Baltic School of High Energy Physics and Accelerator Technologies
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QCD, Higher Orders and Jets

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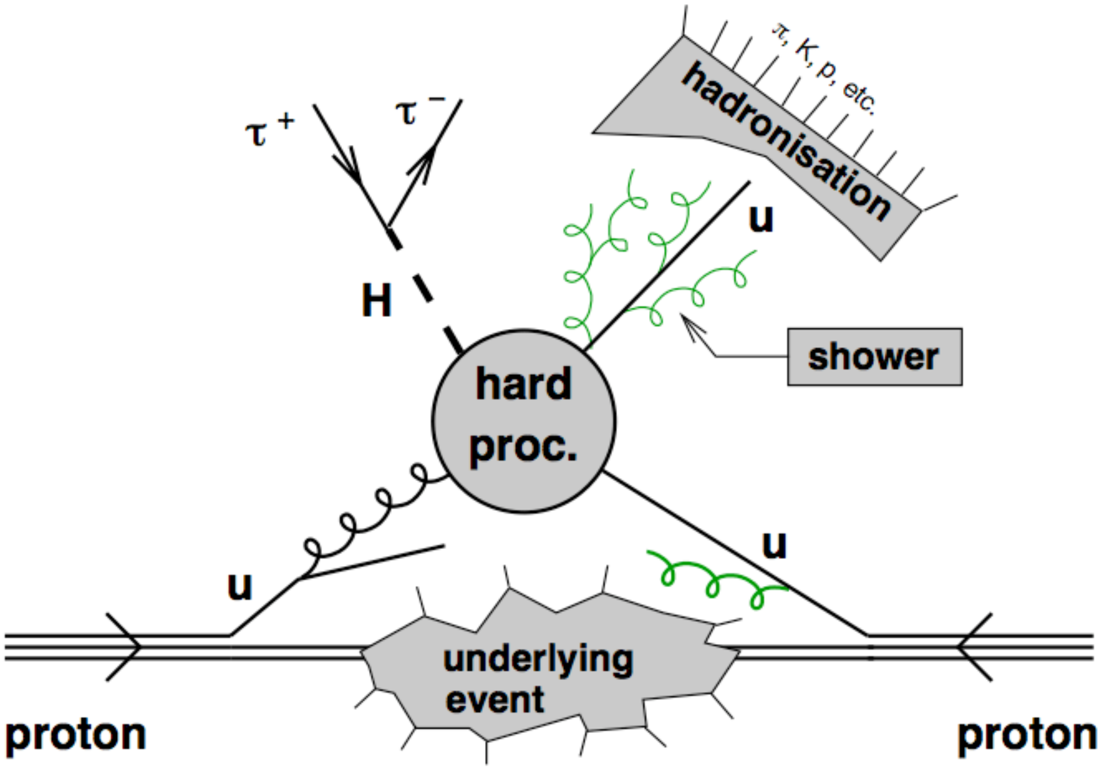
Lecture 2: higher orders and Monte Carlo

Ingredients and tools

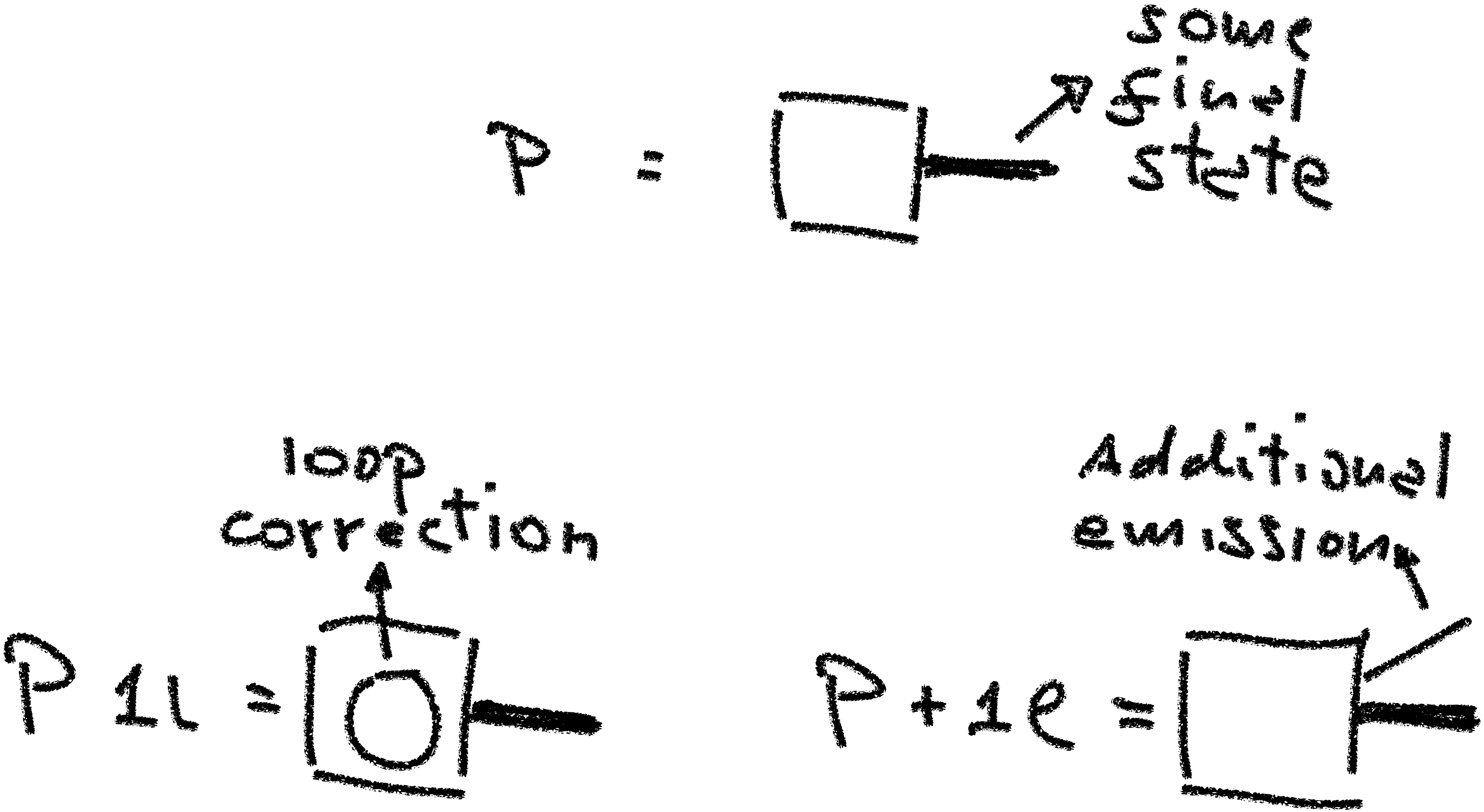


- PDFs
- Hard scattering and shower
- Final state tools

(Higher order) calculations

What goes into them ?

Nomenclature



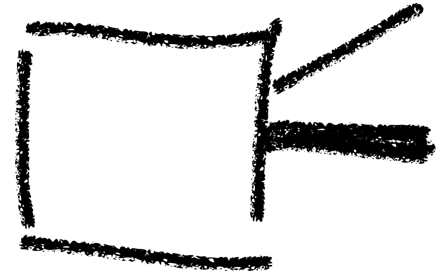
N.B.

$$P + \underline{1}\,e \neq P + \underline{1}\ \text{jet}$$

e = emission

or

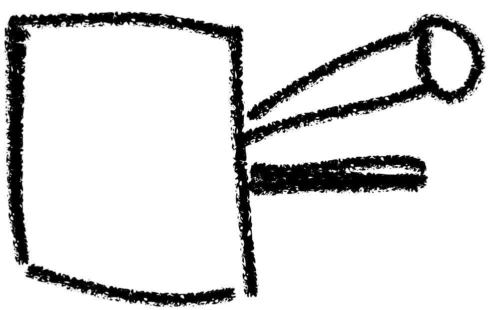
e = eeg

P + 1e = 

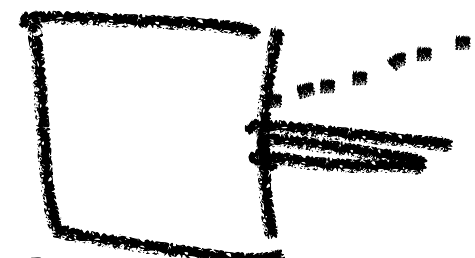
Contributes to:

P + 1 jet

P + X

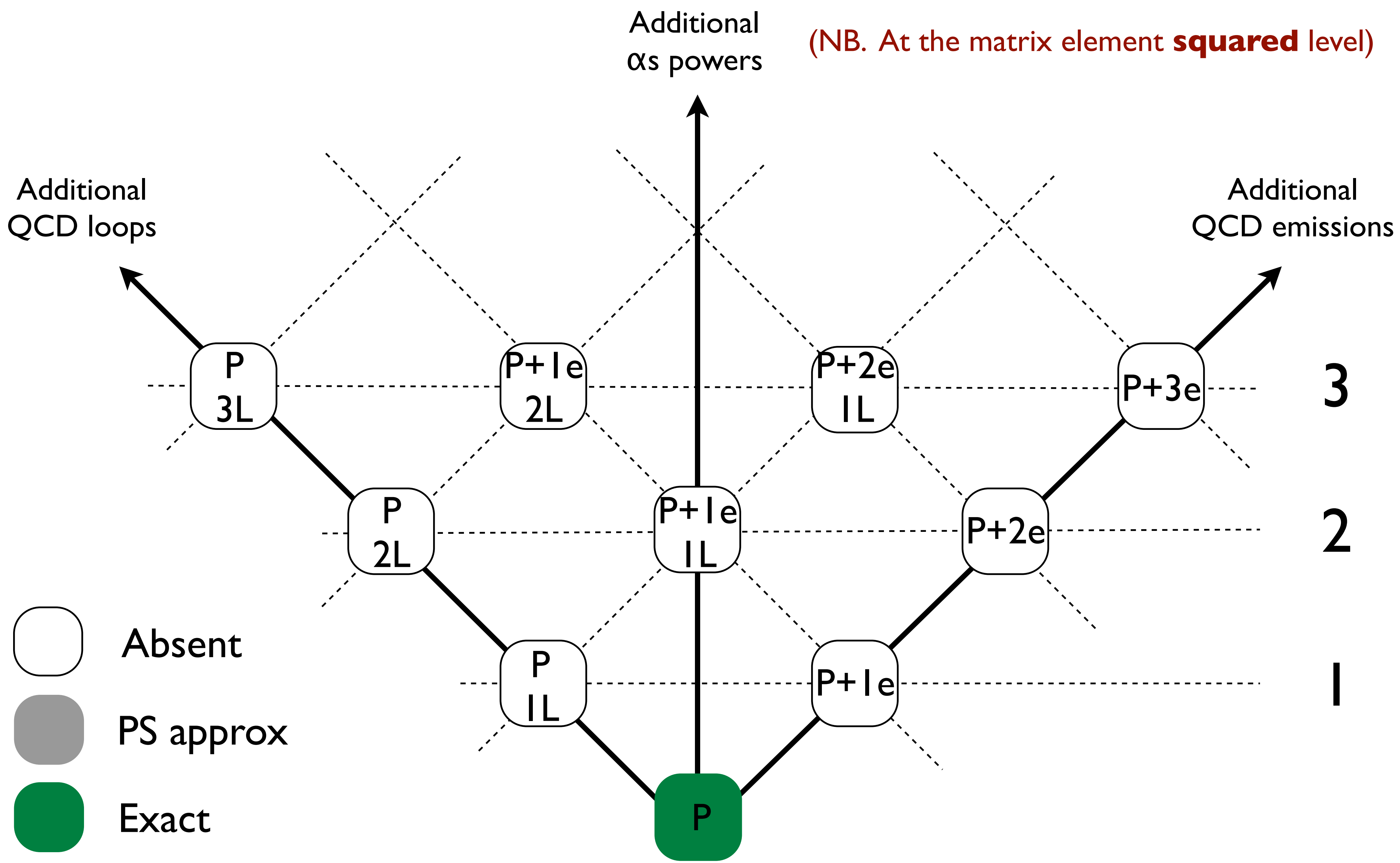


or

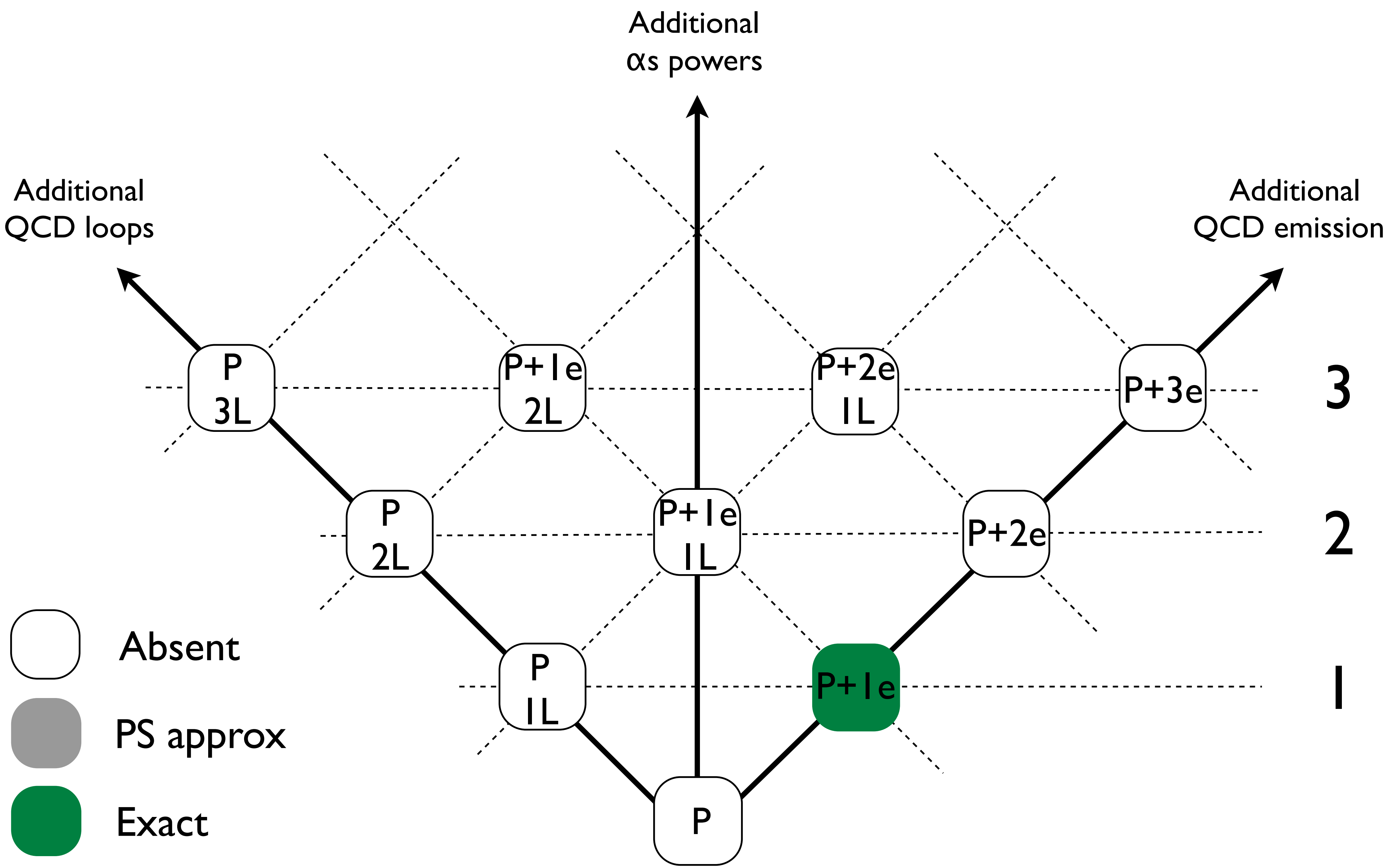


(if e is integrated over)

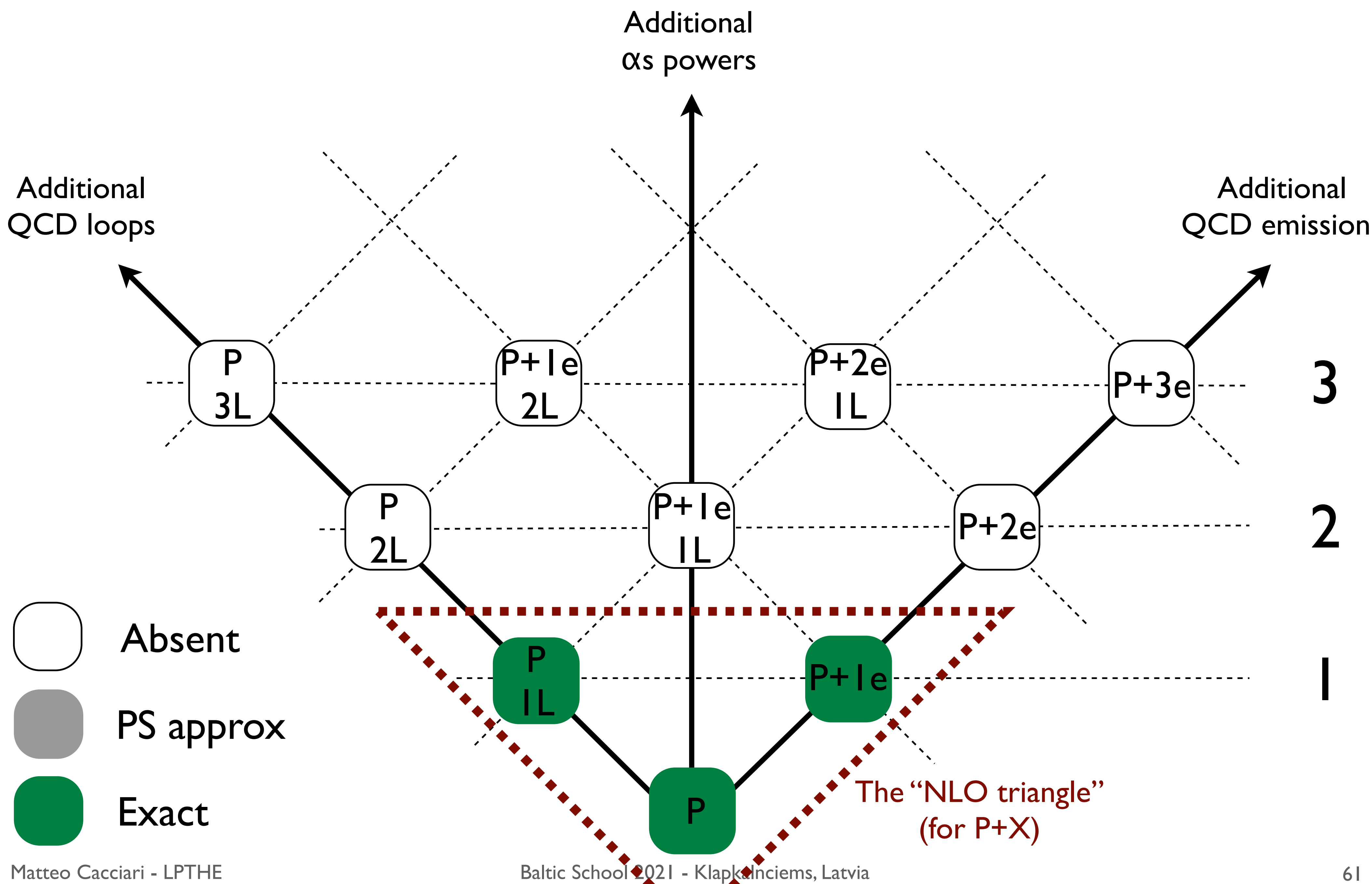
Process P exact at LO, nothing else



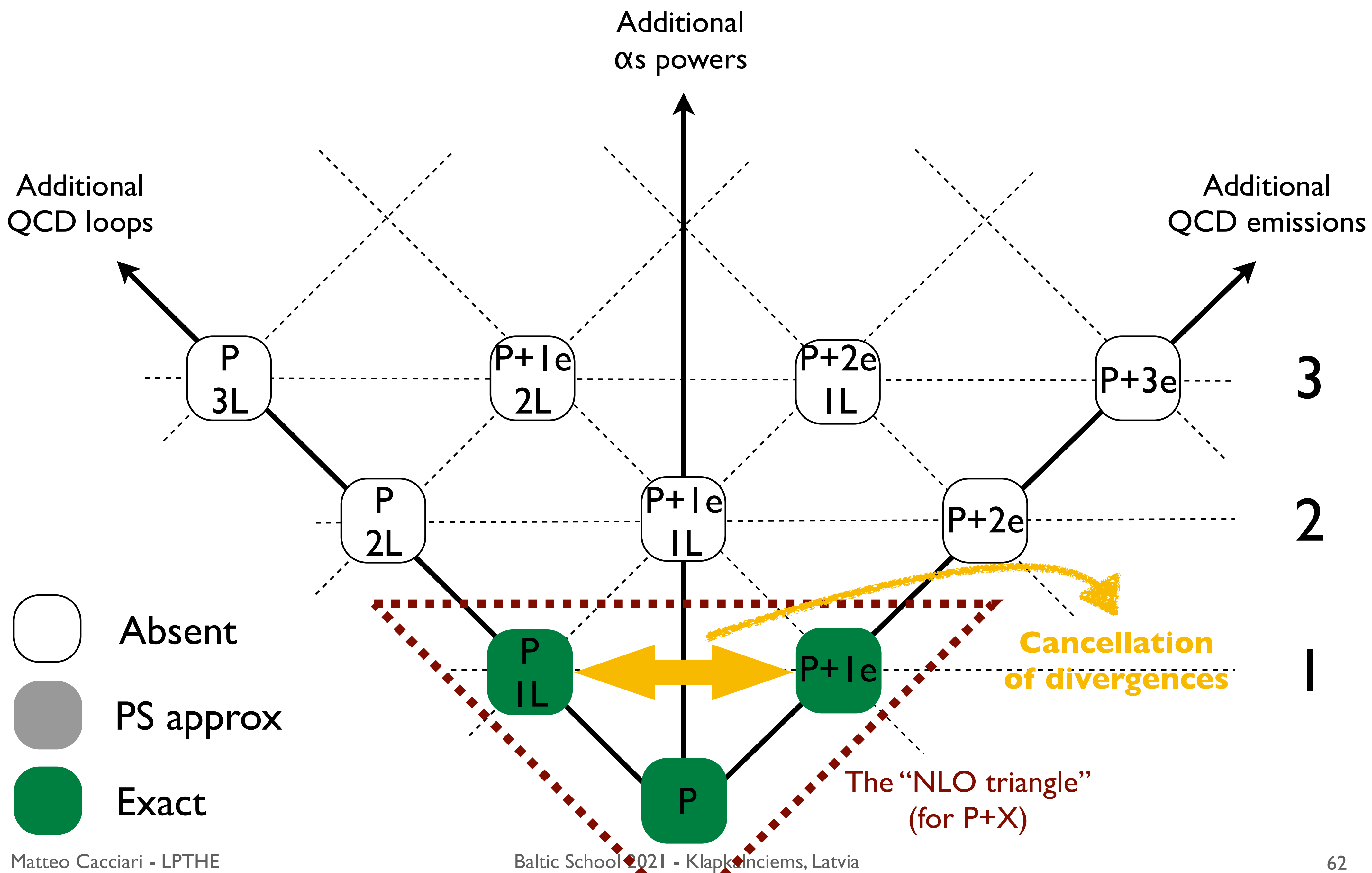
Process P+1j exact at LO, nothing else



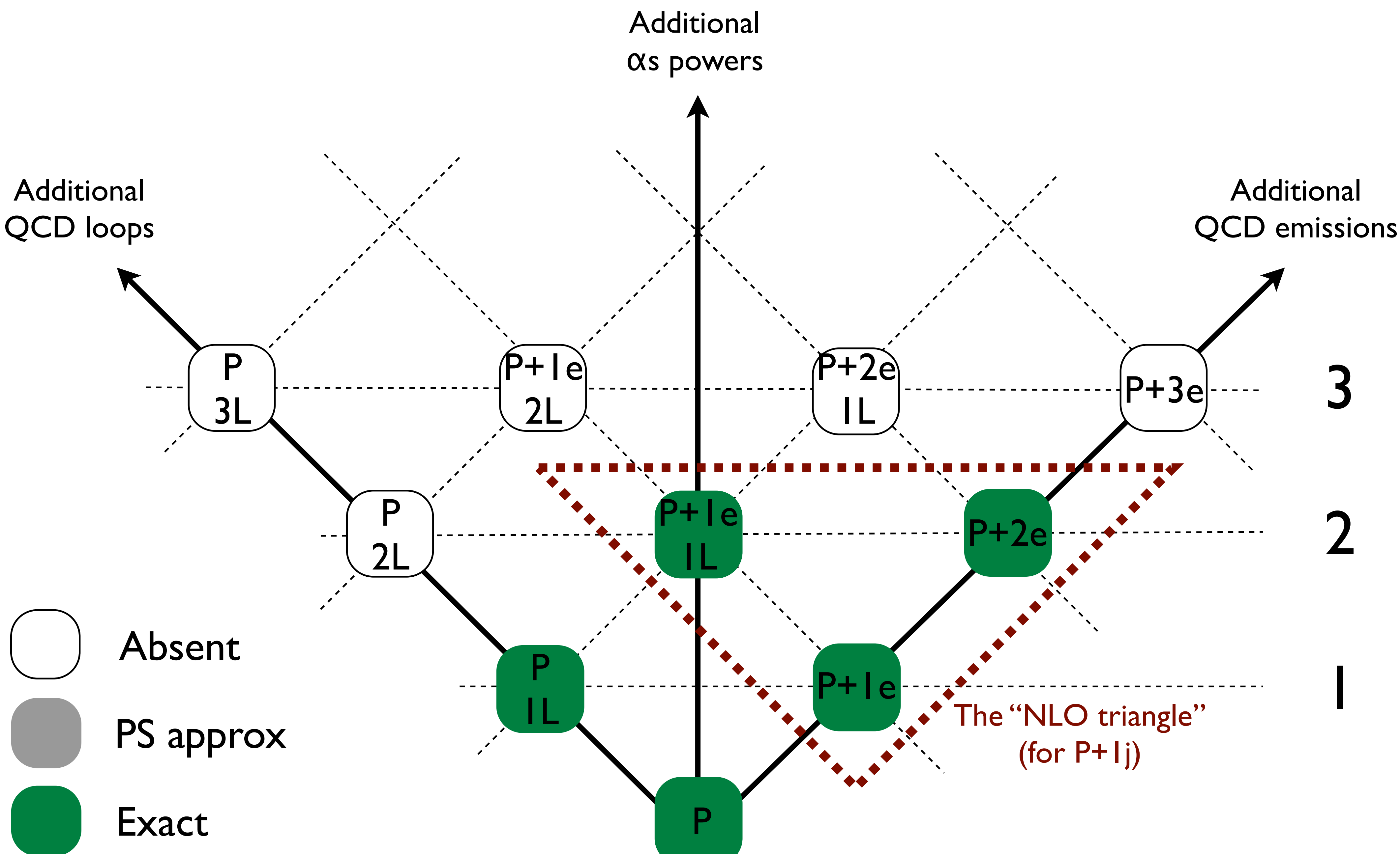
Process P exact at NLO, P+1j exact at LO, nothing else



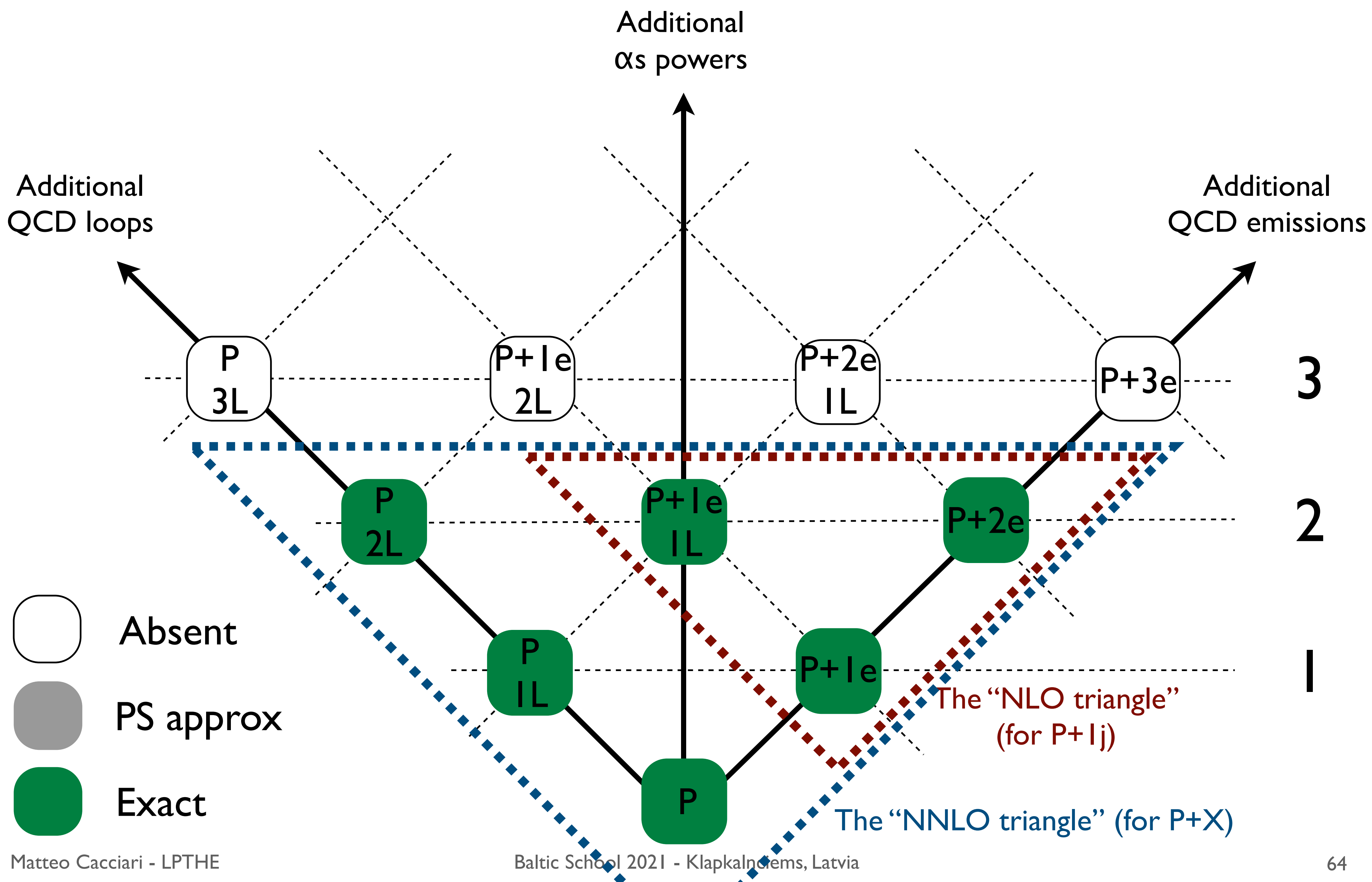
Process P exact at NLO, P+1j exact at LO, nothing else



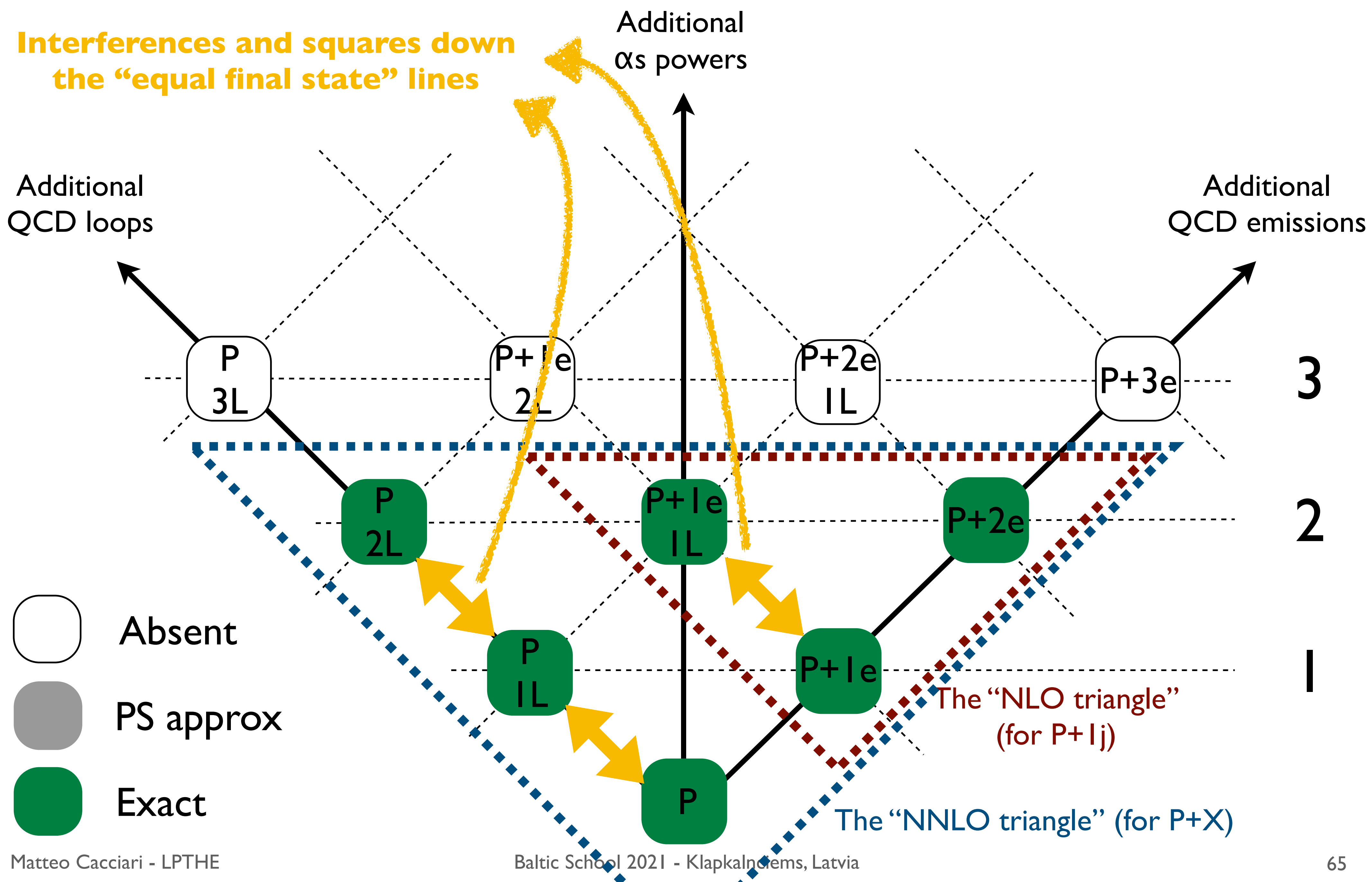
Process P and P+1j exact at NLO, P+2j at LO



Process P exact at NNLO, P+1j exact at NLO, P+2j at LO



Process P exact at NNLO, P+1j exact at NLO, P+2j at LO



Tools for the hard scattering

Can be divided in

- **Integrators**
  - evaluate the (differential) cross section by integrating the calculation over the phase space, yielding (partly) inclusive quantities
  - Produce weighted events (the weight being the value of the cross section)
  - Calculations exist at LO, NLO, NNLO

- **Generators**
  - generate fully exclusive configurations
  - Events are unweighted (i.e. produced with the frequency nature would produce them)
  - Easy at LO, get complicated when dealing with higher orders

Fixed order calculation

Born

$$d\sigma^{Born} = B(\Phi_B) d\Phi_B$$

NLO

$$d\sigma^{NLO} = [B(\Phi_B) + V(\Phi_B)]\, d\Phi_B + R(\Phi_R) d\Phi_R$$

$$d\Phi_R = d\Phi_B \, d\Phi_{rad}$$

$$d\Phi_{rad} = d\cos\theta \, dE \, d\phi$$

Problem:
$V(\Phi_B)$ and $\int R d\Phi_R$ are divergent

Subtraction terms

An observable O is **infrared and collinear safe** if

$$O(\Phi_{\rm R}(\Phi_{\rm B}, \Phi_{\rm rad})) \to O(\Phi_{\rm B})$$

Soft or collinear limit

One can then write, with R→C in the soft/coll limit,

This integration performed analytically

$$\langle O \rangle = \int \left[B(\Phi_B) + V(\Phi_B) + \int C(\Phi_R) d\Phi_{rad} \right] O(\Phi_B) d\Phi_B$$

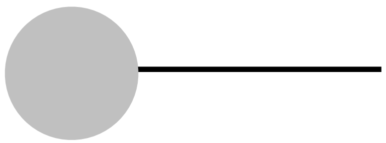
$$+ \left[R(\Phi_R) O(\phi_R) - C(\Phi_R) O(\Phi_B) \right] d\Phi_R$$

Separately finite

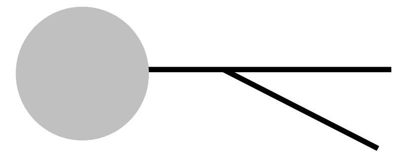
This (or a similar) cancellation will always be implicit in all subsequent equations

Parton Shower Monte Carlo

Exploit factorisation property of soft and collinear radiation



σ_n



σ_{n+1}

Factorisation

$$\mathrm{d}\sigma_{n+1}(\Phi_{n+1}) = \mathcal{P}(\Phi_{\mathrm{rad}})\,\mathrm{d}\sigma_n(\Phi_n)\,\mathrm{d}\Phi_{\mathrm{rad}}$$

Emission probability

$$\mathcal{P}(\Phi_{\mathrm{rad}})\,\mathrm{d}\Phi_{\mathrm{rad}} \approx \frac{\alpha_{\mathrm{S}}(q)}{\pi}\,\frac{\mathrm{d}q}{q}\,P(z,\phi)\,\mathrm{d}z\,\frac{\mathrm{d}\phi}{2\pi}$$

Iterate emissions to generate higher orders (in the soft/collinear approximation)

Parton Shower MC

Based on the **iterative emission of radiation** described in the **soft-collinear limit**

$$d\sigma^{(MC)}(\Phi_R)d\Phi_R = B(\Phi_B)d\Phi_B\mathcal{P}(\Phi_{rad})d\Phi_{rad}$$

Pros: soft-collinear radiation is resummed to all orders in pQCD

Cons: hard large-angle radiation is missing

Overall accuracy will be leading log (LL) for the radiation, and leading order (i.e. Born) for the integrated cross sections

A key ingredient of a parton shower Monte Carlo:

Sudakov form factor
$\Delta(t_1,t_2)$

Probability of **no emission** between the scales t_1 and t_2

Example:

- decay probability per unit time of a nucleus = c_N
- Sudakov form factor $\Delta(t_0,t) = \exp(-c_N(t-t_0))$

Probability that nucleus does **not** decay between t_0 and t

Sudakov form factor: derivation

Decay probability per unit time = $$\frac{dP}{dt} = c_N$$

Probability of **no** decay between t_0 and t = $\Delta(t_0,t)$ [with $\Delta(t_0,t_0) = 1$]

⇒ Probability of decay between t_0 and t = 1- $\Delta(t_0,t)$ [unitarity: either you decay or you don't]

Decay probability per unit time **at time t** can be written in two ways:

1. $$P^{\rm dec}(t) = \frac{d}{dt}\Big(1 - \Delta(t_0,t)\Big) = -\frac{d\Delta(t_0,t)}{dt}$$

2. $$P^{\rm dec}(t) = \Delta(t_0,t)\frac{dP}{dt}$$

No decay until t, probability per unit time to decay at t

Sudakov form factor: derivation

Equating the two expressions for $P^{dec}(t)$ we get

$$-\frac{d\Delta(t_0,t)}{dt} = \Delta(t_0,t)\frac{dP}{dt}$$

We can solve the differential equation using $dP/dt = c_N$ and we get

$$\Delta(t_0,t) = \exp(-c_N(t-t_0))$$

If the decay probability depends on t (and possibly other variables, call them z) this generalises to

$$\Delta(t_0,t) = \exp\left(-\int_{t_0}^{t} dt' \int dz\, c_N(t',z)\right)$$

Sudakov form factor in QCD

Emission probability

$$\mathcal{P}(\Phi_{\rm rad})\,\mathrm{d}\Phi_{\rm rad} \approx \frac{\alpha_{\rm S}(q)}{\pi}\frac{\mathrm{d}q}{q}P(z,\phi)\,\mathrm{d}z\frac{\mathrm{d}\phi}{2\pi}$$

Sudakov form factor = probability of **no emission**

from large scale q_1 to smaller scale q_2

$$\Delta_{\rm S}(q_1,q_2) = \exp\left[-\int_{q_2}^{q_1}\frac{\alpha_{\rm S}(q)}{\pi}\frac{\mathrm{d}q}{q}\int_{z_0}^{1}P(z)\,\mathrm{d}z\right]$$

Conventions for Sudakov form factor

$$\Delta_{\mathrm{S}}(q_1, q_2) = \exp\left[-\int_{q_2}^{q_1} \frac{\alpha_{\mathrm{S}}(q)}{\pi}\frac{\mathrm{d}q}{q}\int_{z_0}^{1} P(z)\,\mathrm{d}z\right]$$

Full expression,with details of soft-collinear radiation probability

$$\Delta(p_{\mathrm{T}}) = \exp\left[-\int_{p_{\mathrm{T}}}^{Q} \frac{\frac{\mathrm{d}\sigma^{(\mathrm{MC})}}{\mathrm{d}y\,\mathrm{d}p'_{\mathrm{T}}}}{\frac{\mathrm{d}\sigma^{(\mathrm{B})}}{\mathrm{d}y}}\mathrm{d}p'_{\mathrm{T}}\right]$$

Dropped upper limit, taken implicitly to be the hard scale Q

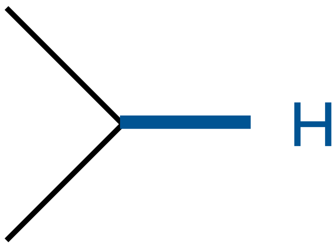
$$\Delta_R(p_T) = \exp\left[-\int \frac{R}{B}\Theta(k_T(\Phi_R) - p_T)d\Phi_{rad}\right]$$

Introduced suffix (R in this case) to indicate expression used to described radiation

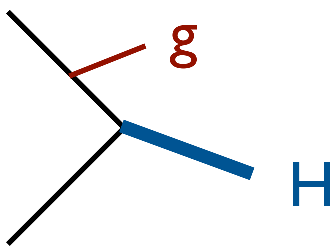
$$\Delta_R(p_T) = \exp\left[-\int_{p_T} \frac{R}{B}d\Phi_{rad}\right]$$

Integration boundaries only implicitly indicated

PS example: Higgs plus radiation



Leading order.
No radiation, Higgs $p_T = 0$



With emission of radiation
Higgs $p_T \neq 0$

Description of hardest emission in PS MC (either event is generated)

$$\frac{d\sigma^{(MC)}}{dy\,dp_T} = \frac{d\sigma^{(B)}}{dy}\delta(p_T)\Delta(Q_0) + \Delta(p_T)\frac{d\sigma^{(MC)}}{dy\,dp_T}$$

$$\Delta(p_T) = \exp\left[-\int_{p_T}^{Q} \frac{\frac{d\sigma^{(MC)}}{dy\,dp'_T}}{\frac{d\sigma^{(B)}}{dy}} dp'_T\right]$$

Sudakov form factor

- x-sect for no emission
- prob. of **no emission** (down to the PS cutoff)
- prob. of no emission down to p_T
- x-sect for **emission at p_T**, as described by the MC

Toy shower for the Higgs p_T

Gavin Salam has made public a 'toy shower' that generates the Higgs transverse momentum via successive emissions controlled by the Sudakov form factor

$$\Delta(p_T) = \exp\left[-\frac{2\alpha_s C_A}{\pi}\ln^2\frac{p_{T,\max}^2}{p_T^2}\right]$$

You can get the code at
https://github.com/gavinsalam/zuoz2016-toy-shower

NB. In order to get more realistic results you need at least at the code in v2

Shower unitarity

It holds

$$\int_0^Q \left[\delta(p_T)\Delta(Q_0) + \frac{\Delta(p_T)\frac{d\sigma^{(MC)}}{dy dp_T}}{\frac{d\sigma^{(B)}}{dy}}\right] dp_T = \Delta(Q_0) + \int_{Q_0}^{Q} \frac{d\Delta(p_T)}{dp_T} dp_T = \Delta(Q) = 1$$

Shower unitarity

so that

$$\int_0^Q dp_T \frac{d\sigma^{(MC)}}{dy dp_T} = \frac{d\sigma^{(B)}}{dy} \int_0^Q \left[\delta(p_T)\Delta(Q_0) + \frac{\Delta(p_T)\frac{d\sigma^{(MC)}}{dy dp_T}}{\frac{d\sigma^{(B)}}{dy}}\right] dp_T = \frac{d\sigma^{(B)}}{dy}$$

A parton shower MC correctly reproduces the Born cross section for integrated quantities

PS MC in different notation

Writing the real cross section as described by the Monte Carlo (i.e. with the parton shower) simply as R^{MC}, we can rewrite

$$d\sigma^{MC} = Bd\Phi_B \left[\Delta_{MC}(Q_0) + \Delta_{MC}(p_T) \frac{R^{MC}}{B} d\Phi_{rad} \right]$$

with $$\Delta_{MC}(p_T) = \exp \left[- \int_{p_T} \frac{R^{MC}}{B} d\Phi_{rad} \right]$$

as our Master Formula for a Parton Shower Monte Carlo.

Thanks to the shower unitarity, it holds

$$\Rightarrow \int d\sigma^{MC} = \int Bd\Phi_B = \sigma^{LO}$$

Matrix Element corrections

In a PS Monte Carlo $R^{(MC)}(\Phi_R) = B(\Phi_B)\mathcal{P}(\Phi_{rad})$ soft-collinear approximation

Replace the MC description of radiation with the **correct** one:

$$\mathcal{P}(\Phi_{rad}) \longrightarrow \frac{R}{B}$$

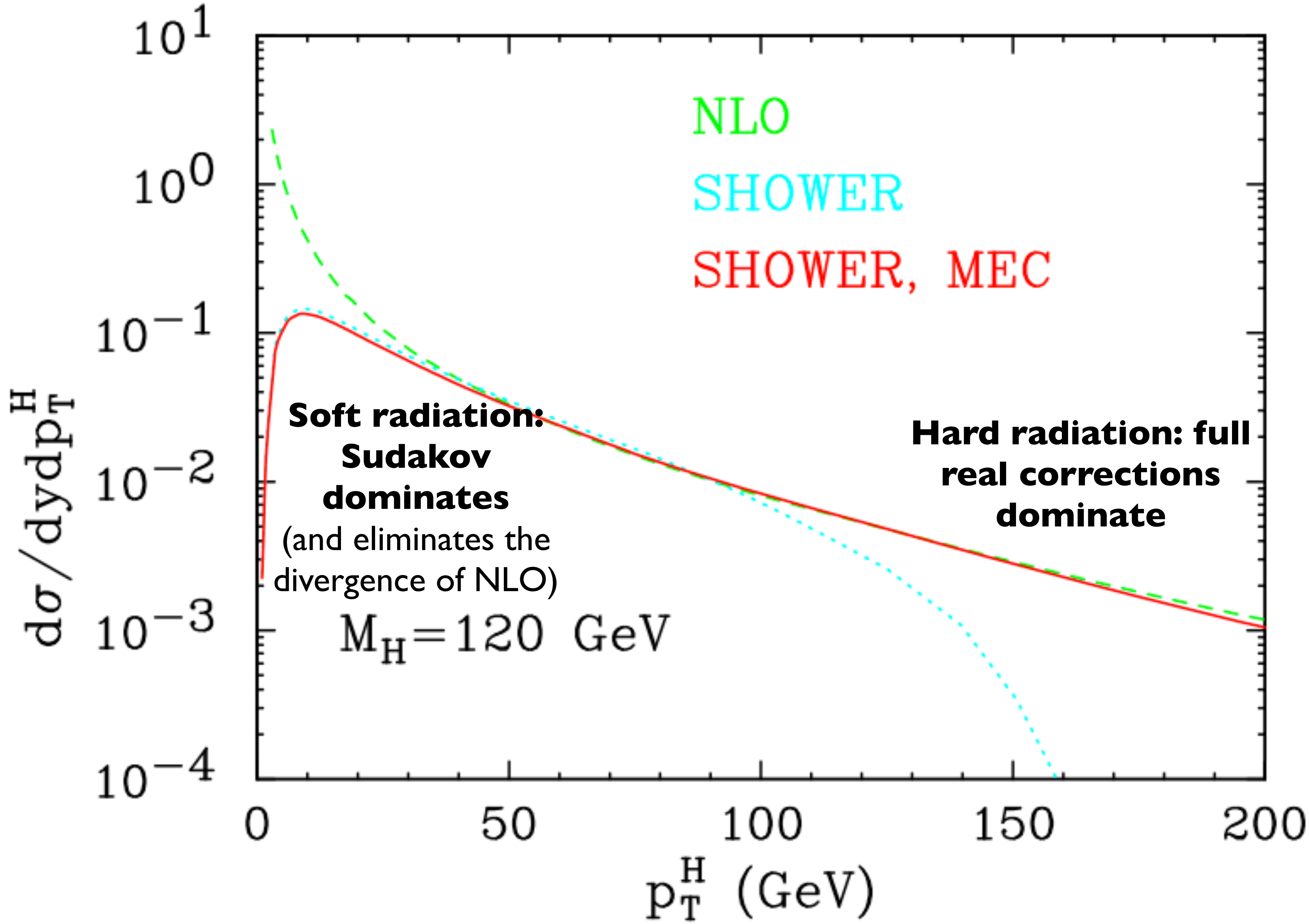
The Sudakov becomes

$$\Delta(p_\mathrm{T}) = \exp\left[-\int_{p_\mathrm{T}}^{Q} \frac{\frac{\mathrm{d}\sigma^{(\mathrm{MC})}}{\mathrm{d}y\,\mathrm{d}p'_\mathrm{T}}}{\frac{\mathrm{d}\sigma^{(\mathrm{B})}}{\mathrm{d}y}} \mathrm{d}p'_\mathrm{T}\right] \longrightarrow \Delta_R(p_T) = \exp\left[-\int \frac{R}{B}\Theta(k_T(\Phi_R) - p_T)d\Phi_{rad}\right]$$

and the x-sect formula for the hardest emission

$$d\sigma^{MEC} = Bd\Phi_B\left[\Delta_R(Q_0) + \Delta_R(p_T)\frac{R}{B}d\Phi_{rad}\right]$$

Matrix Element corrections

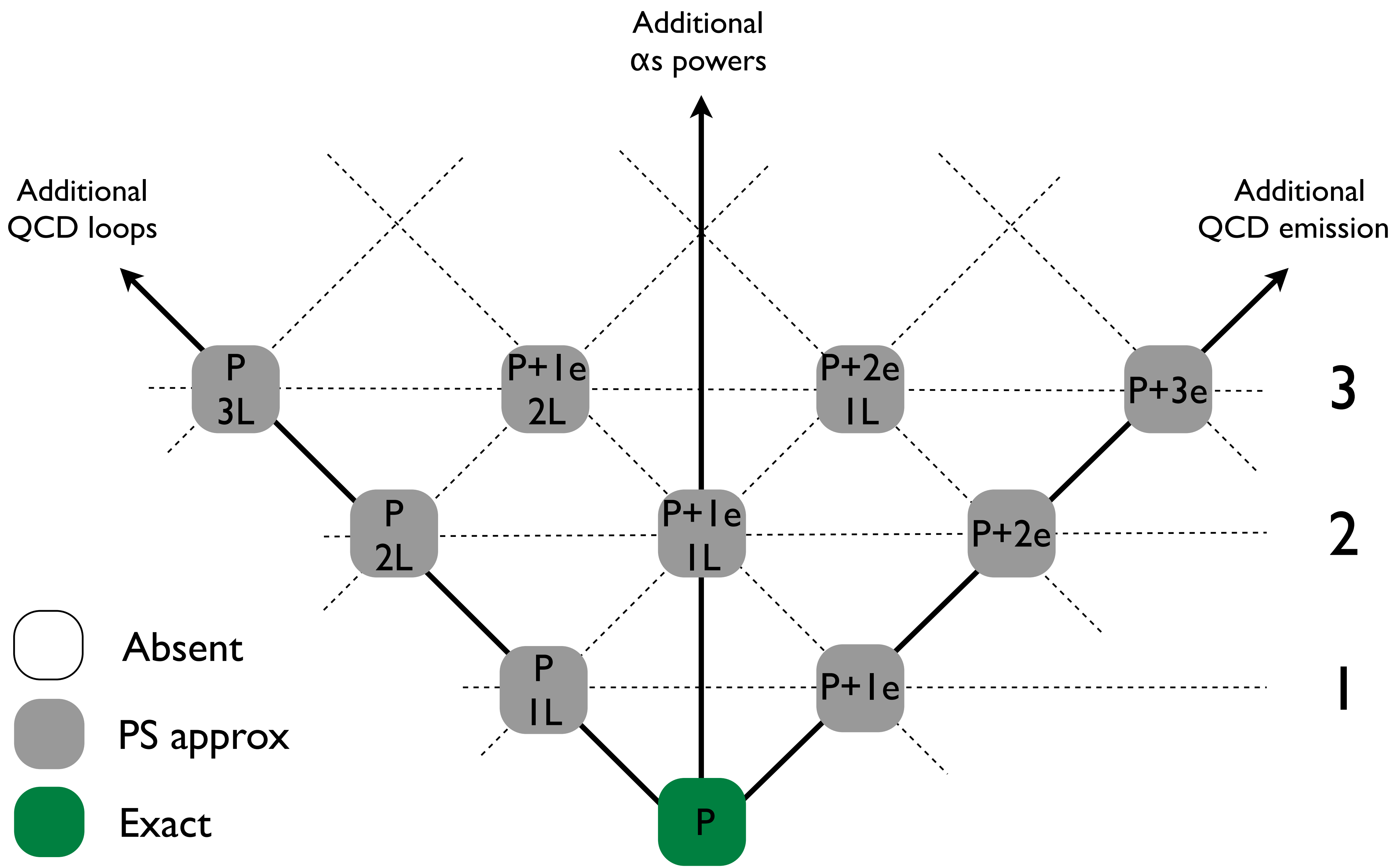


We wish to go beyond a Parton Shower (+MEC) Monte Carlo, so that

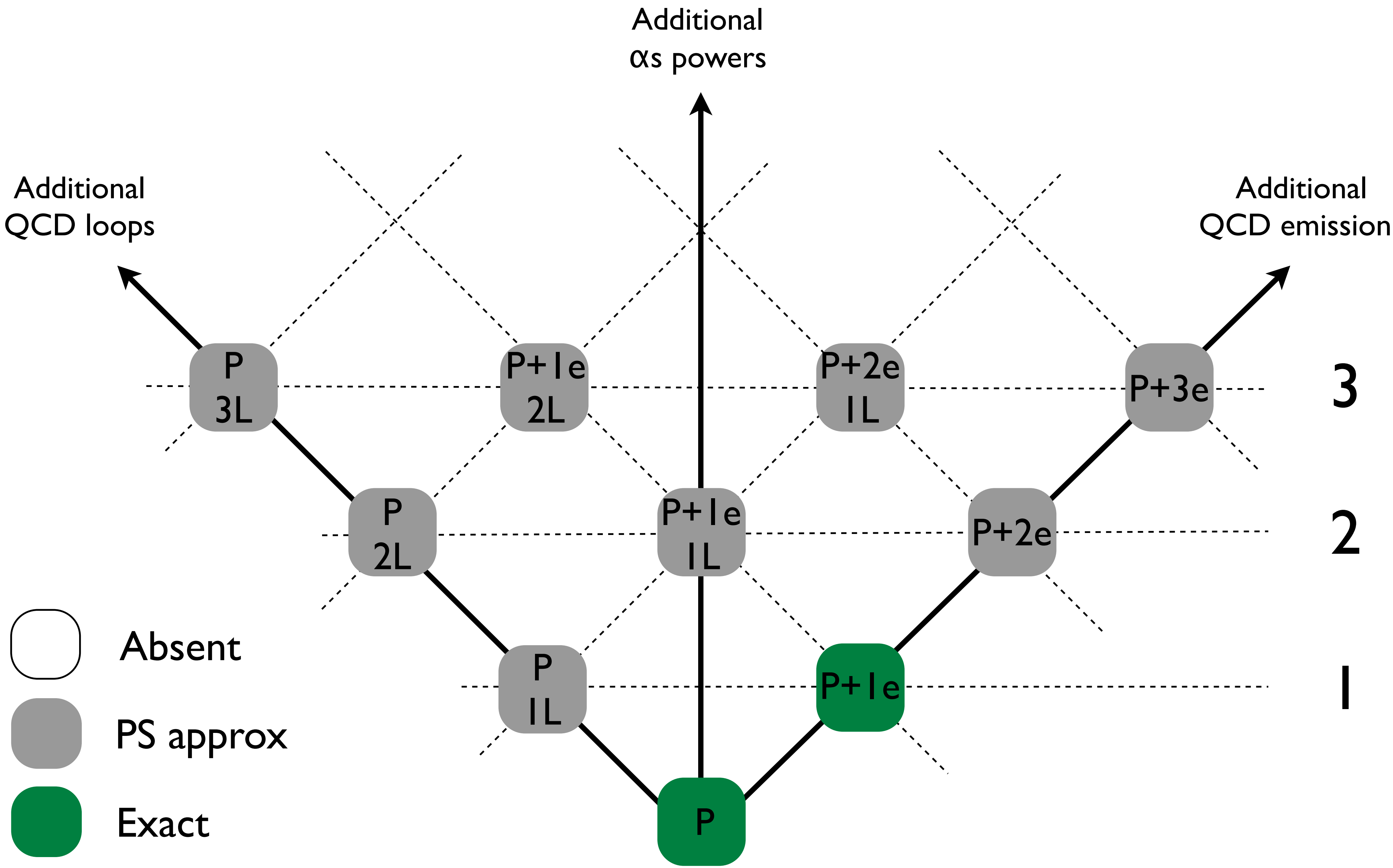
- we can successfully interface **matrix elements for multi-parton production with a parton shower**
- we can successfully interface a **parton shower with a NLO calculation**

It's a **quest for exactness** of ever more complex processes

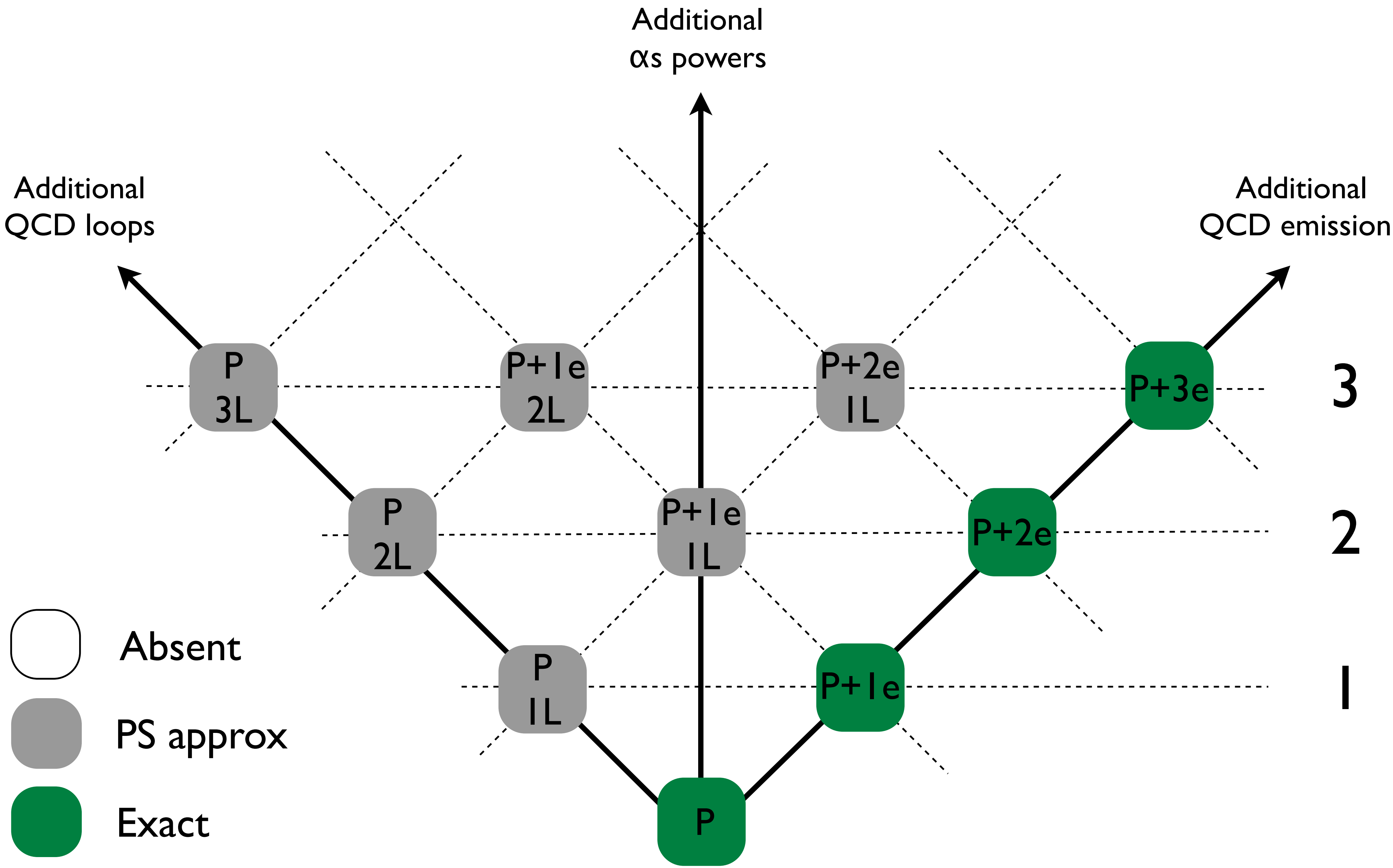
Process P exact at LO, the rest PS approximation



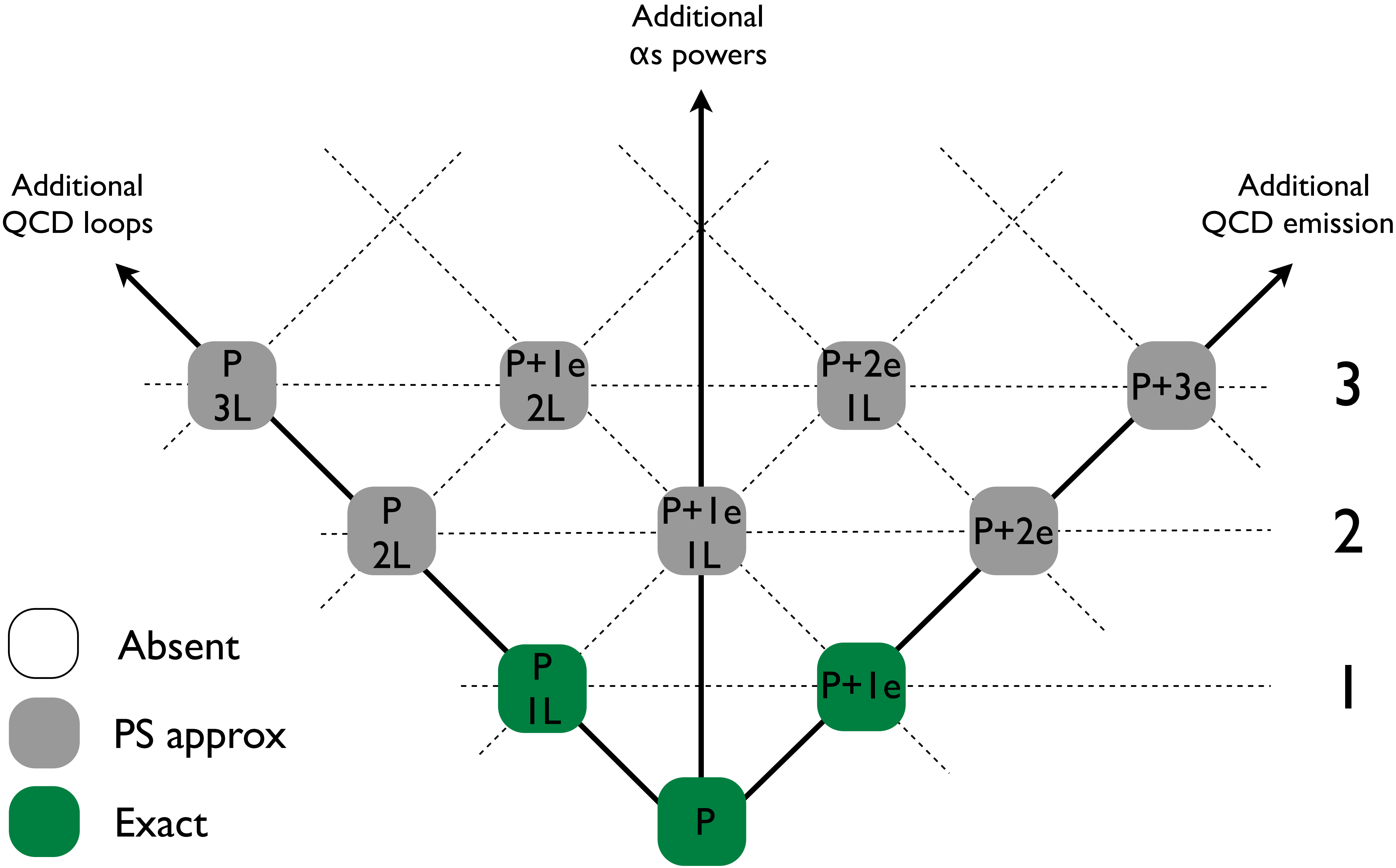
Process P and P+1j exact at LO, the rest PS approximation [PS+MEC or PS from ME for P+1e]



Process P, P+1j, P+2j, ... exact at LO, the rest PS approx. [PS+Matrix Element (CKKW, MLM,....)]

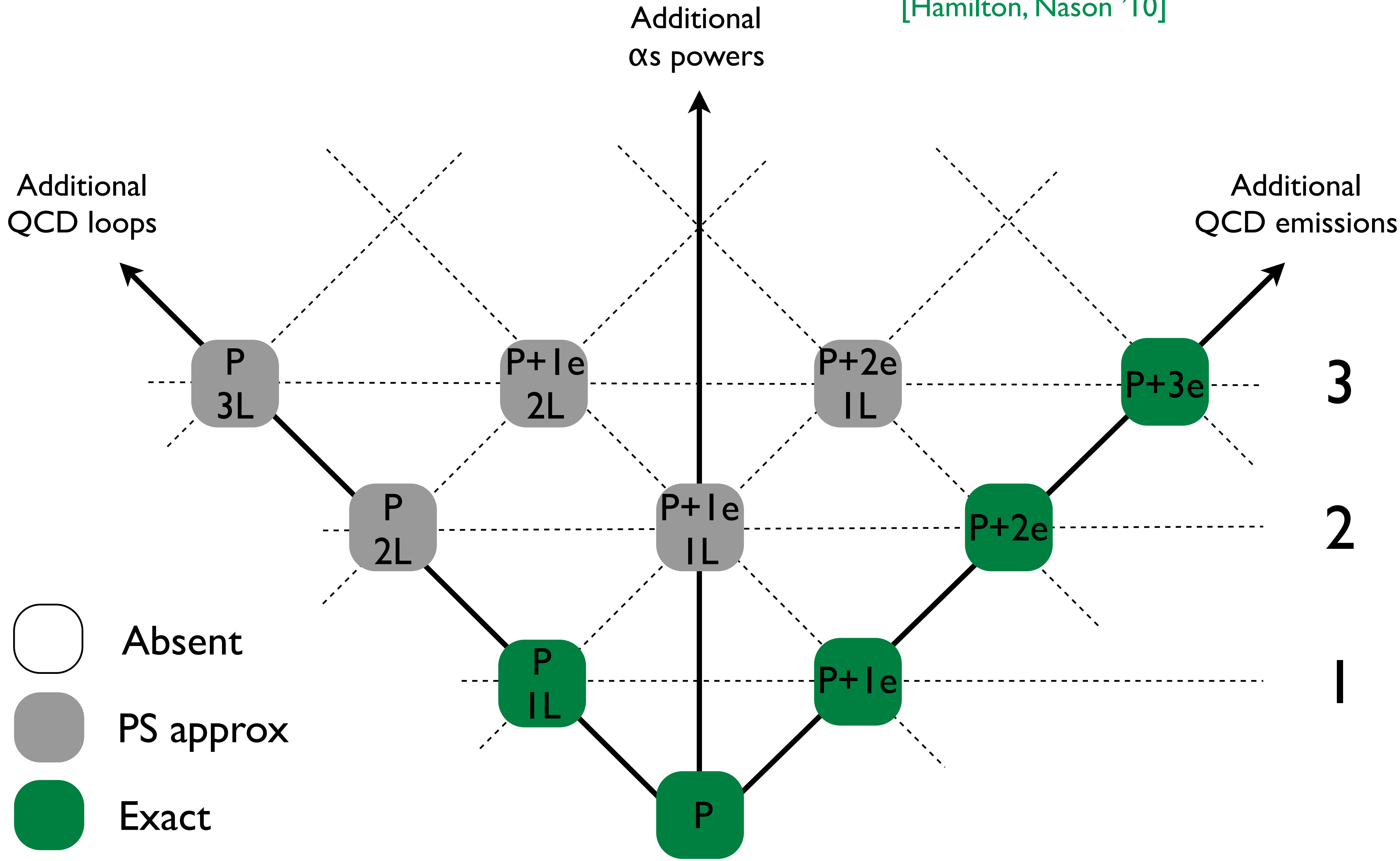


Process P exact at NLO, the rest PS approximation [PS+NLO (MC@NLO, POWHEG,...)]

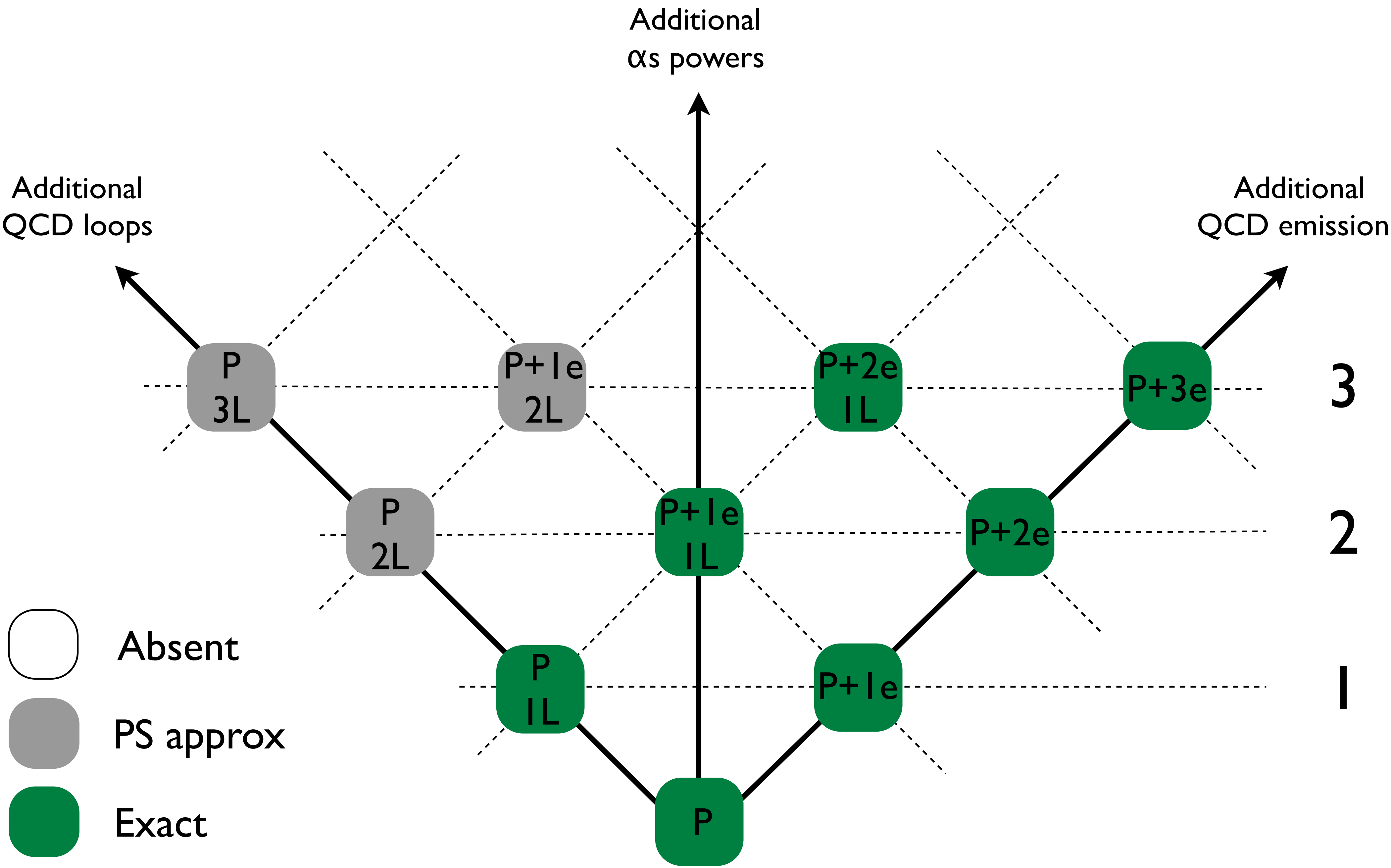


Process P exact at NLO, P+1j, P+2j,... at LO, the rest PS [PS+NLO+ME (MENLOPS,...)]

[Hamilton, Nason '10]



Process P, P+1j, P+2j,... exact at NLO, the rest PS [PS+NLO+ME$_{NLO}$ (MEPS@NLO,...)]



Existing 'MonteCarlos at NLO':

- MC@NLO [Frixione and Webber, 2002]
- POWHEG [Nason, 2004]

NB. MC@NLO is a code, POWHEG is a method

Evolving into (semi)automated forms:

- The POWHEG BOX [powhegbox.mib.infn.it 2010]
- aMC@NLO [amcatnlo.cern.ch 2011]

MCs at NLO

Matrix-element corrected shower Monte Carlos still have leading order accuracy for the total rates

$$d\sigma^{MEC} = Bd\Phi_B \left[\Delta_R(Q_0) + \Delta_R(p_T)\frac{R}{B}d\Phi_{rad}\right] \quad \text{and} \quad \Delta_R(Q_0) + \int \Delta_R(p_T)\frac{R}{B}d\Phi_{rad} = 1$$

$$\Rightarrow \int d\sigma^{MEC} = \int Bd\Phi_B = \sigma^{LO}$$

We want to do better, and **merge** PS and NLO, so that

$$\int d\sigma^{PS+NLO} = \int (B+V)d\Phi_B + \int Rd\Phi_R = \sigma^{NLO}$$

Idea: remove from the NLO the terms that are already generated by the parton shower (NB. MC-specific)

$$d\sigma^{MC@NLO} = \bar{B}_{MC} d\Phi_B \left[\Delta_{MC}(Q_0) + \Delta_{MC}(p_T)\frac{R^{MC}}{B} d\Phi_{rad}\right] + [R - R^{MC}] d\Phi_R$$

$$\bar{B}_{MC} = B + \left[V + \int R^{MC} d\Phi_{rad}\right]$$

'soft' event — MC shower — 'hard' event

It is easy to see that, as desired,

$$\int d\sigma^{MC@NLO} = \int d\sigma^{NLO}$$

POWHEG

Idea: generate hardest radiation first, then pass event to MC for generation of subsequent, softer radiation

$$d\sigma^{POWHEG} = \bar{B} d\Phi_B \left[\Delta_R(Q_0) + \Delta_R(p_T) \frac{R}{B} d\Phi_{rad} \right]$$

$$\bar{B} = B + \left[V + \int R \, d\Phi_{rad} \right]$$

NLO x-sect

MC shower

It is easy to see that, as desired,

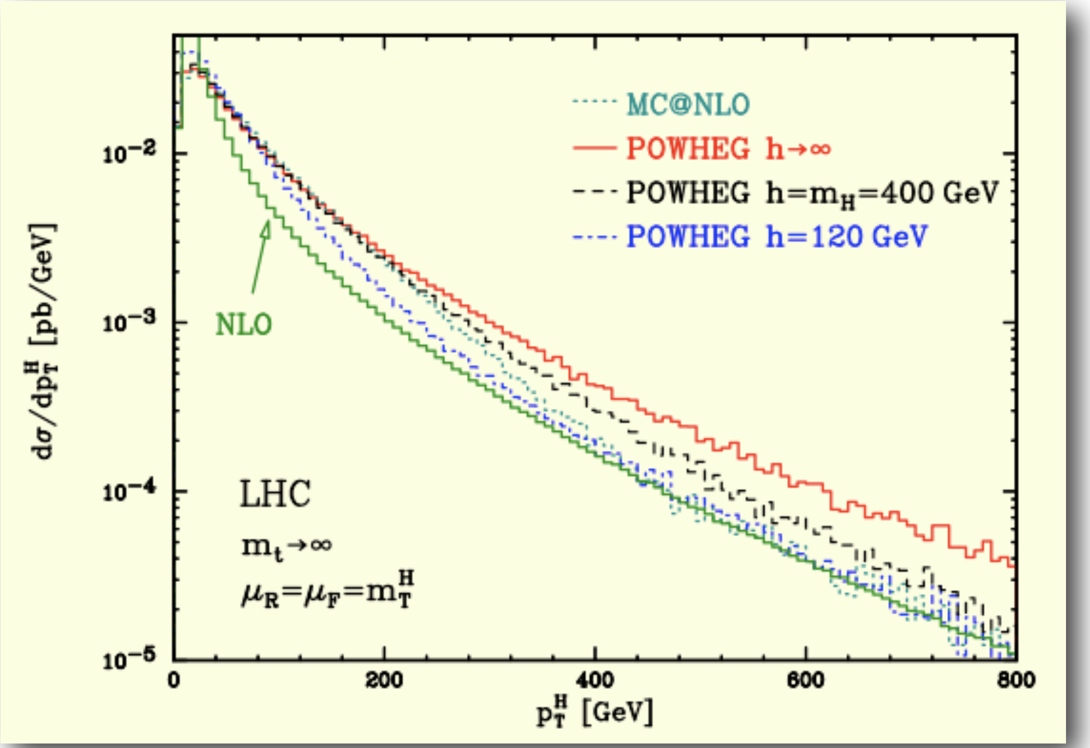
$$\int d\sigma^{POWHEG} = \int d\sigma^{NLO}$$

Large p_T enhancement in POWHEG

The 'naive' formulation for POWHEG is

$$d\sigma^{POWHEG} = \bar{B} d\Phi_B \left[\Delta_R(Q_0) + \Delta_R(p_T) \frac{R}{B} d\Phi_{rad} \right]$$

In this form $\bar{B} d\Phi_B$ provides the NLO K-factor (order $1+O(\alpha_s)$) , but also associates it to large p_T radiation, where the calculation is already $O(\alpha_s)$ (but only LO accuracy).



This generates an effective (but not necessarily correct) $O(\alpha_s^2)$ term (i.e. NNLO for the total cross section)

OK because beyond nominal accuracy, but one may feel uncomfortable with such large numerical factors

Modified POWHEG

The ‘problem’ with the naive POWHEG comes from the hard radiation being enhanced by spurious higher orders. In order to suppress this effect, we split

$$R = R^S + R^F \qquad R^S \equiv \frac{h^2}{h^2 + p_T^2} R \qquad R^F \equiv \frac{p_T^2}{h^2 + p_T^2} R$$

R^S: Contains singularities

R^F: Regular in small p_T region

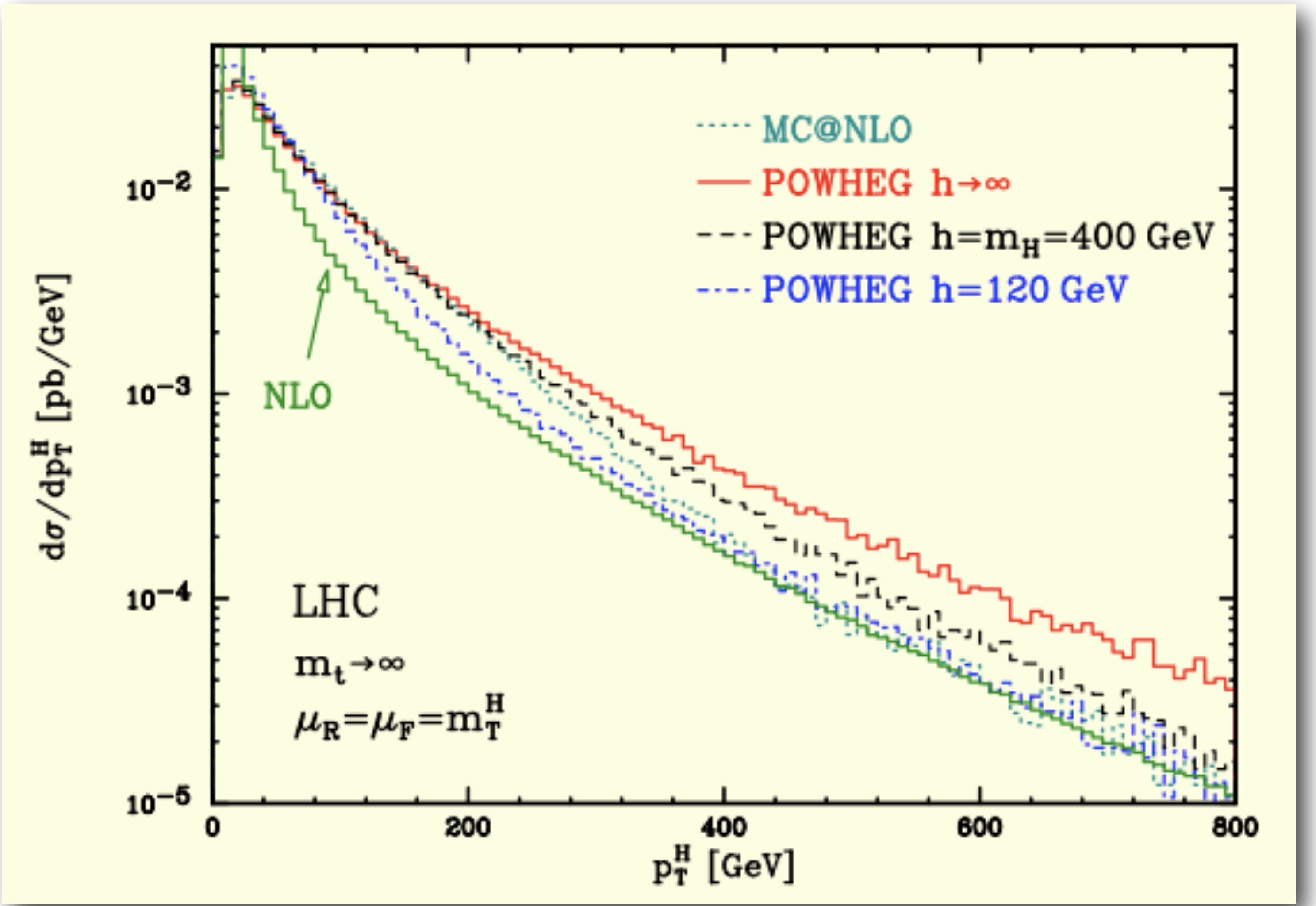
$$d\sigma^{POWHEG} = \bar{B}^S d\Phi_B \left[\Delta_S(Q_0) + \Delta_S(p_T)\frac{R^S}{B} d\Phi_{rad}\right] + R^F d\Phi_R$$

$$\bar{B}^S = B + \left[V + \int R^S \, d\Phi_{rad}\right]$$

$$\Delta_S(p_T) = \exp\left[-\int_{p_T} \frac{R^S}{B} d\Phi_{rad}\right]$$

Modified POWHEG

In the $h \to \infty$ limit the exact NLO result is recovered



Comparisons

$$d\sigma^{MC} = Bd\Phi_B \left[\Delta(Q_0) + \Delta(p_T)\frac{R^{MC}}{B}d\Phi_{rad}\right]$$

$$d\sigma^{MEC} = Bd\Phi_B \left[\Delta_R(Q_0) + \Delta_R(p_T)\frac{R}{B}d\Phi_{rad}\right]$$

$$d\sigma^{NLO} = [B+V]\, d\Phi_B + Rd\Phi_R$$

$$d\sigma^{MC@NLO} = \bar{B}_{MC}d\Phi_B \left[\Delta_{MC}(Q_0) + \Delta_{MC}(p_T)\frac{R^{MC}}{B}d\Phi_{rad}\right] + [R - R^{MC}]d\Phi_R$$

$$d\sigma^{POWHEG} = \bar{B}^S d\Phi_B \left[\Delta_S(Q_0) + \Delta_S(p_T)\frac{R^S}{B}d\Phi_{rad}\right] + R^F d\Phi_R$$

POWHEG approaches MC@NLO if $R^S \to R^{MC}$

Take home messages

Monte Carlos in QCD are complicated. I only scratched the surface here and gave almost no details. If interested, check lectures of real MC people (Sjostrand, Skands, Nason, Maltoni, Frixione, Krauss, Richardson, Webber,.....)

Monte Carlos exploit property of universality of soft/collinear radiation to resum its effects to all orders (within some approximations)

Effects of multi-parton, hard, large-angle radiation can be included via exact calculations and proper (and delicate) mergings

The result is a detailed description of the final state, covering as much phase space as possible. Accurate descriptions of data are usually achieved