr/~yuri
  For the brave ones
- Dissertori, Knowles, Schmelling, *Quantum Chromodynamics: High Energy Experiments and Theory,* Oxford Science Publications
- Campbell, Huston, Krauss, *The Black Book of Quantum Chromodynamics*, Oxford University Press
  Perhaps the most recent QCD book
- M.L. Mangano, *Introduction to QCD*, http://doc.cern.ch//archive/cernrep//1999/99-04/p53.pdf
- S. Catani, *Introduction to QCD*, CERN Summer School Lectures 1999

...and more recent lectures, slides and...videos

- Gavin Salam,
  - "Elements of QCD for Hadron Colliders", http://arxiv.org/abs/arXiv:1011.5131
  - http://gsalam.web.cern.ch/gsalam/teaching/PhD-courses.html
- Peter Skands
  - 2015 CERN-Fermilab School lectures, http://skands.physics.monash.edu/slides/
  - "Introduction to QCD", http://arxiv.org/abs/arXiv:1207.2389
- Fabio Maltoni
  - "QCD and collider physics", GGI lectures,
    https://www.youtube.com/playlist?list=PL1CFLtxeIrQqvt-e8C5pwBKG4PljSyouP
- Search YouTube for "GGI Thaler", "GGI Soyez", "GGI Catani" "GGI Peskin"
- Search You Tube/web for "ICTP particle physics summer school"

Outline of ‘Basics of QCD’

- strong interactions
- QCD lagrangian, colour, ghosts
- running coupling
- radiation
- calculations of observables
  - theoretical uncertainties estimates
  - power corrections
  - infrared divergencies and IRC safety
  - factorisation

QED v. QCD

QED has a wonderfully simple lagrangian, determined by local gauge invariance

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}A\!\!\!/\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

In the same spirit, we build **QCD**:
a non abelian local gauge theory, based on SU(3)$_{\text{colour}}$,
with 3 quarks (for each flavour) in the fundamental representation of the group and 8 gluons in the adjoint

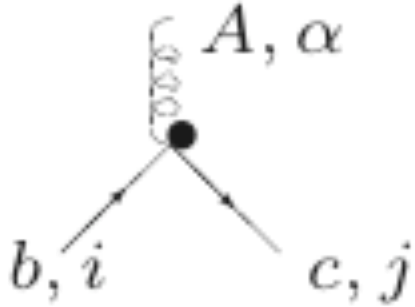
$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + \sum_f \bar{\psi}_i^{(f)}(i\partial\!\!\!/ - m_f)\psi_i^{(f)} - \bar{\psi}_i^{(f)}(g_s t^a_{ij} A\!\!\!/_a)\psi_j^{(f)}$$

Gauge Fields and their interact. | Matter | Interaction

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

1. Colour

quark-gluon interaction



$$-ig\,(t^A)_{cb}\,(\gamma^\alpha)_{ji}$$

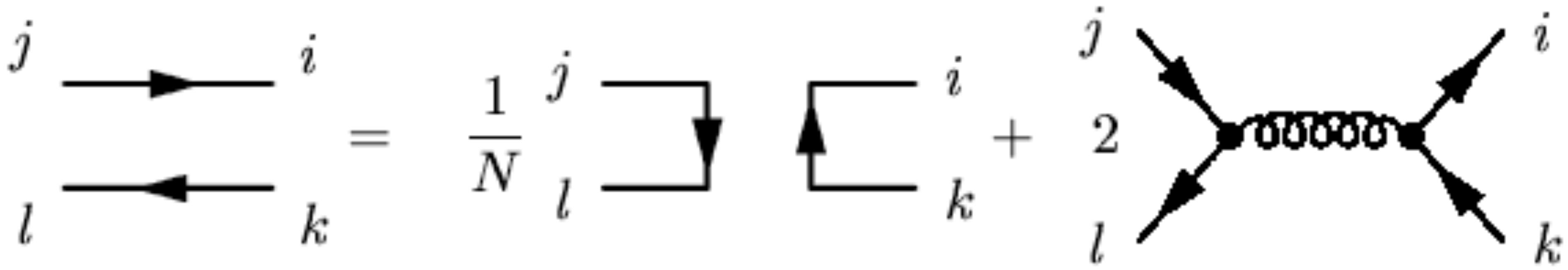
colour matrix (generator of $SU(3)_{\text{colour}}$)

$$t^A{}_{cb}$$

Index of the **adjoint** representation

Indices of the **fundamental** representation

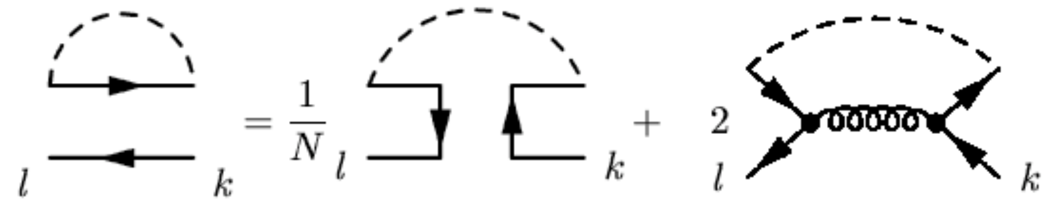
A fundamental colour relation



$$\delta_{ij}\delta_{lk} = \frac{1}{N}\delta_{ik}\delta_{lj} + 2\,t^A_{ik}t^A_{lj}$$

Take i=j in

$$\delta_{ij}\delta_{lk} = \frac{1}{N}\delta_{ik}\delta_{lj} + 2\,t^A_{ik}t^A_{lj}$$



$$\Downarrow$$

$$N\delta_{lk} = \frac{1}{N}\delta_{lk} + 2\,t^A_{ik}t^A_{li}$$

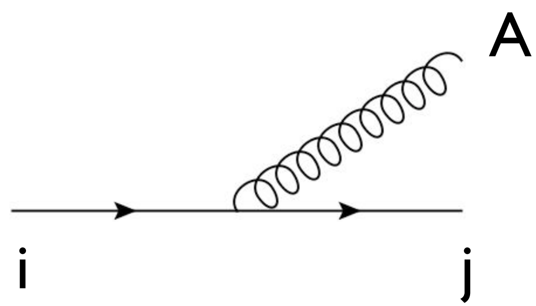
$$\Downarrow$$

$$(t^At^A)_{lk} = \frac{1}{2}\left(N - \frac{1}{N}\right)\delta_{lk} = \frac{N^2-1}{2N}\delta_{lk} \equiv C_F\delta_{lk}$$

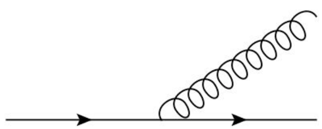
This defines C_F.
It is the Casimir of the fundamental representation of SU(N).
What is it, physically?

C_F

Gluon emission from a quark

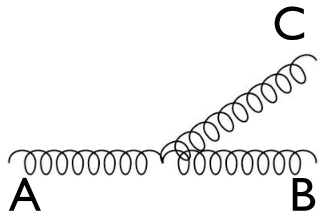


$$\sim \quad t^A_{ji}$$

Prob ~ $\sum_{jA} \Big|$  $\Big|^2 \sim \sum_{jA} t^A_{ij} t^A_{ji} = \sum_A (t^A t^A)_{ii} = C_F \delta_{ii}$

$C_F = (N^2-1)/(2N)$ is therefore the ‘colour charge’ of a quark, i.e. its probability of emitting a gluon (except for the strong coupling, of course)

Analogously, one can show that

Prob $\sim \sum_{BC} \Big|$  $\Big|^2 \sim C_A \delta_{AA}$

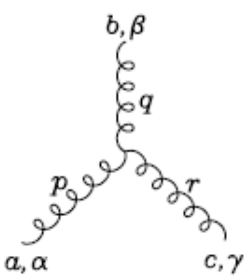
C_A = N is the 'colour charge' of a gluon, i.e.
its probability of emitting a gluon (except for the strong coupling, of course).
It is also the Casimir of the adjoint representation.

2. Gauge bosons self couplings

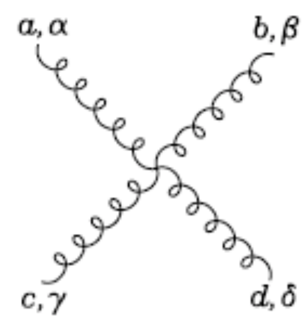
In QCD the gluons interact among themselves:

$$\mathcal{L}_{YM} = -\frac{1}{4}\sum_a F^a_{\mu\nu} F^{a\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$



$$= g f^{abc}\left[g^{\alpha\beta}(p-q)^\gamma + g^{\beta\gamma}(q-r)^\alpha + g^{\gamma\alpha}(r-p)^\beta\right]$$



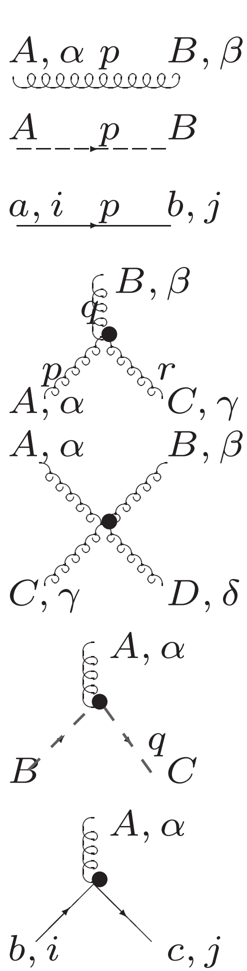
$$\begin{aligned} = & -ig^2 f^{xac} f^{xbd}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \\ & -ig^2 f^{xad} f^{xbc}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right) \\ & -ig^2 f^{xab} f^{xcd}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \end{aligned}$$

New Feynman diagrams, in addition to the ‘standard’ QED-like ones

Direct consequence of non-abelianity of theory

3. Need for ghosts

Cancel unphysical degrees of freedom that would otherwise propagate in covariant gauges



ghost propagator

gluon-ghost vertex

gauge parameter

$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^\alpha p^\beta}{p^2+i\varepsilon}\right]\frac{i}{p^2+i\varepsilon}$$

$$\delta^{AB}\frac{i}{p^2+i\varepsilon}$$

$$\delta^{ab}\frac{i}{(\not{p}-m+i\varepsilon)_{ji}}$$

$$-gf^{ABC}\left[g^{\alpha\beta}(p-q)^\gamma + g^{\beta\gamma}(q-r)^\alpha + g^{\gamma\alpha}(r-p)^\beta\right]$$

(all momenta incoming)

$$\begin{aligned}&-ig^2 f^{XAC}f^{XBD}(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\delta}g_{\beta\gamma})\\&-ig^2 f^{XAD}f^{XBC}(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\gamma}g_{\beta\delta})\\&-ig^2 f^{XAB}f^{XCD}(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})\end{aligned}$$

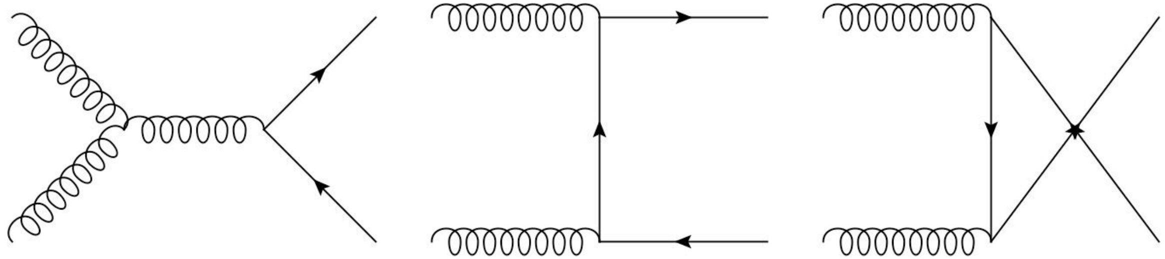
$$gf^{ABC}q^\alpha$$

$$-ig\left(t^A\right)_{cb}\left(\gamma^\alpha\right)_{ji}$$

Table 1: Feynman rules for QCD in a covariant gauge.

Ghosts: an example

$gg \to q\bar{q}$



In QED we would sum over the (photon) polarisations using

$$\sum_{pol} \epsilon_i^{\mu} \epsilon_i^{*\nu} = -g_{\mu\nu}$$

In QCD this would give the wrong result

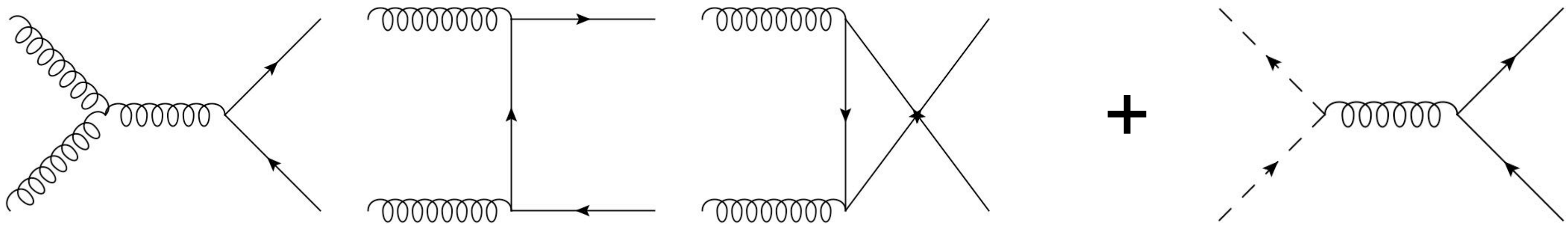
We must use instead

$$\sum_{phys\,pol} \epsilon_i^{\mu} \epsilon_i^{*\nu} = -g_{\mu\nu} + \frac{k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu}{k \cdot \bar{k}}$$

$\bar{k}$ is a light-like vector, we can use $(k_0,0,0,-k_0)$

Ghosts: an example

An **alternative** approach is to include the ghosts in the calculation



Now we can safely use $$\sum_{pol} \epsilon_i^{\mu} \epsilon_i^{*\nu} = -g_{\mu\nu}$$

Macroscopic differences

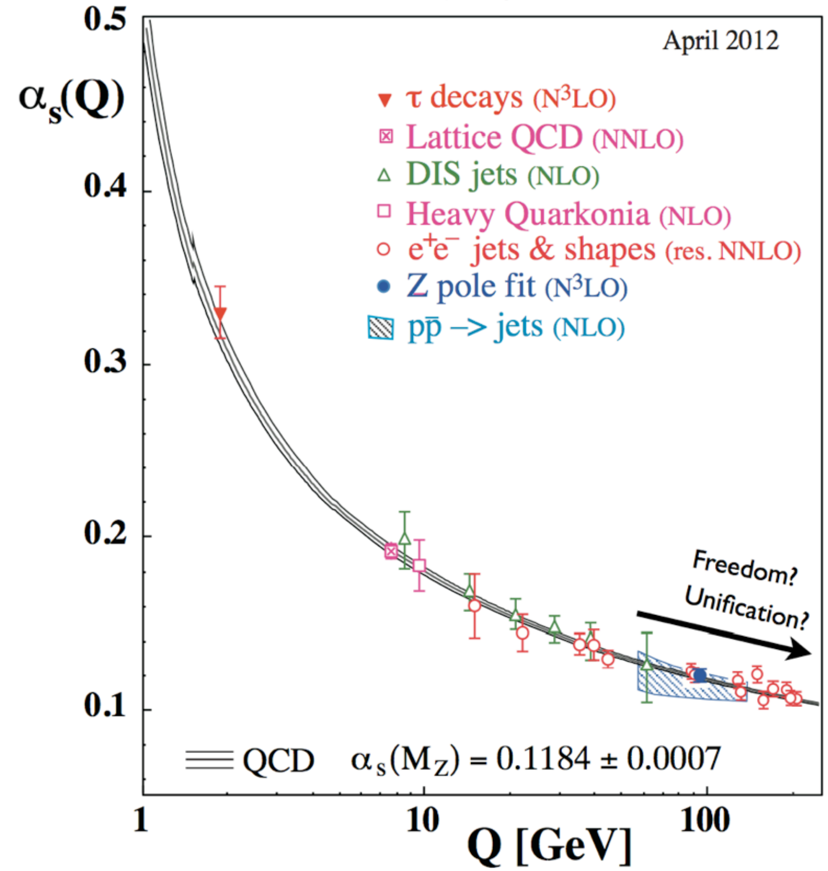
1. Confinement (probably -- no proof in QCD)

We never observe the fundamental degrees of freedom (quarks and gluons). They are always confined into hadrons.

2. Asymptotic Freedom

The running coupling of the theory, α_s, **decreases** at large energies

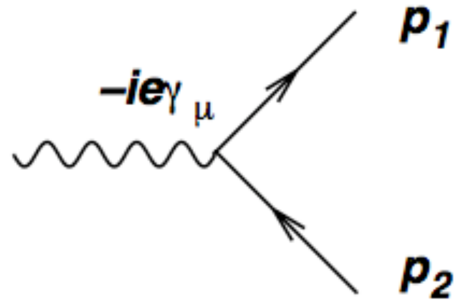
From S. Bethke, Nucl.Phys.Proc.Suppl. 234 (2013) 229
See also PDG Review on QCD. by Dissertori & Salam



QCD radiation

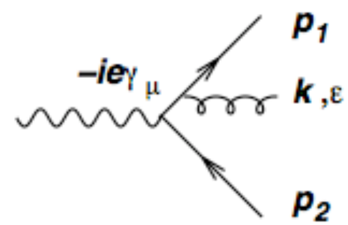
Start with $\gamma^* \to q\bar{q}$:

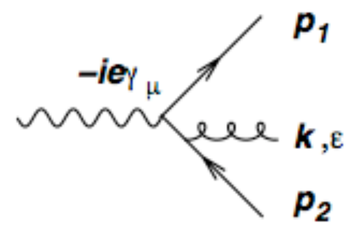
$$\mathcal{M}_{q\bar{q}} = -\bar{u}(p_1) i e_q \gamma_\mu v(p_2)$$



Emit a gluon:

$$\mathcal{M}_{q\bar{q}g} = \bar{u}(p_1) i g_s \not{\epsilon} t^A \frac{i}{\not{p}_1 + \not{k}} i e_q \gamma_\mu v(p_2) - \bar{u}(p_1) i e_q \gamma_\mu \frac{i}{\not{p}_2 + \not{k}} i g_s \not{\epsilon} t^A v(p_2)$$





In the **soft** limit , $k << p_{1,2}$

$$\mathcal{M}_{q\bar{q}g} \simeq \bar{u}(p_1) i e_q \gamma_\mu t^A v(p_2)\, g_s \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right)$$

QCD radiation

Squared amplitude, including phase space

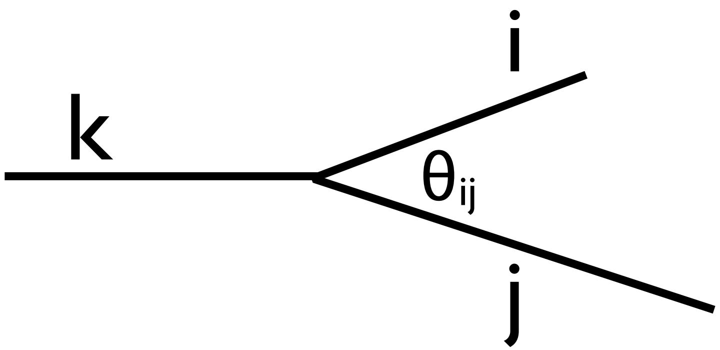
$$d\Phi_{q\bar{q}g}|M^2_{q\bar{q}g}| \simeq (d\Phi_{q\bar{q}}|M^2_{q\bar{q}}|)\, \frac{d^3\vec{k}}{2E(2\pi)^3} C_F g_s^2 \frac{2p_1.p_2}{(p_1.k)(p_2.k)}$$

Factorisation: Born × radiation

Changing variables (use energy of gluon E and emission angle θ) we get for the radiation part

$$dS = \frac{2\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

QCD emission probability



$$\frac{dP_{k\to ij}}{dE_i d\theta_{ij}} \sim \frac{\alpha_s}{\min(E_i, E_j)\theta_{ij}}$$

Singular in the soft ($E_{i,j} \to 0$) and in the collinear ($\theta_{ij} \to 0$) limits.

Logarithmically divergent upon integration

The divergences can be cured by the addition of virtual corrections and/or **if** the definition of an observable is appropriate

Altarelli-Parisi kernel

Using the variables E=(1-z)p and k_t = Eθ we can rewrite

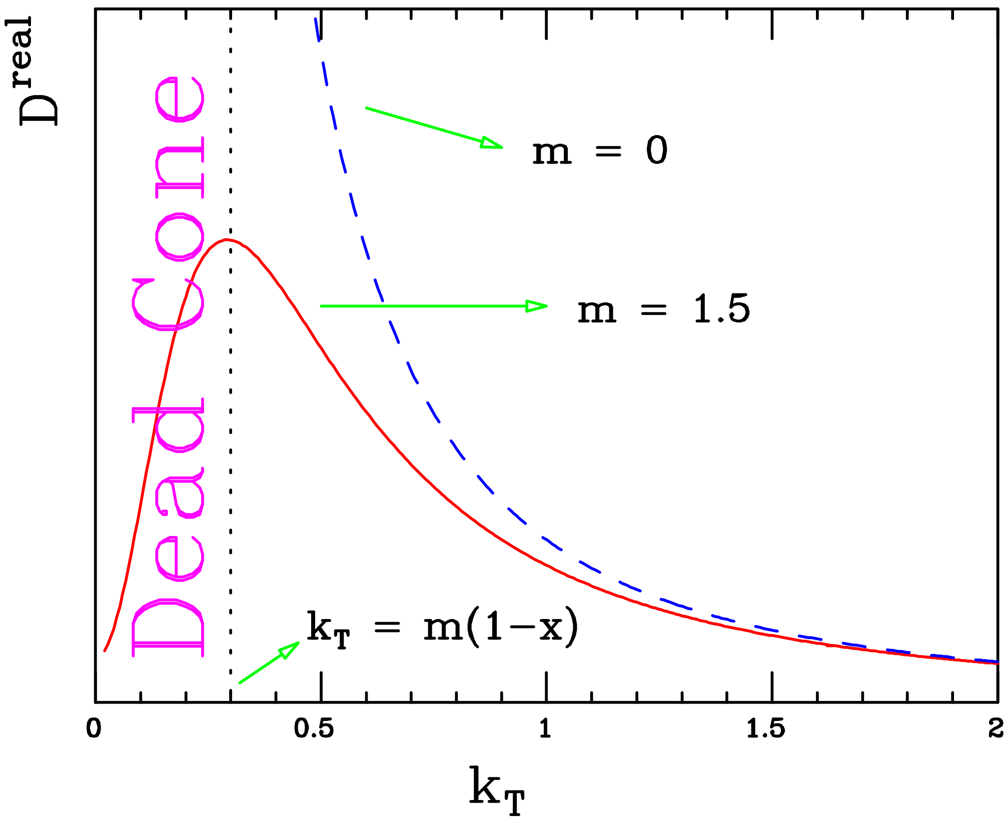
$$dS = \frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\sin\theta}\frac{d\phi}{2\pi} \quad \longrightarrow \quad \frac{\alpha_s C_F}{\pi}\frac{1}{1-z}dz\frac{dk_t^2}{k_t^2}\frac{d\phi}{2\pi}$$

'almost' the Altarelli-Parisi splitting function P_{qq}

Massive quarks

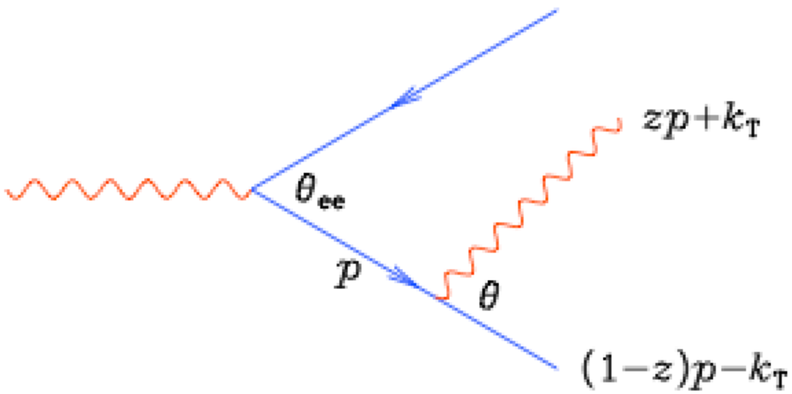
If the quark is massive the collinear singularity is **screened**

$$\frac{\alpha_s C_F}{\pi}\frac{1}{1-z}dz\frac{dk_t^2}{k_t^2}\frac{d\phi}{2\pi} \to \frac{\alpha_s C_F}{\pi}\frac{1}{1-z}dz\frac{dk_t^2}{k_t^2+(1-z)^2m^2}\frac{d\phi}{2\pi}+\cdots$$



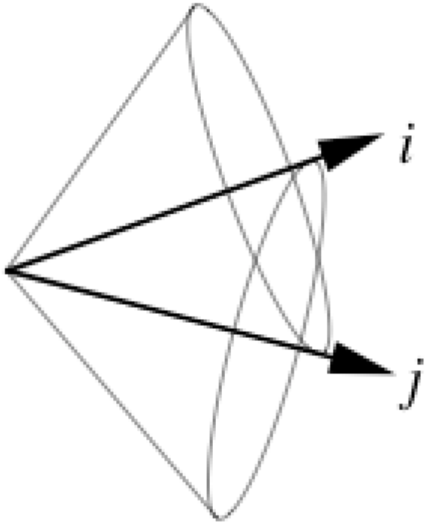
Angular ordering

The universal soft and collinear spectrum is not the only relevant characteristic of radiation. **Angular ordering** is another



Angular ordering means $\theta < \theta_{ee}$

Soft radiation emitted by a dipole is restricted to cones smaller than the angle of the dipole



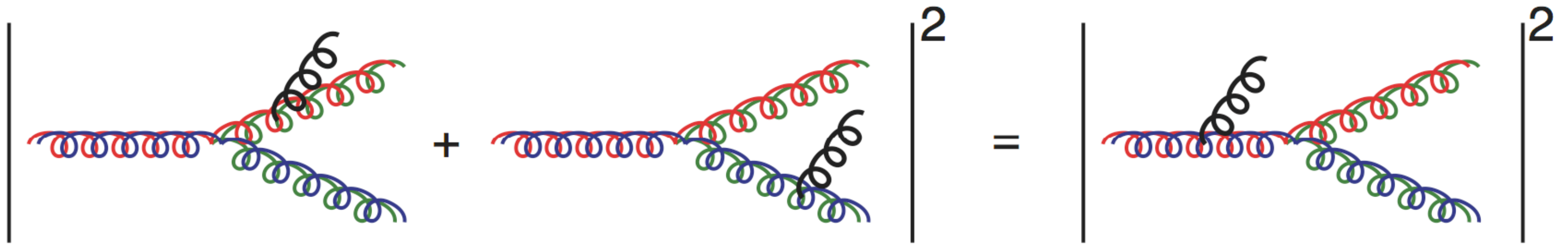
Coherence

Angular ordering is a manifestation of **coherence**, a phenomenon typical of gauge theories

Coherence leads to the **Chudakov effect**, suppression of soft bremsstrahlung from an e^+e^- pair.

"Quasi-classical" explanation: a soft photon cannot resolve a small-sized pair, and only sees its total electric charge (i.e. zero)

The phenomenon of coherence is preserved also in QCD. Soft guon radiation off a coloured pair can be described as being emitted coherently by the colour charge of the parent of the pair

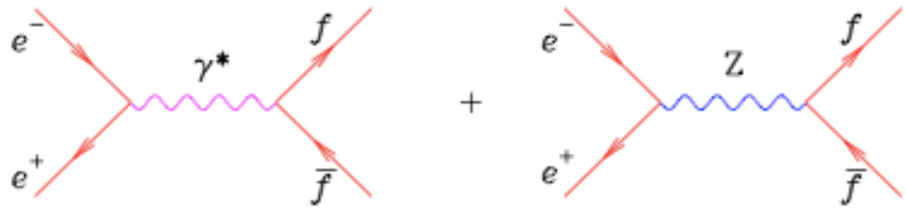


Drawing: P. Skands

$e^+e^- \to$ hadrons

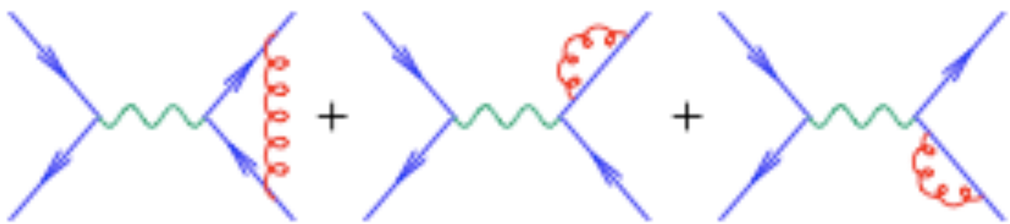
Easiest higher order calculation in QCD. Calculate $e^+e^- \to$ qqbar+X in pQCD

Born



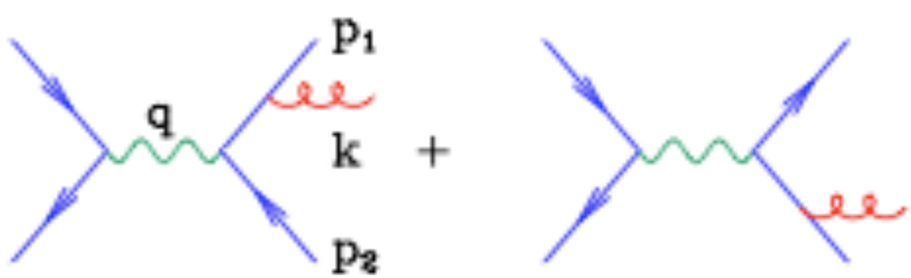
α_s^0

Virtual



α_s^1

Real



$e^+e^- \rightarrow$ hadrons

Regularize with dimensional regularization, expand in powers of ε

$$\sigma^{q\bar{q}g} = 2\sigma_0 \frac{\alpha_S}{\pi} H(\epsilon) \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon)\right]$$

Real

$$\sigma^{q\bar{q}} = 3\sigma_0 \left\{1 + \frac{2\alpha_S}{3\pi} H(\epsilon) \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon)\right]\right\}$$

Virtual

$$R = 3 \sum_q Q_q^2 \left\{1 + \frac{\alpha_S}{\pi} + \mathcal{O}(\alpha_S^2)\right\}$$

Sum

Real and virtual, separately divergent, ‘conspire’ to make total cross section finite

Cancellation of singularities

In fact, the 'conspiration' is not accidental

Block-Nordsieck theorem

IR singularities cancel in sum over soft unobserved photons in final state

(formulated for massive fermions ⇒ no collinear divergences)

Kinoshita-Lee-Nauenberg theorem

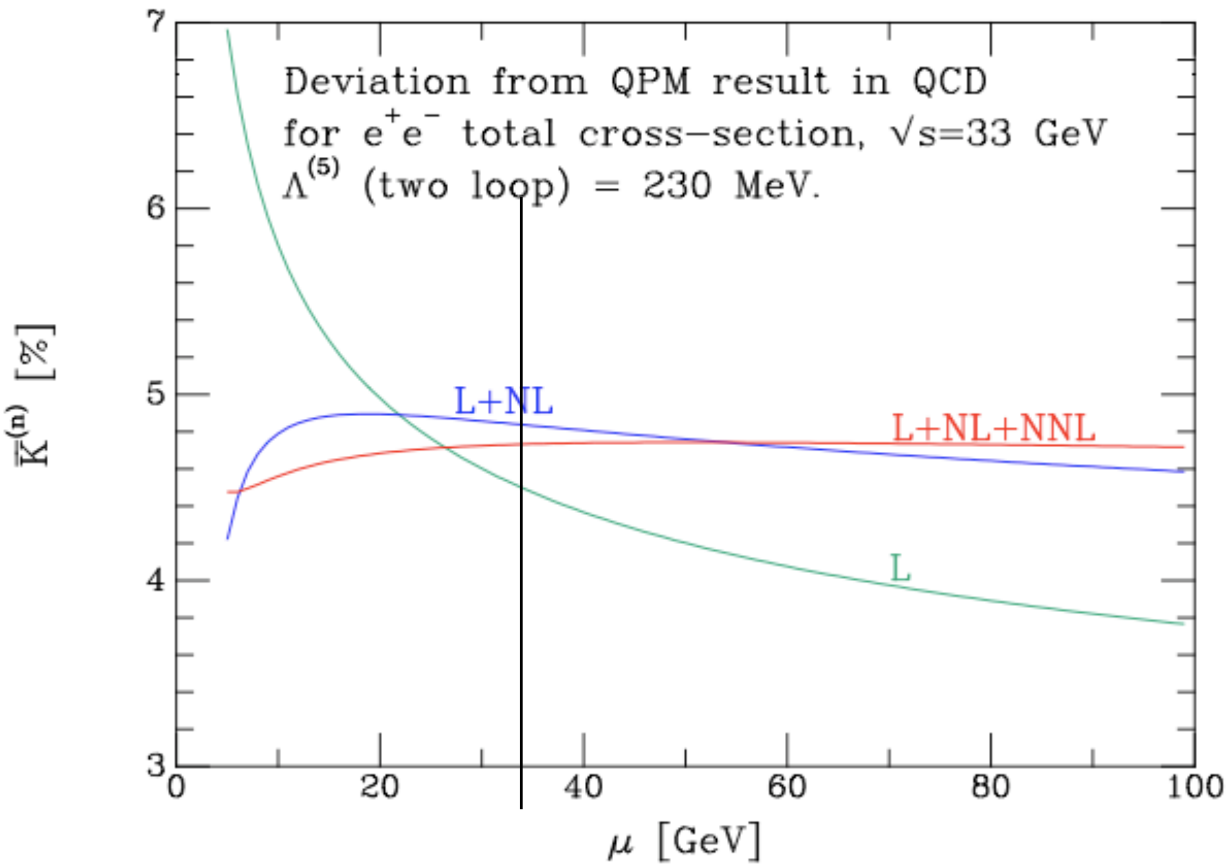
IR and collinear divergences cancel in sum over degenerate initial and final states

These theorems suggest that the observable must be crafted in a proper way for the cancellation to take place

Scale dependence

$$K_{QCD} = 1 + \frac{\alpha_S(\mu^2)}{\pi} + \sum_{n\geq 2} C_n \left(\frac{s}{\mu^2}\right) \left(\frac{\alpha_S(\mu^2)}{\pi}\right)^n$$

C_n known up to C_3



Cross section prediction varies with renormalisation scale choice. **Which value do we pick for μ?**

None.

μ cannot be uniquely fixed. It can however be exploited to **estimate the theoretical uncertainty of the calculation**

Theoretical uncertainties

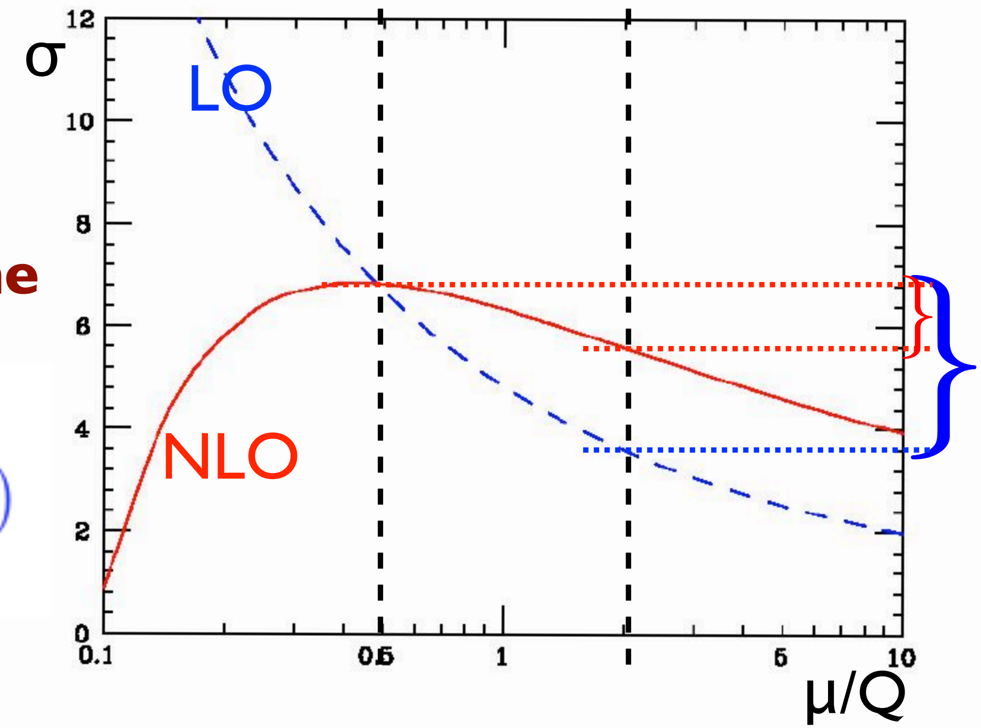
We wrote before: $$\frac{d}{d\ln\mu^2}\ln\sigma^{phys} = 0$$

i.e. independence of cross sections on artificial scales

Would only hold for all-orders calculations.

In real life: residual dependence at one order higher than the calculation

$$\frac{d}{d\log\mu}\sum_{n=1}^{N} c_n(\mu)\alpha_S^n(\mu) \sim \mathcal{O}\left(\alpha_S^n(\mu)^{N+1}(\mu)\right)$$



Vary scales (around a physical one) to **ESTIMATE** the uncalculated higher order

Non-perturbative contributions

We have calculated $\sum_q \sigma(e^+e^- \to q\bar{q})$ in **perturbative** QCD

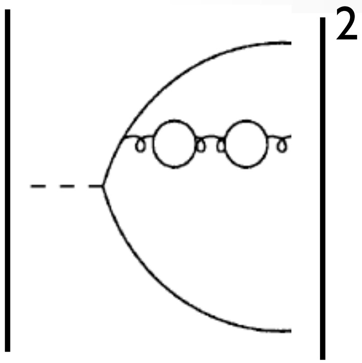
However

$$\sum_q \sigma(e^+e^- \to q\bar{q}) \neq \sigma(e^+e^- \to \text{hadrons})$$

The (small) difference is due to hadronisation corrections, and is of non-perturbative origin

We cannot calculate it in pQCD, but in some cases we can get an idea of its behaviour from the incompleteness of pQCD itself

Suppose we keep calculating to higher and higher orders:



$$\rightarrow \alpha_s^{n+1} \beta_{0f}^n \, n!$$

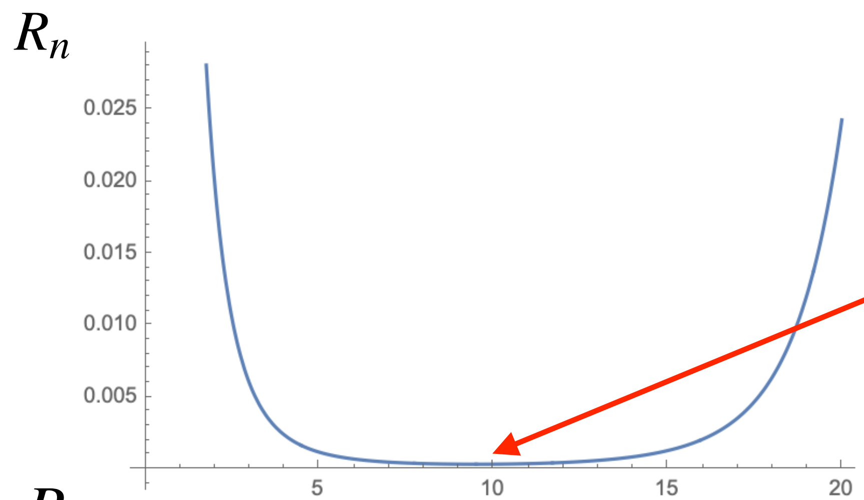
Factorial growth

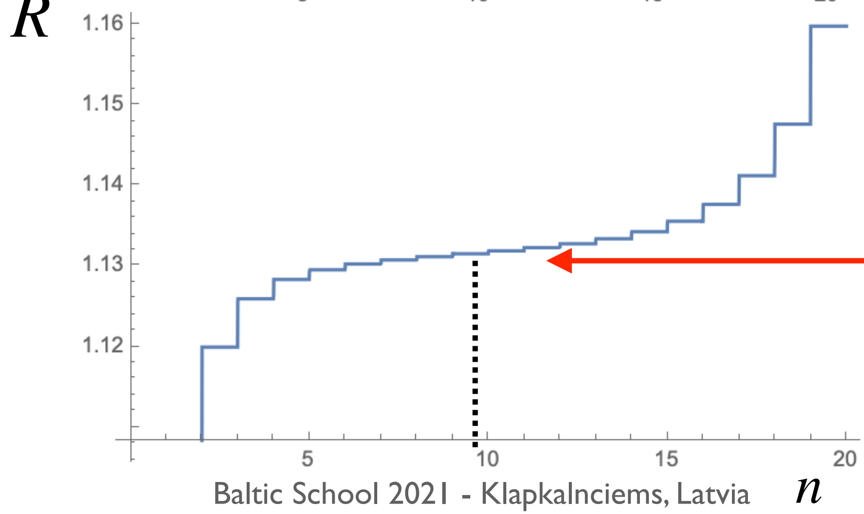
This is big trouble: the series is not convergent, but only asymptotic

Evidence: try summing

$$R = \sum_{n=0}^{\infty} \alpha^n \, n!$$

$$(\alpha = 0.1)$$





Power corrections

The renormalons signal the incompleteness of perturbative QCD

One can only define what the sum of a perturbative series is (like truncation at the minimal term)

The rest is a genuine ambiguity, to be eventually lifted by non-perturbative corrections:

$$R^{true} = R^{pQCD} + R^{NP}$$

In QCD these non-perturbative corrections take the form of power suppressed terms:

$$R^{NP} \sim \exp\left(-\frac{p}{\beta_0 \alpha_s}\right) = \exp\left(-p \ln \frac{Q^2}{\Lambda^2}\right) = \left(\frac{\Lambda^2}{Q^2}\right)^p$$

The value of p depends on the process, and can sometimes be predicted by studying the perturbative series: pQCD - NP physics bridge

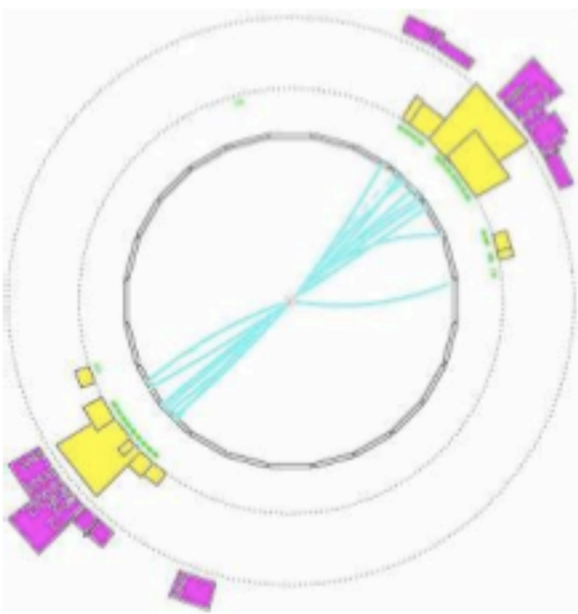
Event shapes: Thrust

Another example of a calculable e^+e^- observable

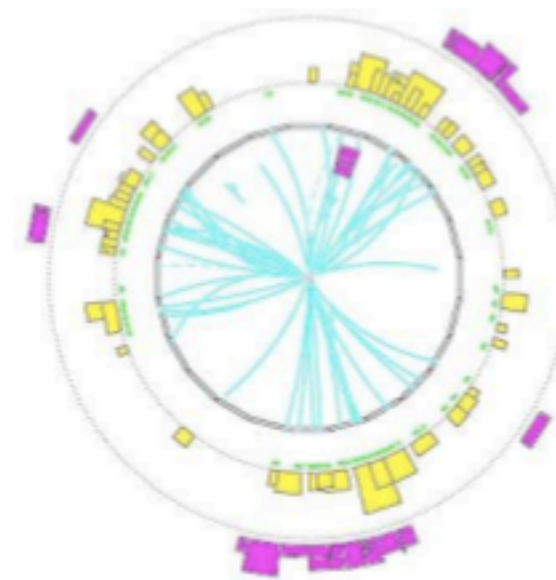
One of the simplest examples of an '**event shape**'

$$T = \max_{\vec{n}_T} \frac{\sum_i |\vec{p}_i . \vec{n}_T|}{\sum_i |\vec{p}_i|} ,$$

Measures the 'pencilness' of an event



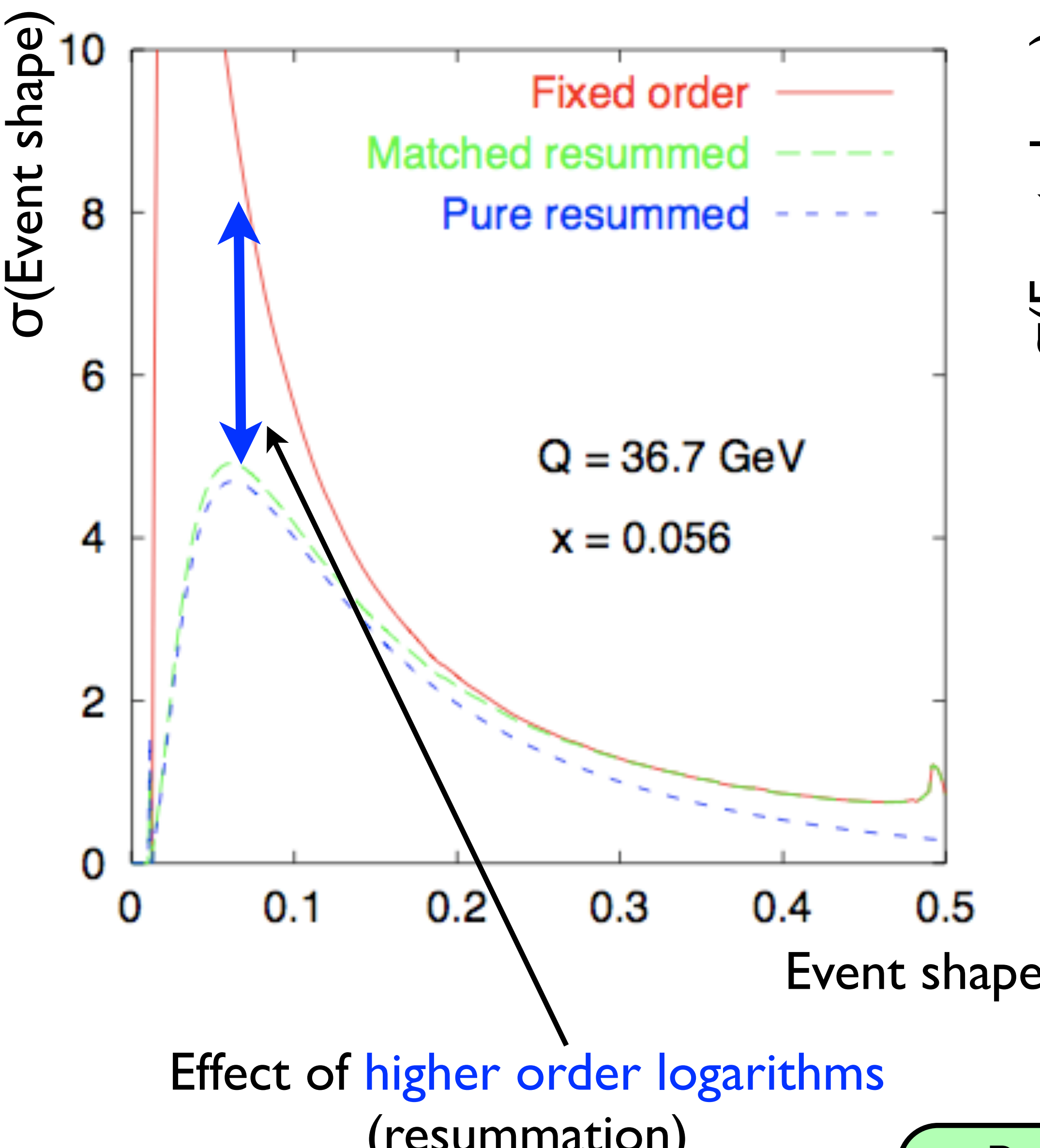
2-jet event: $T \simeq 1$

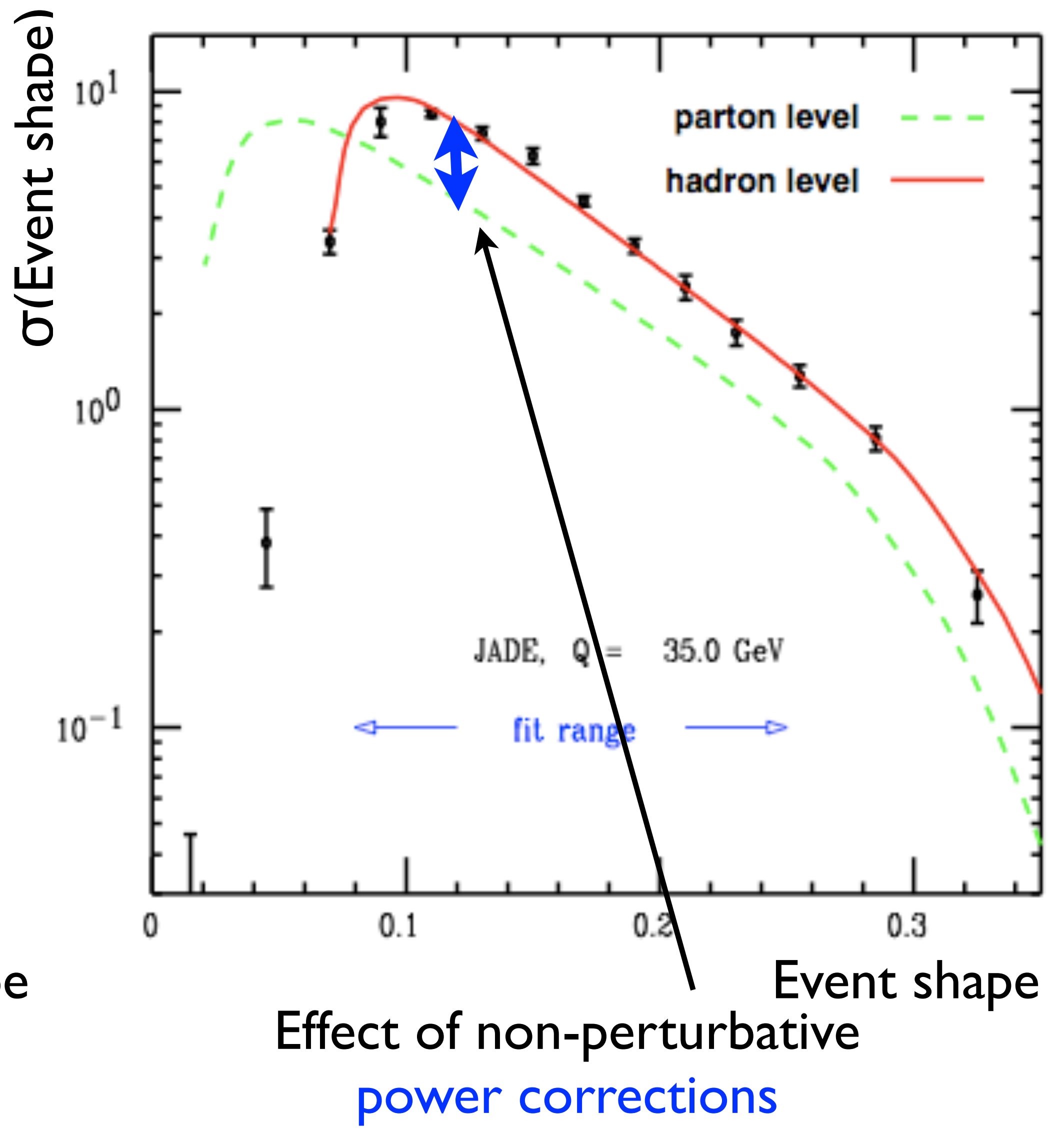


3-jet event: $T \simeq 2/3$

from G. Salam

Perturbative (and NP) QCD predictions

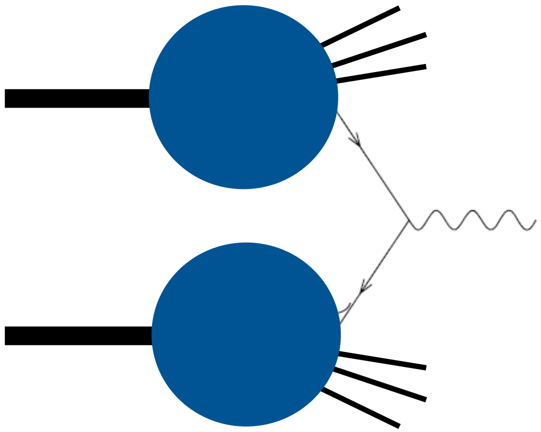




Power corrections models (i.e. Monte Carlo hadronisation) can be built and tested on data

pQCD calculations: hadrons

Turn hadron production in e+e- collisions around: Drell-Yan.



Still easy in Parton Model: just a convolution of probabilities

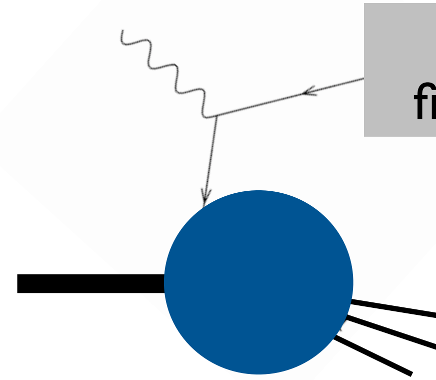
$$\frac{d\sigma_{NN\to\mu\bar{\mu}+X}(Q,p_1,p_2)}{dQ^2 d\ldots} \sim \int d\xi_1 d\xi_2 \sum_{a=q\bar{q}} \frac{d\sigma^{\mathrm{EW,\,Born}}_{a\bar{a}\to\mu\bar{\mu}}(Q,\xi_1 p_1,\xi_2 p_2)}{dQ^2 d\ldots}$$

$$\times(\text{probability to find parton a}(\xi_1)\text{ in }N)$$

$$\times(\text{probability to find parton }\bar{\text{a}}(\xi_2)\text{ in }N)$$

This isn't anymore an **inclusive process** as far as hadrons are concerned: I find them in the initial state, **I can't 'sum over all of them'**

Still, the picture holds at tree level (parton model)
The parton distribution functions can be roughly equated to those extracted from DIS



Sum over all final state hadrons

The non-inclusiveness of a general strong interaction process is a threat to calculability.

What do we do if we can't count on Bloch-Nordsieck and Kinoshita-Lee-Nauenberg?

QCD calculations adopt two strategies:

- Infrared and collinear safe observables
  - less inclusive but still calculable in pQCD
- Factorisation
  - trade divergencies for universal measurable quantities

A generic (not fully inclusive) observable *O* is **infrared and collinear safe** if

$$O(X; p_1, \ldots, p_n, p_{n+1} \to 0) \to O(X; p_1, \ldots, p_n)$$
$$O(X; p_1, \ldots, p_n \parallel p_{n+1}) \to O(X; p_1, \ldots, p_n + p_{n+1})$$

Infrared and collinear safety demands that, in the limit of a collinear splitting, or the emission of an infinitely soft particle, the observable remain **unchanged**

IRC safety: proof

Cancellation of singularities in **total cross section** (KLN)

$$\sigma_{tot} = \int_n |M_n^B|^2 d\Phi_n + \int_n |M_n^V|^2 d\Phi_n + \int_{n+1} |M_{n+1}^R|^2 d\Phi_{n+1}$$

A generic observable

$$\begin{aligned}\frac{dO}{dX} &= \int_n |M_n^B|^2 O(X; p_1, \ldots, p_n) d\Phi_n \\ &+ \int_n |M_n^V|^2 O(X; p_1, \ldots, p_n) d\Phi_n + \int_{n+1} |M_{n+1}^R|^2 O(X; p_1, \ldots, p_n, p_{n+1}) d\Phi_{n+1}\end{aligned}$$

In order to ensure the same cancellation existing in σ_{tot}, the definition of the observable must not affect the soft/collinear limit of the real emission term, because it is there that the real/virtual cancellation takes place

Drell-Yan: factorisation

Non fully inclusive process (hadrons in initial state): non cancellation of collinear singularities in pQCD

Same procedure used for renormalising the coupling: reabsorb the divergence into bare non-perturbative quantities, the parton probabilities (collinear factorisation)

The factorisation theorem

$$\sigma^{phys} = F^{bare} F^{bare} \sigma^{divergent}(\varepsilon) = F(\mu) F(\mu) \hat{\sigma}(\mu)$$

infrared regulator

Parton Distribution Function

factorisation scale

short-distance cross section

and (schematically) $$F(\mu) = F^{bare}\left(1 + \alpha_s P \log\frac{\mu^2}{\mu_0^2}\right)$$

This factor universal

Drell-Yan: NLO result

$$\frac{d^2\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma^* g}(z,Q^2,\mu^2)}{dQ^2} = \sigma_0(Q^2)\left(\frac{\alpha_s(\mu)}{\pi}\right)\left\{2(1+z^2)\left[\frac{\ln(1+z^2)}{1-z}\right]_+ - \frac{\left[(1+z^2)\ln z\right]}{(1-z)} + \left(\frac{\pi^2}{3}-4\right)\delta(1-z)\right\} + \sigma_0(Q^2)\, C_F\, \frac{\alpha_s}{\pi}\left[\frac{1+z^2}{1-z}\right]_+ \ln\left(\frac{Q^2}{\mu^2}\right)$$

→ soft and collinear large log

→ residual of collinear factorisation

A prototype of QCD calculations: many finite terms but, more importantly, a few characteristic large logarithms

In many circumstances and kinematical situations the logs are much more important than the finite terms: hence in pQCD resummations of these terms are often phenomenologically more relevant than a full higher order calculation

Factorisation

$$\sigma^{phys} = F(\mu)\hat{\sigma}(\mu)$$

Evolution

$$F(\mu) = F^{bare}\left(1 + \alpha_s P \log\frac{\mu^2}{\mu_0^2}\right)$$

$$\frac{d}{d\ln\mu^2}\ln\sigma^{phys} = 0 \quad \Rightarrow \quad \frac{d\ln\hat{\sigma}(\mu)}{\ln\mu^2} = -\frac{d\ln F(\mu)}{\ln\mu^2} = -\alpha_s P$$

DGLAP evolution equations for PDF's

Resummation

Solution of evolution equations resums higher order terms

Responsible for scaling violations

(for instance in DIS structure functions)

DGLAP equations

[Dokshitzer, Gribov, Lipatov, Altarelli, Parisi]

$$\frac{df_q(x,t)}{dt} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}(z)\, f_q(\frac{x}{z},t) + P_{qg}(z) f_g(\frac{x}{z},t)\right]$$

$$\frac{df_g(x,t)}{dt} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{gq}(z)\sum_{i=q,\bar{q}} f_i\left(\frac{x}{z},t\right) + P_{gg}(z) f_g\left(\frac{x}{z},t\right)\right]$$

The Altarelli-Parisi kernels control the evolution of the Parton Distribution Functions

Altarelli-Parisi kernels

[Altarelli-Parisi, 1977, Dokshitzer, 1977]

$$P_{gg} \to 2C_A \left\{ \frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) \right\} + \delta(1-x) \left[\frac{11C_A - 2n_f}{6} \right]$$

$$P_{qq}(z) \to \left(\frac{1+z^2}{1-z} \right)_+ \equiv \frac{1+z^2}{1-z} - \delta(1-z) \int_0^1 dy \left(\frac{1+y^2}{1-y} \right)$$

$$P_{qg} = \frac{1}{2} \left[z^2 + (1-z)^2 \right]$$

$$P_{gq}(z) = \quad C_F \frac{1+(1-z)^2}{z}$$

Higher orders: Curci-Furmansky-Petronzio (1980), Moch, Vermaseren, Vogt (2004)

Altarelli-Parisi kernels: NLO

$$P^{(1)}_{\rm ps}(x) = 4\,C_F n_f\left(\frac{20}{9}\frac{1}{x} - 2 + 6x - 4H_0 + x^2\left[\frac{8}{3}H_0 - \frac{56}{9}\right] + (1+x)\left[5H_0 - 2H_{0,0}\right]\right)$$

$$P^{(1)}_{\rm qg}(x) = 4\,C_A n_f\left(\frac{20}{9}\frac{1}{x} - 2 + 25x - 2p_{\rm qg}(-x)H_{-1,0} - 2p_{\rm qg}(x)H_{1,1} + x^2\left[\frac{44}{3}H_0 - \frac{218}{9}\right]\right.$$
$$\left.+4(1-x)\left[H_{0,0} - 2H_0 + xH_1\right] - 4\zeta_2 x - 6H_{0,0} + 9H_0\right) + 4\,C_F n_f\left(2p_{\rm qg}(x)\left[H_{1,0} + H_{1,1} + H_2\right.\right.$$
$$\left.\left.-\zeta_2\right] + 4x^2\left[H_0 + H_{0,0} + \frac{5}{2}\right] + 2(1-x)\left[H_0 + H_{0,0} - 2xH_1 + \frac{29}{4}\right] - \frac{15}{2} - H_{0,0} - \frac{1}{2}H_0\right)$$

$$P^{(1)}_{\rm gq}(x) = 4\,C_A C_F\left(\frac{1}{x} + 2p_{\rm gq}(x)\left[H_{1,0} + H_{1,1} + H_2 - \frac{11}{6}H_1\right] - x^2\left[\frac{8}{3}H_0 - \frac{44}{9}\right] + 4\zeta_2 - 2\right.$$
$$\left.-7H_0 + 2H_{0,0} - 2H_1 x + (1+x)\left[2H_{0,0} - 5H_0 + \frac{37}{9}\right] - 2p_{\rm gq}(-x)H_{-1,0}\right) - 4\,C_F n_f\left(\frac{2}{3}x\right.$$
$$\left.-p_{\rm gq}(x)\left[\frac{2}{3}H_1 - \frac{10}{9}\right]\right) + 4\,C_F^{\,2}\left(p_{\rm gq}(x)\left[3H_1 - 2H_{1,1}\right] + (1+x)\left[H_{0,0} - \frac{7}{2} + \frac{7}{2}H_0\right] - 3H_{0,0}\right.$$
$$\left.+1 - \frac{3}{2}H_0 + 2H_1 x\right)$$

$$P^{(1)}_{\rm gg}(x) = 4\,C_A n_f\left(1 - x - \frac{10}{9}p_{\rm gg}(x) - \frac{13}{9}\left(\frac{1}{x} - x^2\right) - \frac{2}{3}(1+x)H_0 - \frac{2}{3}\delta(1-x)\right) + 4\,C_A^{\,2}\left(27\right.$$
$$+(1+x)\left[\frac{11}{3}H_0 + 8H_{0,0} - \frac{27}{2}\right] + 2p_{\rm gg}(-x)\left[H_{0,0} - 2H_{-1,0} - \zeta_2\right] - \frac{67}{9}\left(\frac{1}{x} - x^2\right) - 12H_0$$
$$\left.-\frac{44}{3}x^2 H_0 + 2p_{\rm gg}(x)\left[\frac{67}{18} - \zeta_2 + H_{0,0} + 2H_{1,0} + 2H_2\right] + \delta(1-x)\left[\frac{8}{3} + 3\zeta_3\right]\right) + 4\,C_F n_f\left(2H_0\right.$$
$$\left.+\frac{2}{3}\frac{1}{x} + \frac{10}{3}x^2 - 12 + (1+x)\left[4 - 5H_0 - 2H_{0,0}\right] - \frac{1}{2}\delta(1-x)\right)\ .$$

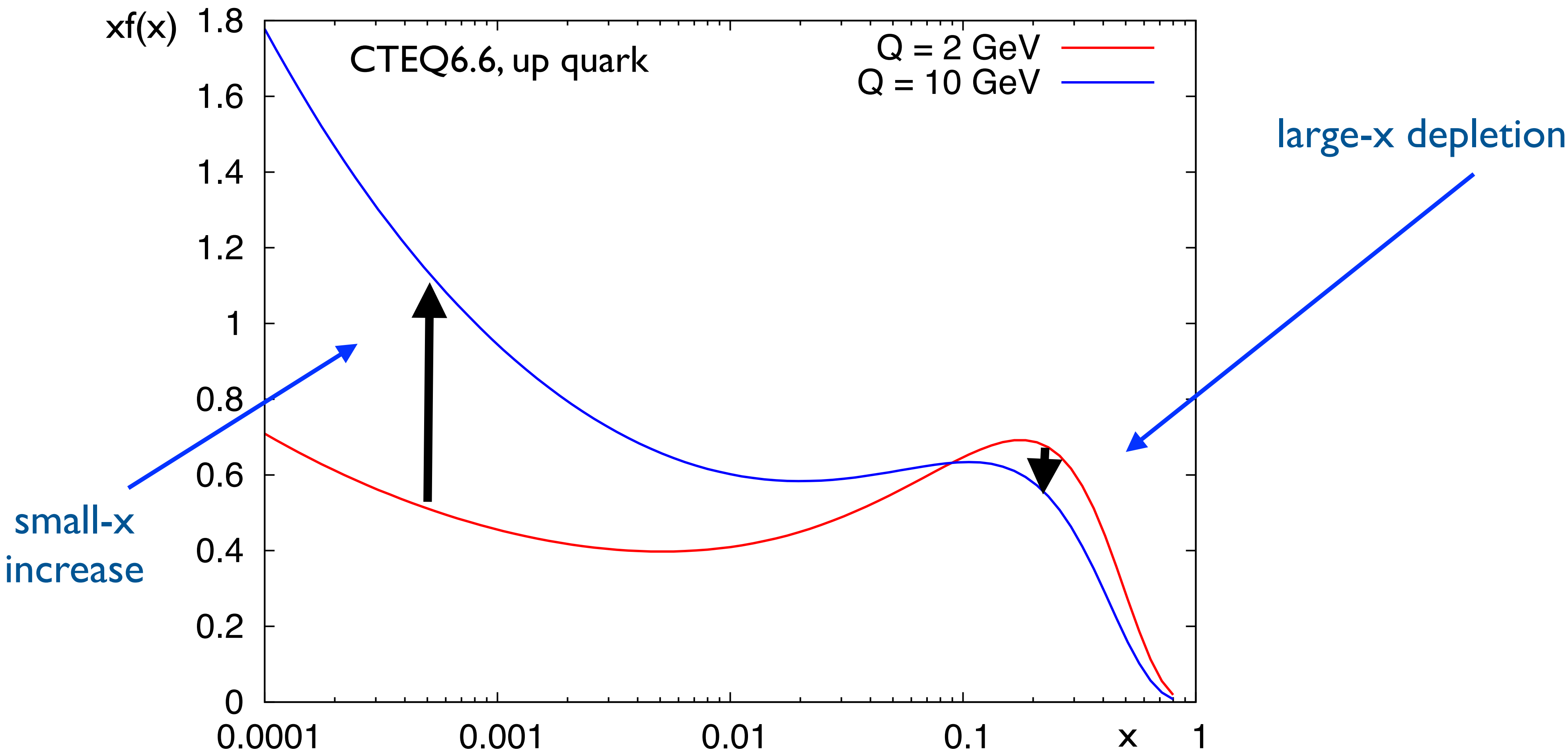
$$P_{ab} = \frac{\alpha_s}{2\pi}P^{(0)} + \frac{\alpha_s^2}{16\pi^2}P^{(1)}$$

Curci, Furmanski & Petronzio '80

Altarelli-Parisi kernels: NNLO

NNLO, $P^{(2)}_{ab}$: Moch, Vermaseren & Vogt '04

DGLAP evolution of PDFs



Evolution (i.e. higher momentum scale) produces more partons at small momentum fraction (because they lose energy by radiating)

As for the coupling, one can't predict PDF's values in pQCD, but only their evolution

Take-home points

- universal character of soft/collinear emission
- both real and virtual diagrams usually contribute to an observable (and are both needed to cancel divergencies)
- not everything is calculable. Restrict to IRC-safe observables and/or employ factorisation