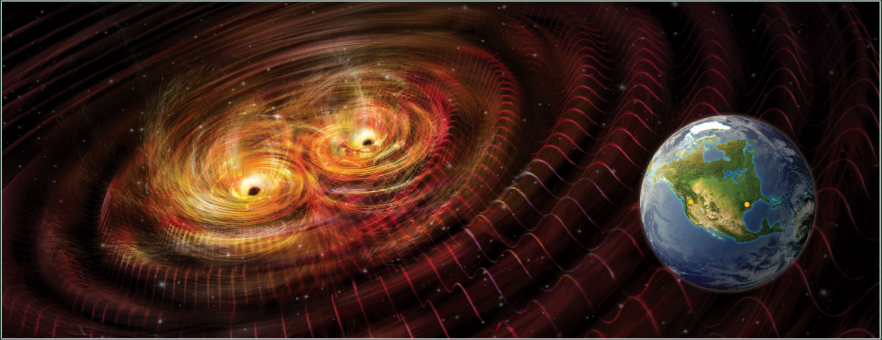
Gravitational Waves from the Early Universe

Lecture 1A: Gravitational Waves, Theory

Kai Schmitz (CERN)

Chung-Ang University, Seoul, South Korea | June 2 – 4

First direct detection of gravitational waves



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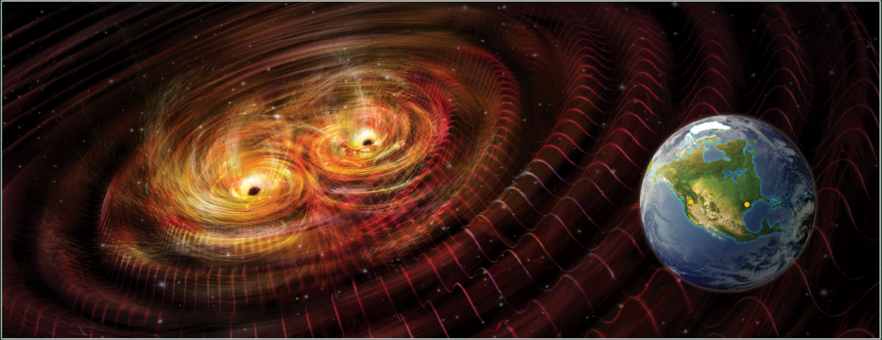

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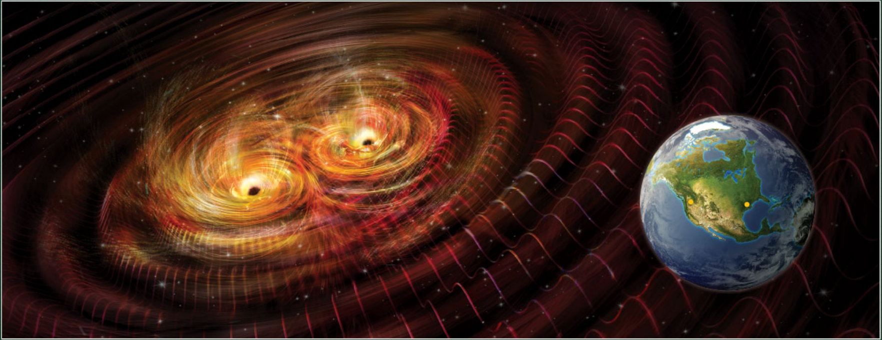

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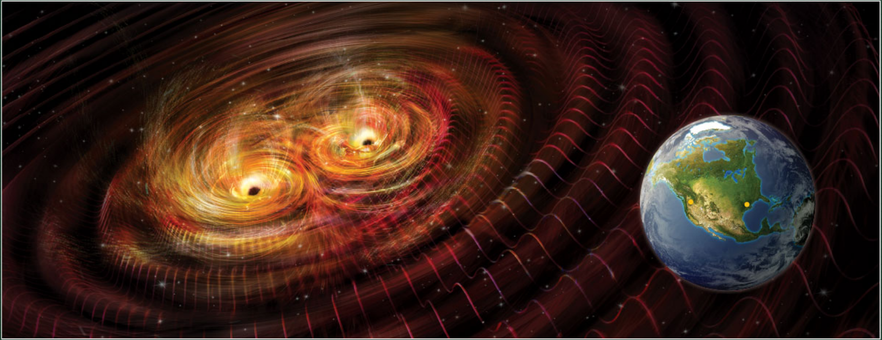

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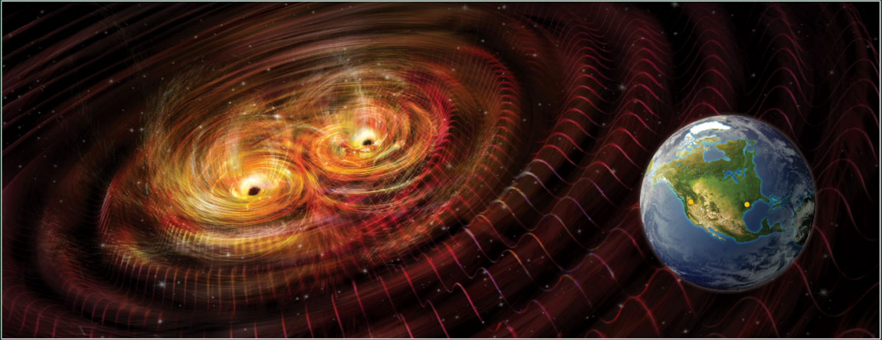

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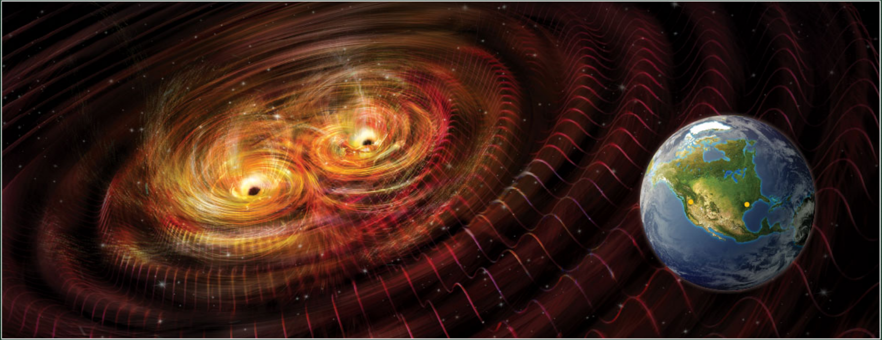

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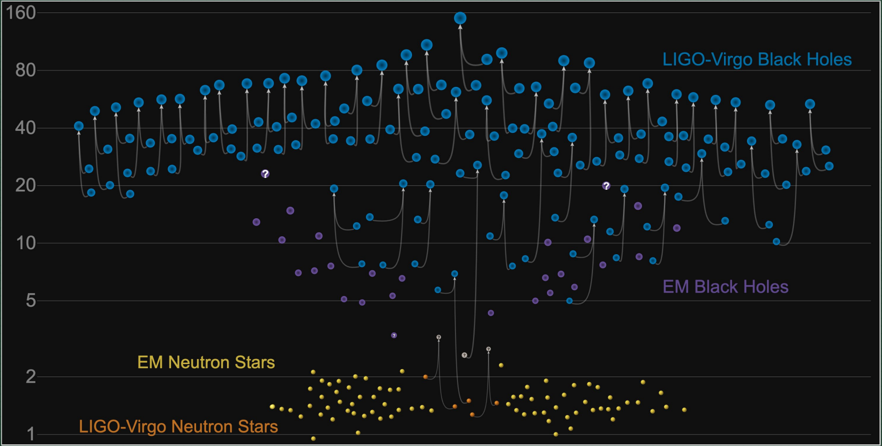
→ Discovery of a new class of astrophysical objects: heavy black holes in binary systems

Gravitational-Wave Transient Catalog 2 (GWTC-2)



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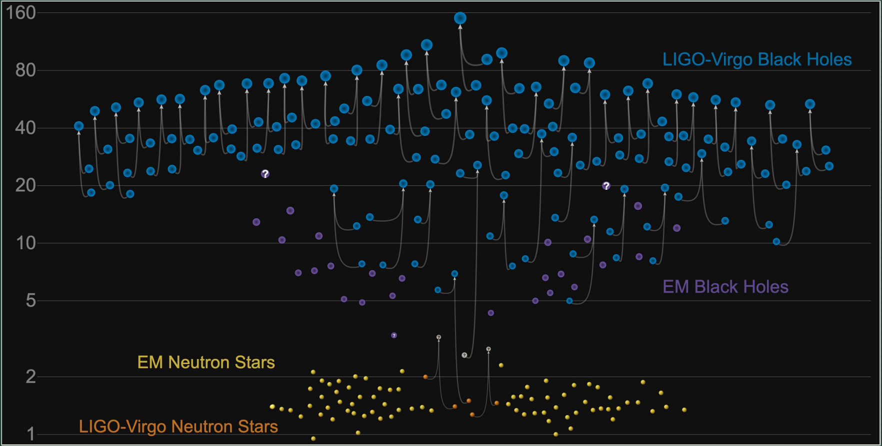

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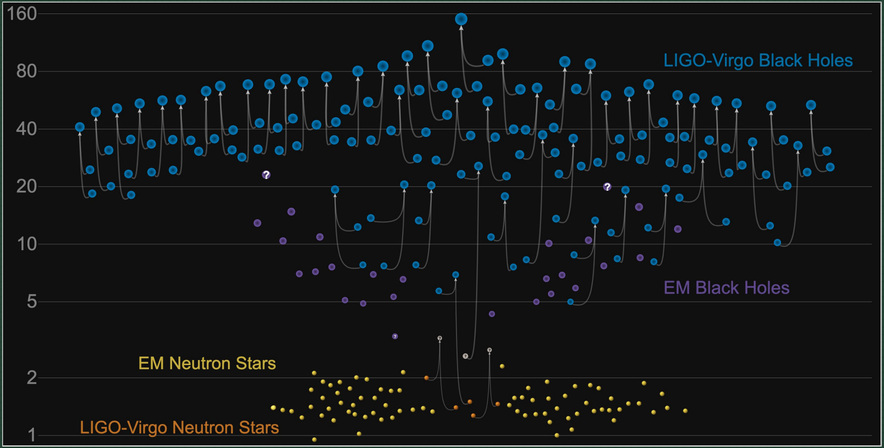

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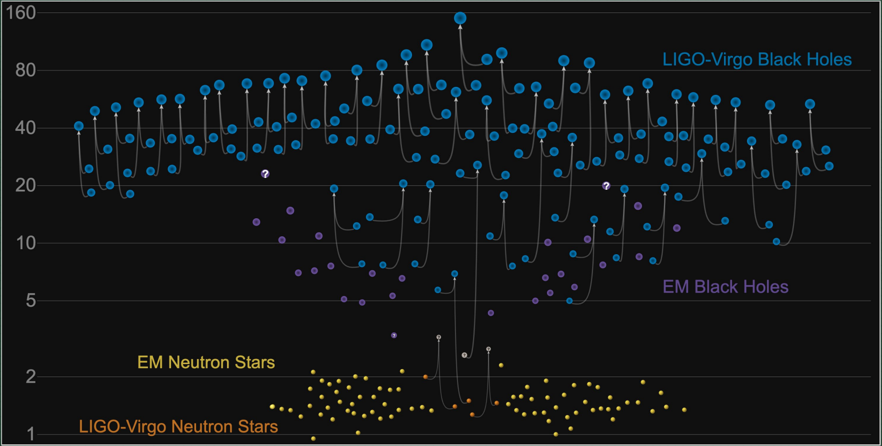

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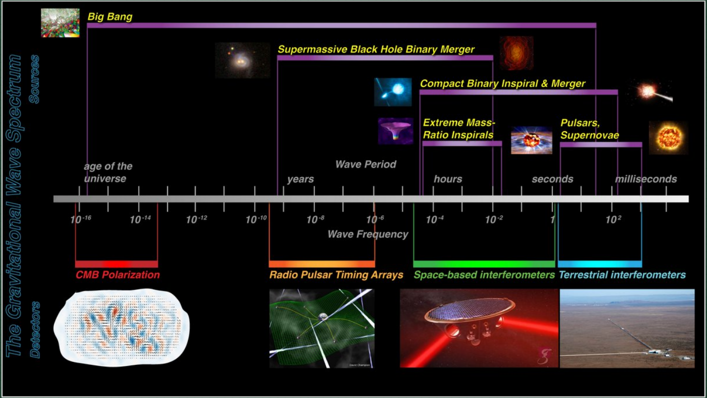
- Rapid and impressive progress: 50 events during observing runs O1, O2, and O3a
- Mergers: 47 binary black holes, 2 binary neutron stars, 1 black hole + ?
- 1 primary BH in the supernova mass gap, 1 remnant intermediate-mass black hole

A new era in astrophysics and cosmology



[NASA]

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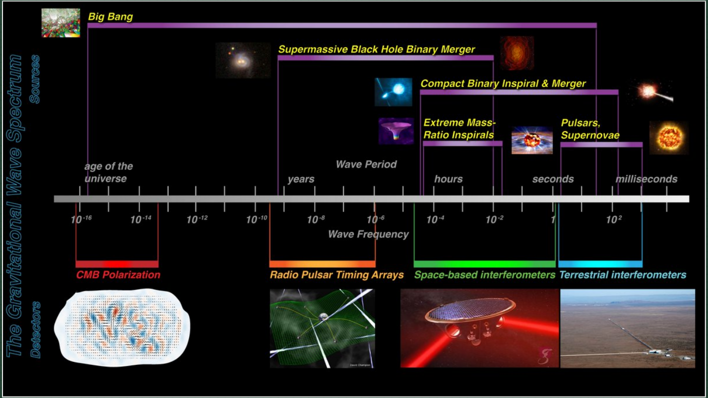

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- GWs propagate freely after their production; new window onto the (early) Universe

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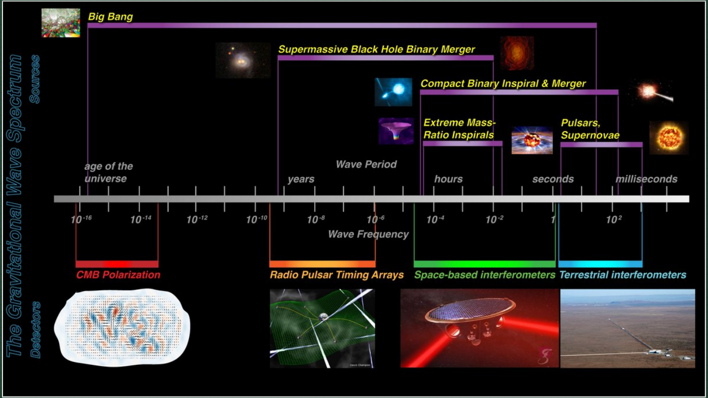

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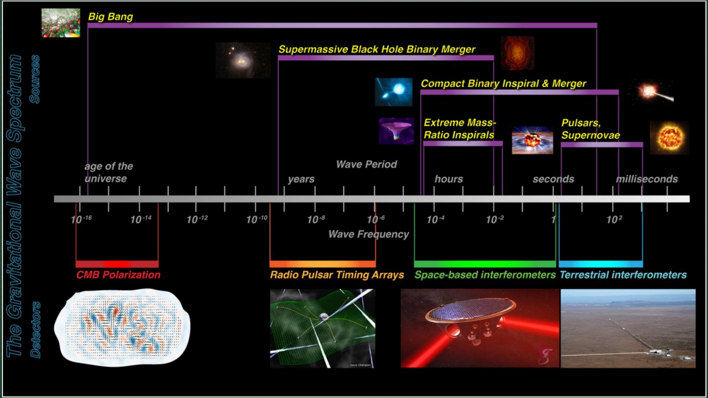

[NASA]

- GWs propagate freely after their production; new window onto the (early) Universe
- Allow us to probe energies far beyond the reach of other experiments / observations

The journey has just begun: GWs will turn into indispensable tool for astrophysics and cosmology and advance to a primary probe of fundamental physics in the 21st century.

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Aim of this set of lectures: Highlight some of the exciting new physics scenarios that we might be able to discover in the GW sky, with a focus on GWs from the early Universe

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Slides (slightly longer version): `https://doi.org/10.5281/zenodo.4678779`

Notation and conventions

Natural units: $c = \hbar = 1$

Minkowski metric: $\eta_{\mu\nu} = \mathrm{diag}\,(-1, +1, +1, +1)$

Physical constants:

- Planck mass $M_{\mathrm{Pl}} = 1/\sqrt{G} \simeq 1.22 \times 10^{19}\,\mathrm{GeV}$
- Reduced Planck mass $m_{\mathrm{Pl}} = M_{\mathrm{Pl}}/\sqrt{8\pi} \simeq 2.44 \times 10^{18}\,\mathrm{GeV}$

Abbreviations:

ALP	Axion-like particle	MD	Matter domination
BBN	Big-bang nucleosynthesis	PBH	Primordial black holes
BOS	Blanco-Pillado–Olum–Shlaer	PISC	Peak-integrated sensitivity curve
BSM	Beyond the Standard Model	PLISC	Power-law-integrated sensitivity curve
CCR	Canonical commutation relation	QCD	Quantum chromodynamics
CMB	Cosmic microwave background	QFT	Quantum field theory
DM	Dark matter	RD	Radiation domination
DOF	Degree of freedom	SFOPT	Strong first-order phase transition
EOM	Equation of motion	SGWB	Stochastic gravitational-wave background
EOS	Equation of state	SM	Standard Model
EW	Electroweak	SMBH	Supermassive black hole
FLRW	Friedmann–Lemaître–Robertson–Walker	SNR	Signal-to-noise ratio
GW	Gravitational wave	SVT	Scalar–vector–tensor
KAGRA	Kamioka Gravitational Wave Detector	TOA	Time of arrival
LHS	Left-hand side	TT	Transverse-traceless
LIGO	Laser Interferometer Gravitational-Wave Observatory	UTC	Unequal-time correlator
LRS	Lorenz–Ringeval–Sakellariadou	VD	Vacuum domination
LUNA	Laboratory for Underground Nuclear Astrophysics	VOS	Velocity-dependent one-scale

Outline Lecture 1A

1. GWs in linerarized gravity

2. Propagation of GWs in the expanding Universe

3. Stochastic backgrounds of cosmological GWs

4. Summary

Definition of gravitational waves

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In linearized gravity over a fixed, flat Minkowski background:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}| \ll 1\,, \qquad h_{\mu\nu} = h_{\nu\mu} \tag{1}$$

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GR is invariant under general coordinate transformations. Transformation behavior of $h_{\mu\nu}$:

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \qquad \Rightarrow \qquad h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_\mu \xi_\nu(x) - \partial_\nu \xi_\mu(x) \tag{2}$$

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Einstein tensor $G_{\mu\nu}$ (LHS of Einstein's field equations) to first order in $h_{\mu\nu}$:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = \frac{1}{2}\left(\partial_\alpha \partial_\nu \bar{h}^{\alpha}{}_{\mu} + \partial^\alpha \partial_\mu \bar{h}_{\nu\alpha} - \Box \bar{h}_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial^\beta \bar{h}^{\alpha}{}_{\beta}\right) \tag{3}$$

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in terms of the trace-reversed metric perturbation:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu}\, h\,, \qquad h = h^{\mu}{}_{\mu} = \eta^{\mu\nu} h_{\nu\mu} \tag{4}$$

Lorentz gauge

Under infinitesimal coordinate transformations:

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Einstein tensor of the linearized theory in Lorentz gauge:

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} \tag{8}$$

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Residual gauge freedom in Lorentz gauge: four functions ξ_μ satisfying $\Box \xi_\mu = 0$.

Transverse-traceless gauge

In Lorentz gauge, the field equations obtain the form of wave equations:

$$G_{\mu\nu} = \frac{1}{m_{\mathrm{Pl}}^2} T_{\mu\nu} \qquad \Rightarrow \qquad \Box \bar{h}_{\mu\nu} = -\frac{2}{m_{\mathrm{Pl}}^2} T_{\mu\nu} \tag{9}$$

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which are invariant under Lorentz-gauge-preserving coordinate transformations in vacuum:

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Use this residual gauge freedom to impose four conditions and thus completely fix the gauge:

$$\bar{h} = 0 \qquad \Rightarrow \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} \tag{11}$$

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We can therefore choose $h_{00} = V(\boldsymbol{x}) = 0$, which leads to the transverse-traceless gauge:

$$h_{\mu 0} = 0\,, \qquad h = 0\,, \qquad \partial_i h_{ij} = 0 \tag{13}$$

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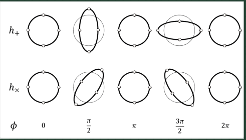
$$\bar{h} = 0 \qquad \Rightarrow \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} \tag{11}$$

$$\partial^\mu h_{\mu 0} = 0\,, \qquad h_{i0} = 0 \qquad \Rightarrow \qquad \partial^0 h_{00} = 0 \tag{12}$$

We can therefore choose $h_{00} = V(\boldsymbol{x}) = 0$, which leads to the transverse-traceless gauge:

$$h_{\mu 0} = 0\,, \qquad h = 0\,, \qquad \partial_i h_{ij} = 0 \tag{13}$$



Physical DOFs:
10 components of the symmetric metric perturbation $h_{\mu\nu}$
$-\,4$ gauge DOFs to fix Lorentz gauge
$-\,4$ residual gauge DOFs to fix transverse-traceless gauge
$=\,2$ physical transverse polarization states h_+ and $h_\times$

Scalar–vector–tensor (SVT) decomposition

Next: Identify physical DOFs independent of a particular gauge choice, allow for $\delta T_{\mu\nu} \neq 0$.

[1] Caution: New notation, $h_{ij} \to \delta g_{ij}$. h_{ij} are now the tensor components of the general spatial perturbations δg_{ij}.

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$$\delta g_{00} = -2\phi \tag{14}$$

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$$\delta g_{ij} = -2\,\psi\,\delta_{ij} + (\partial_i\partial_j - {}^1\!/_3\,\delta_{ij}\Delta)E + \partial_i F_j + \partial_j F_i + h_{ij} \tag{16}$$

$$\delta T_{00} = \rho \tag{17}$$

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$$\delta T_{ij} = p\,\delta_{ij} + (\partial_i\partial_j - {}^1\!/_3\,\delta_{ij}\Delta)\sigma + \partial_i v_j + \partial_j v_i + \Pi_{ij} \tag{19}$$

plus a set of constraint equations. E.g., for the tensor perturbations: $\partial_i h_{ij} = h_{ii} = 0$.

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Gauge-invariant linear combinations:

$$\Phi = \phi + \dot{B} - {}^1\!/_2\,\ddot{E}\,, \qquad \Theta = -2\phi - {}^1\!/_3\,\Delta E\,, \qquad \Sigma_i = S_i - \dot{F}_i\,, \qquad h_{ij} \tag{20}$$

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$$\Delta\Phi = \frac{\rho + 3p - 3\dot{u}}{2m_{\rm Pl}^2}\,, \qquad \Delta\Theta = -\frac{\rho}{m_{\rm Pl}^2}\,, \qquad \Delta\Sigma_i = -\frac{2S_i}{m_{\rm Pl}^2}\,, \qquad \Box h_{ij} = -\frac{2\Pi_{ij}}{m_{\rm Pl}^2} \tag{21}$$

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The tensor perturbations h_{ij} are the only gauge-invariant DOFs satisfying a wave equation.

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Curved background

Separation of scales: Consider background metric $\bar{g}_{\mu\nu}$ that varies on length (time) scales L_B (T_B). GWs then defined in terms of perturbations $\delta g_{\mu\nu}$ on scales $\Delta L \ll L_B$ ($\Delta T \ll T_B$):

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \qquad |\delta g_{\mu\nu}| \ll |\bar{g}_{\mu\nu}| \tag{22}$$

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$$T_{\mu\nu}^{\mathrm{GW}} = \frac{1}{4} m_{\mathrm{Pl}}^2 \left\langle \nabla_\mu h_{\alpha\beta} \nabla_\nu h^{\alpha\beta} \right\rangle \sim \frac{h^2}{\Delta L^2} \tag{23}$$

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Energy density contained in GWs in transverse-traceless gauge:

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Equation of motion on a curved background without any GW source and in Lorentz gauge:

$$\bar{g}^{\alpha\beta} \nabla_\alpha \nabla_\beta \bar{h}_{\mu\nu} - 2 R^\lambda{}_{\mu\nu}{}^\sigma \bar{h}_{\lambda\sigma} = 0 \tag{25}$$

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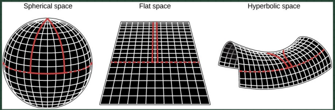
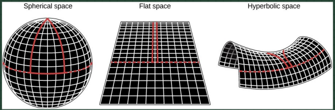


Cosmological background solutions:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \quad (26)$$

Homogeneous, isotropic (cosmological principle!) and expanding solutions of the field equations

Friedmann–Lemaître–Robertson–Walker (FLRW) background



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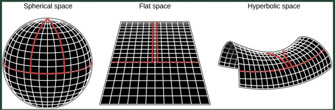

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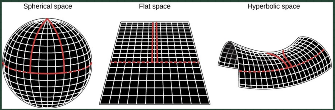

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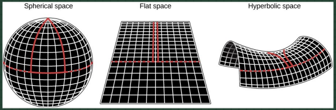

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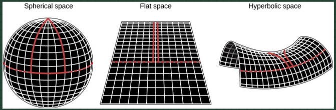

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- No coupling between scalar, vector, tensor modes at first order in perturbation theory.

Equation of motion in an FLRW background

EOM for the transverse-traceless (TT) piece of the metric perturbation $(\partial_i h_{ij} = h_{ii} = 0)$:

$$\boxed{\ddot{h}_{ij}(t,\boldsymbol{x}) + 3\,H\,\dot{h}_{ij}(t,\boldsymbol{x}) - \frac{1}{a^2}\Delta\,h_{ij}(t,\boldsymbol{x}) = \frac{2}{m_{\rm Pl}^2}\,\Pi_{ij}^{\rm TT}(t,\boldsymbol{x})} \tag{28}$$

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$$\Pi_{ij}(t,\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\Pi_{ij}(t,\boldsymbol{k})\,e^{-i\boldsymbol{k}\boldsymbol{x}} \tag{29}$$

$$\Pi_{ij}^{\mathrm{TT}}(t,\boldsymbol{k}) = \mathcal{O}_{ijlm}(\boldsymbol{n})\,\Pi_{lm}(t,\boldsymbol{k}) = \left[P_{il}(\boldsymbol{n})P_{jm}(\boldsymbol{n}) - {}^1\!/_2\,P_{ij}(\boldsymbol{n})P_{lm}(\boldsymbol{n})\right]\Pi_{lm}(t,\boldsymbol{k}) \tag{30}$$

$$P_{ij}(\boldsymbol{n}) = \delta_{ij} - n_i\,n_j\,, \qquad \boldsymbol{n} = \boldsymbol{k}/\|\boldsymbol{k}\| \tag{31}$$

such that $k_i\Pi_{ij}^{\mathrm{TT}} = \Pi_{ii}^{\mathrm{TT}} = 0$ in Fourier space and $\partial_i\Pi_{ij}^{\mathrm{TT}} = \Pi_{ii}^{\mathrm{TT}} = 0$ in position space.

Decomposition into Fourier modes

Decompose TT piece into Fourier modes with definite momentum $\boldsymbol{k}$ and polarization p:

$$h_{ij}(t,\boldsymbol{x}) = \sum_{p=+,\times} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, h_p(t,\boldsymbol{k})\, e^p_{ij}(\boldsymbol{n})\, e^{-i\boldsymbol{k}\boldsymbol{x}} \tag{32}$$

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Orthonormal and completeness relations:

$$\sum_{ij} e^p_{ij}(\boldsymbol{n})\, e^q_{ij}(\boldsymbol{n}) = 2\,\delta_{pq}\,, \qquad \sum_p e^p_{ij}(\boldsymbol{n})\, e^p_{lm}(\boldsymbol{n}) = P_{il}\,P_{jm} + P_{im}\,P_{jl} - P_{ij}\,P_{lm} \tag{34}$$

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Explicit expressions in terms of $\{\boldsymbol{u},\boldsymbol{v}\}$, which span the 2D vector space perpendicular to $\boldsymbol{n}$:

$$e^+_{ij}(\boldsymbol{n}) = u_i u_j - v_i v_j\,, \qquad e^\times_{ij}(\boldsymbol{n}) = u_i v_j + v_i u_j\,, \qquad P_{ij} = u_i u_j + v_i v_j \tag{35}$$

Mode equations in Fourier space

Perturbed FLRW metric in terms of conformal time η and comoving spatial coordinates x:

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j\right], \qquad a(\eta)d\eta = dt\,, \qquad X' = \frac{dX}{d\eta} \tag{36}$$

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EOMs for $H_{ij}(\eta, \boldsymbol{k}) = a(\eta)\sum_p h_p(\eta, \boldsymbol{k})\, e_{ij}^p(\boldsymbol{n})$ in the presence of a source term:

$$H_{ij}''(\eta, \boldsymbol{k}) + \left(k^2 - \frac{a''}{a}\right) H_{ij}(\eta, \boldsymbol{k}) = \frac{2a^3}{m_{\mathrm{Pl}}^2}\, \Pi_{ij}^{\mathrm{TT}}(\eta, \boldsymbol{k}) \tag{37}$$

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General solution in terms of spherical Bessel functions, assuming $a(\eta) \propto \eta^n$,

$$\boxed{h_p(\eta, \boldsymbol{k}) = \frac{\eta}{a(\eta)} \left[A_p(\boldsymbol{k})\, j_{n-1}(k\eta) + B_p(\boldsymbol{k})\, y_{n-1}(k\eta)\right]} \tag{39}$$

Solutions far inside / outside the Hubble radius

For $a(\eta) \propto \eta^n$, we have ($n = 1$ in RD, $n = 2$ in MD, $n = -1$ in dS):

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Sum of a constant and a (quickly) decaying contribution outside the Hubble radius.

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$$\langle h_+(\eta,\boldsymbol{k}) h_\times(\eta,\boldsymbol{k})\rangle = 0 \qquad \Leftrightarrow \qquad \langle h_L(\eta,\boldsymbol{k}) h_L(\eta,\boldsymbol{k})\rangle = \langle h_R(\eta,\boldsymbol{k}) h_R(\eta,\boldsymbol{k})\rangle \tag{44}$$

with the left- and right-handed tensor modes h_L and h_R in the helicity basis:

$$e_{ij}^{L,R} = \frac{1}{2}\left(e_{ij}^{+} \mp i\, e_{ij}^{\times}\right) \tag{45}$$

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- Mostly Gaussian statistics, in consequence of $N \ggg 1$ (central limit theorem).

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Characteristic strain

Statistical properties described by power spectrum of the mode functions in Fourier space:

$$\Big\langle h_p(\eta, \boldsymbol{k})\, h_q^*(\eta, \boldsymbol{\ell}) \Big\rangle = \frac{8\pi^5}{k^3}\, \delta^{(3)}(\boldsymbol{k} - \boldsymbol{\ell})\, \delta_{pq}\, h_c^2(\eta, k) \tag{46}$$

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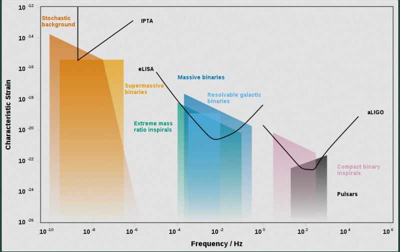

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Characteristic strain h_c describes typical GW amplitude per logarithmic k interval.

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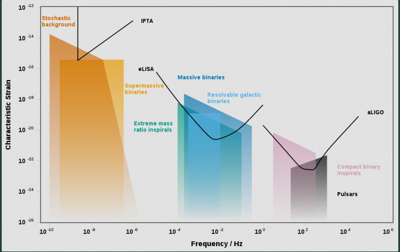

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Related to strain power spectrum S_h:

$$S_h(f) = \frac{h_c^2(f)}{f} \tag{48}$$

$h_c(f) = h_c(\eta_0, f)$ and $f = 1/(2\pi)\, k/a_0$.
Note: $S_h = h_c^2/(2f)$ in some references.

Energy density spectrum

GW energy density per logarithmic k interval $d\rho_{\rm GW}/d\ln k$:

$$\rho_{\rm GW} = \frac{1}{4}\, m_{\rm Pl}^2 \left\langle \dot h_{ij} \dot h^{ij} \right\rangle = \frac{1}{4\, a^2(\eta)}\, m_{\rm Pl}^2 \left\langle h'_{ij} h'^{ij} \right\rangle = \int_0^\infty \frac{dk}{k}\, \frac{d\rho_{\rm GW}}{d\ln k} \tag{49}$$

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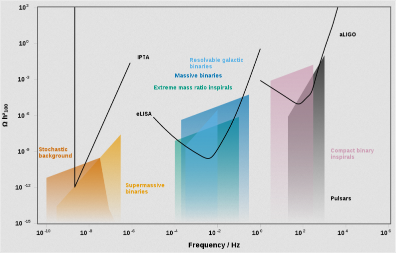

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Evaluate $d\rho_{\rm GW}/d\ln k$ today, in units of the critical energy density $\rho_c = 3\, m_{\rm Pl}^2\, H_0^2$:

$$\Omega_{\rm GW}(f) = \frac{1}{\rho_c}\,\frac{d\rho_{\rm GW}}{d\ln k} \tag{51}$$

$$= \frac{2\pi^2}{3\,H_0^2}\, f^2 h_c^2(f) = \frac{2\pi^2}{3\,H_0^2}\, f^3 S_h(f) \tag{52}$$

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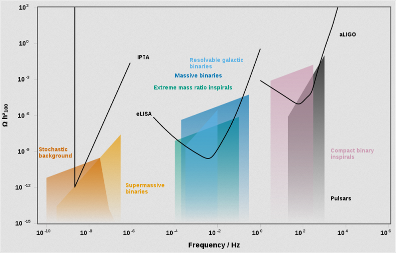

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Multiply by h^2, where $H_0 = 100\,h\,{\rm km/s/Mpc}$, to remove dependence on Hubble parameter.

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- Signal today can be represented in terms of h_c, S_h, or $h^2\Omega_{\rm GW}$ as function of f.

Summary Lecture 1A

Take-home messages:

- GWs open a new window onto the Universe, unique tool for fundamental physics.
- GWs are radiative and gauge-invariant tensor perturbations of the spacetime metric.
- The physical DOFs correspond to two transverse polarization states h_+ and $h_\times$.
- Manifest after complete gauge fixing (TT gauge in vacuum) or SVT decomposition.
- GWs also well-defined on curved backgrounds, in particular, in FLRW cosmology.
- Source-free EOM in FLRW can be solved exactly / inside / outside Hubble radius.
- Cosmological GWs are typically homogeneous, isotropic, unpolarized, and Gaussian.
- Signal today can be represented in terms of h_c, S_h, or $h^2\Omega_{\rm GW}$ as function of f.

End of Lecture 1A. Thanks a lot for your attention!