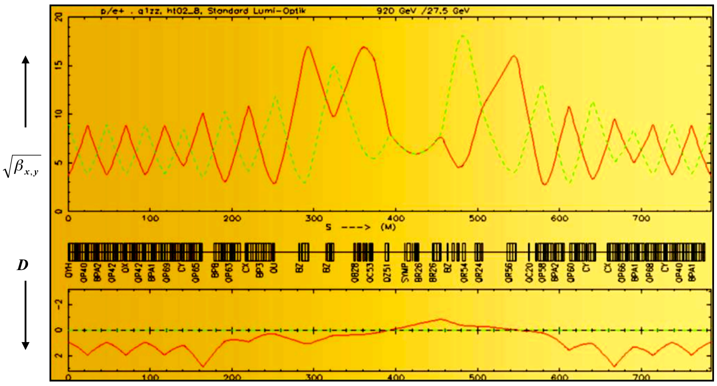
Lattice Design in Particle Accelerators

Bernhard Holzer



... in the context of space charge dominated beams

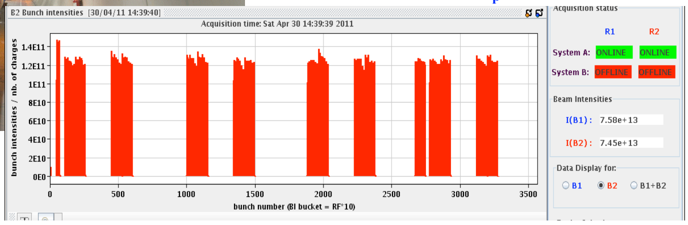
The Non-Problem

LHC lattice structure

768 x 768 colliding bunches

goal: 2808 bunches

N_p =1.2*10^{11} per bunch



Acquisition time: Sat Apr 30 14:39:39 2011

System A: ONLINE ONLINE

System B: OFFLINE OFFLINE

I(B1) : 7.58e+13

I(B2) : 7.45e+13

○ B1 ● B2 ○ B1+B2

$$x'' + K(s)x = 0$$

Hills equation without ...

... and with space charge

$$x'' + (K_q(s)x + K_{sc}(s))\,x = 0$$

$$K_{sc} = \frac{-2r_0 I}{ea^2 \boxed{c(\beta\gamma)^3}}$$

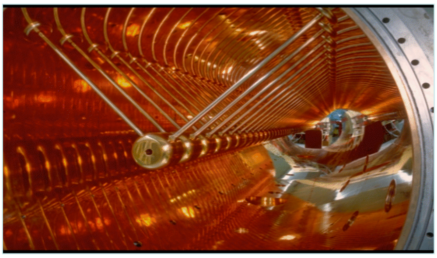
$$\beta = 1, \quad \gamma \approx 7500$$

$$K_{sc} \approx 3.6 * 10^{-11}$$

$$K_q \approx 9 * 10^{-3}$$

I like high energy accelerators !!!

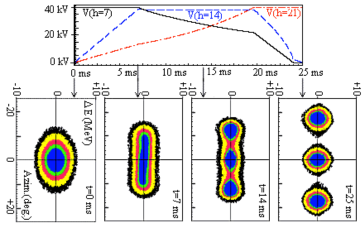
The Problem: low energy part

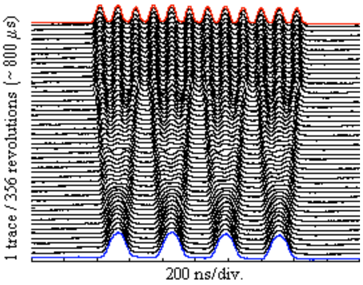


Alvarez structure „Drift Tube Linac“

(GSI Unilac)

Bunch Splitting in the PS





CERN: Linac 2 injection into PSB

$N_p \approx 1.5*10^{13}$ protons per bunch, ***$E_{inj} = 50\ MeV$***

$\beta = 0.31$

$\gamma = 1.05$

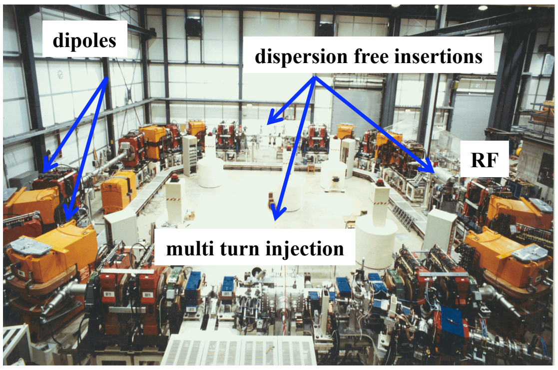
$$1/(\beta\gamma)^3 \approx \infty$$

1.) Lattice Design: ***„… how to build an accelerator“***

circular machines: somewhere in the lattice we need a number of dipole magnets, that are bending the design orbit to a closed ring

Geometry of the ring:

centrifugal force = Lorentz force



typical low energy storage ring (TSR)
8 dipole magnets of equal bending strength

$$e * v * B = \frac{mv^2}{\rho}$$

$$\rightarrow e * B = \frac{mv}{\rho} = p / \rho$$

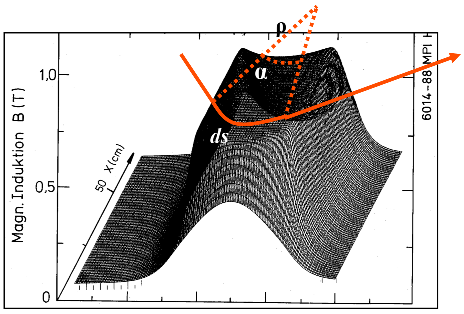
$$\rightarrow B * \rho = p / e$$

p = momentum of the particle,
ρ = curvature radius

Bρ= beam rigidity

Circular Orbit:

„… defining the geometry“



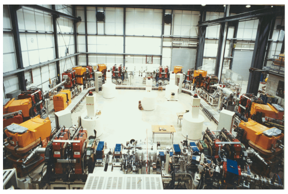
field map of a storage ring dipole magnet

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{B\,dl}{B\rho}$$

The angle swept out in one revolution must be 2π, so

$$\int Bdl = 2\pi \frac{p}{q} \quad \textit{... for a full circle}$$

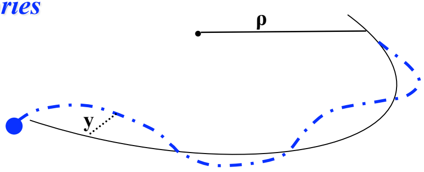
Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ ***is usually required !!***



$$\left.\begin{array}{l} E_{kin} = 60\,MeV \\ E^2 = p^2c^2 + m^2c^4 \end{array}\right\}$$

... 8 dipoles, l=1m
B=0.95T

2.) Focusing forces … single particle trajectories



$$y'' + K * y = 0$$

$K = -k + 1/\rho^2$ **hor. plane**

$K = k$ **vert. plane**

dipole magnet $\frac{1}{\rho} = \frac{B}{p/q}$

quadrupole magnet $k = \frac{g}{p/q}$

Example: HERA Ring:
Bending radius: ρ = 580 m
Quadrupol Gradient: g = 110 T/m

$k = 33.64*10^{-3}/m^2$
$1/\rho^2 = 2.97*10^{-6}/m^2$

For estimates in large accelerators the weak focusing term 1/ρ² can in general be neglected

Solution for a focusing magnet

$$y(s) = y_0 * \cos(\sqrt{K} * s) + \frac{y'_0}{\sqrt{K}} * \sin(\sqrt{K} * s)$$

$$y'(s) = -y_0 * \sqrt{K} * \sin(\sqrt{K} * s) + y'_0 * \cos(\sqrt{K} * s)$$

Or written more convenient in matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_0 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

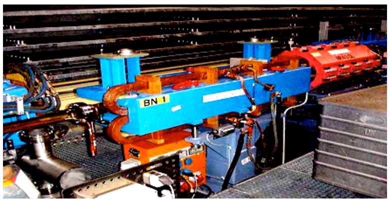
Hor. focusing Quadrupole Magnet

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

Hor. defocusing Quadrupole Magnet

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} * l) \\ \sqrt{K} \sinh(\sqrt{K} * l) & \cosh(\sqrt{K} * l) \end{pmatrix}$$

Drift space $M_{Drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$



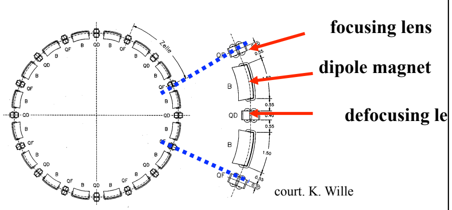
$$M_{lattice} = M_{QF1} * M_{D1} * M_{QD} * M_{D1} * M_{QF2} \dots$$

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

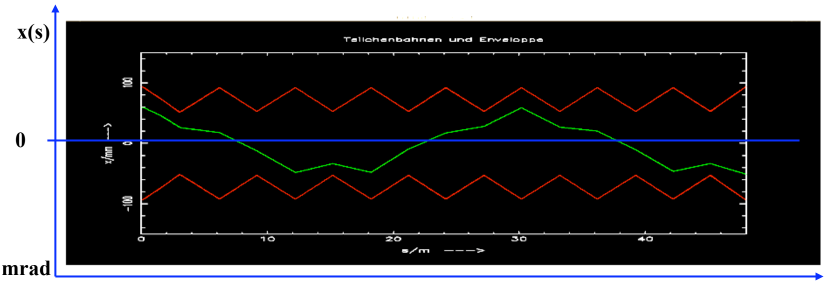
$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



court. K. Wille

in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „

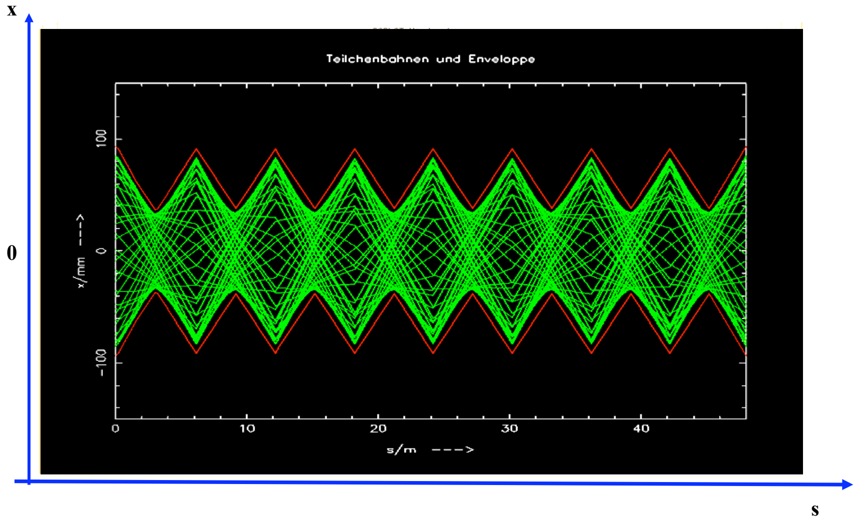


typical values in a strong foc. machine:

x ≈ mm, x´ ≤ mrad

Question: what will happen, if the particle performs a second turn ?

... or a third one or ... 10^{10} turns



3.) The Beta Function

General solution of Hill´s equation:

(i) $$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

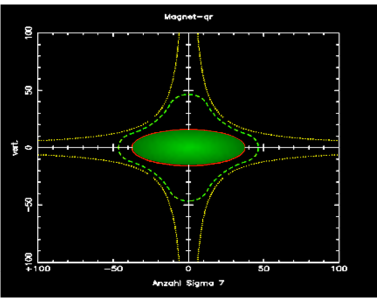
Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“

$$Q_y = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

The Beta Function

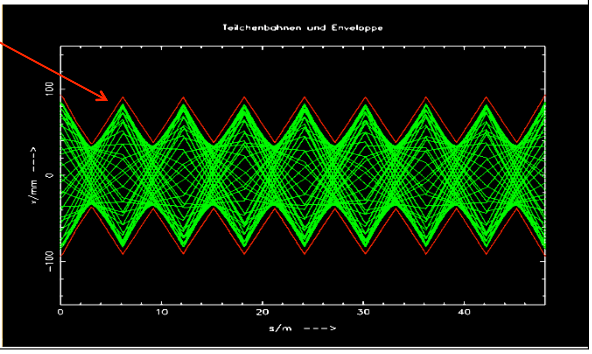


Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$

Maximum size of a particle amplitud

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

4.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\begin{cases} (1) \quad x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi) \\ (2) \quad x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\,\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\} \end{cases}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

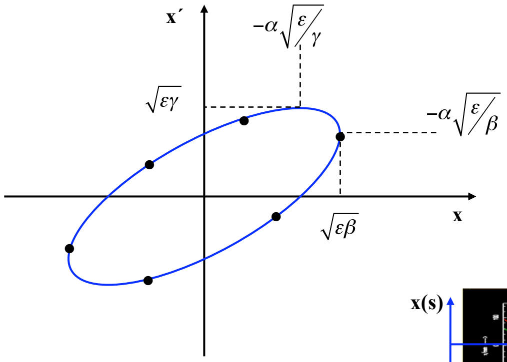
Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

- **ε is a constant of the motion … it is independent of „s"**
- **parametric representation of an ellipse in the x x' space**
- **shape and orientation of ellipse are given by α, β, γ**

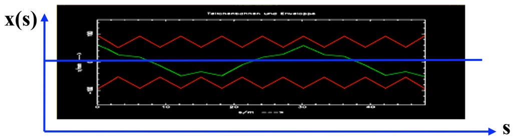
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$



Liouville: **in reasonable storage rings**
area in phase space is constant.

A = π*ε=const



ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
Scientifiquely speaking: area covered in transverse x, x´ phase space … and it is constant !!!

5.) Transfer Matrix M

for the transformation of particle trajectories ...
... can be expressed ... for convenience ... as a function of the opticsl parameners α,β,Φ

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

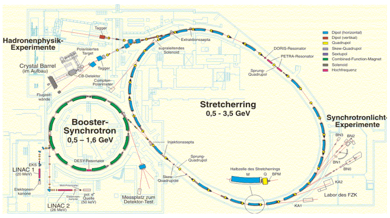
$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}(\cos\psi_s - \alpha_s \sin\psi_s) \end{pmatrix}$$

- ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***
- ***and nothing but the α β γ at these positions.***
- ***… !***

Periodic Lattices

transfer matrix for particle trajectories as a function of the lattice parameters

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$



Delta Electron Storage Ring

„This rather formidable looking matrix simplifies considerably if we consider one complete turn …“

$$\boldsymbol{M(s)} = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

ψ_{turn} = phase advance per period

$$Q = \frac{1}{2\pi} * \oint \frac{ds}{\beta(s)}$$

Tune: Phase advance per turn in units of 2π

The new parameters α, β, γ can be transformed through the lattice via the matrix elements defined above.

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s1}$$

the optical parameters depend on the focusing properties of the lattice,
... and can be optimised accordingly !!!

... and here starts the lattice design !!!

Most simple example: drift space

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \qquad \rightarrow \qquad \boxed{\begin{aligned} x(l) &= x_0 + l * x_0' \\ x'(l) &= x_0' \end{aligned}}$$

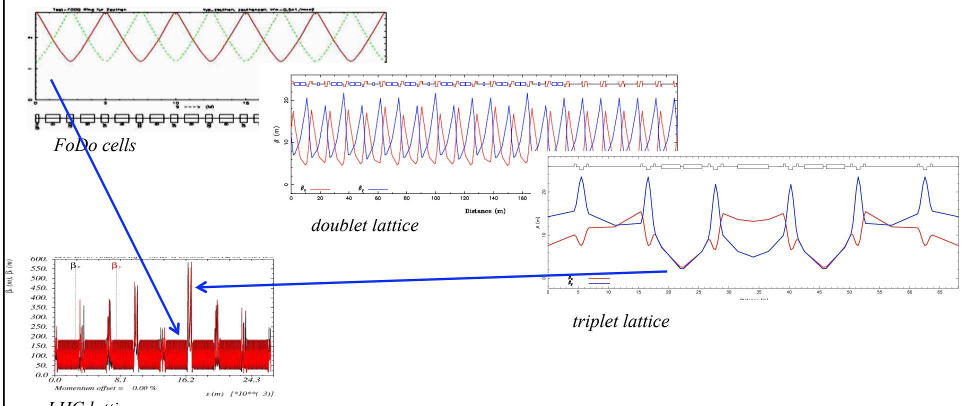
transformation of twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0 \qquad \boxed{\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0}$$

6.) Lattice Cells

***accelerator lattice:* magnet structure for the accelerator,**
in general built out of basic building blocks "the cells"
lattice cell* - *basic building block for the lattice

a number of cell types has been developed and optimised for different particle beams
in the case of space charge dominated beams you find in general:



FoDo cells

doublet lattice

triplet lattice

LHC lattice

Lattice Cells and space charge: a practical example

Linac 4: injector for the PSB ring

critical parameter: space charge tune shift at booster injection

$$\Delta Q \propto \frac{I_p}{\varepsilon_{x,y}\beta^2\gamma^3}$$

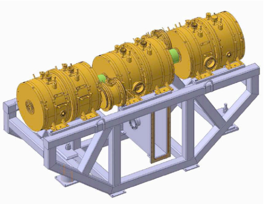
-> higher injection energy in the ring

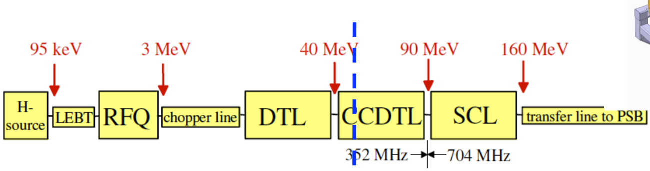
-> faster acceleration in the Linac

-> Linac-Lattice: *"don't leave the particles unattended"*

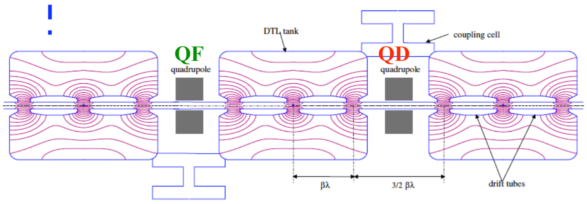
constant regular focusing required

→ again the *FoDo*

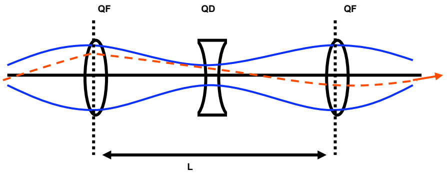


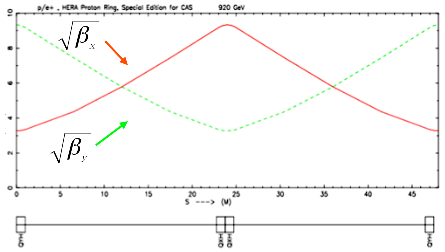


CERN Linac 4 design



periodic solution of a FoDo cell





Output of the optics program:

Nr	*Type*	*Length* *m*	*Strength* *1/m2*	β_x *m*	α_x	ψ_x *1/2π*	β_y *m*	α_y	ψ_y *1/2π*
0	*IP*	*0,000*	*0,000*	*11,611*	*0,000*	*0,000*	*5,295*	*0,000*	*0,000*
1	*QFH*	*0,250*	*-0,541*	*11,228*	*1,514*	*0,004*	*5,488*	*-0,781*	*0,007*
2	*QD*	*3,251*	*0,541*	*5,488*	*-0,781*	*0,070*	*11,228*	*1,514*	*0,066*
3	*QFH*	*6,002*	*-0,541*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*
4	*IP*	*6,002*	*0,000*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*

Q_X= *0,125* Q_Y= *0,125* → **$0.125*2*\pi = 45^0$**

can we understand what the optics code is doing ?

matrices

$$M_{QF}=\begin{pmatrix}\cos(\sqrt{K}*l_q) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}*l_q)\\ -\sqrt{K}\sin(\sqrt{K}*l_q) & \cos(\sqrt{K}*l_q)\end{pmatrix}, \qquad M_{Drift}=\begin{pmatrix}1 & l\\ 0 & 1_d\end{pmatrix}$$

strength and length of the FoDo elements

K = +/- 0.54102 m^{-2}
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo}=M_{qfh}*M_{ld}*M_{qd}*M_{ld}*M_{qf}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo}=\begin{pmatrix}0.707 & 8.206\\ -0.061 & 0.707\end{pmatrix}$$

The transfer matrix for 1 period gives us all the information that we need !

Phase advance per cell

$$M(s)=\begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu\\ -\gamma\sin\mu & \cos(\mu)-\alpha\sin\mu\end{pmatrix}\rightarrow$$

$$\cos(\mu)=\frac{1}{2}*trace(M)=0.707$$

$$\mu=arc\cos(\frac{1}{2}*trace(M))=\underline{\underline{45°}}$$

hor β-function

$$\beta=\frac{M(1,2)}{\sin(\mu)}=\underline{\underline{11.611\ \ m}}$$

hor α-function ♪

$$\alpha=\frac{M(1,1)-\cos(\mu)}{\sin(\mu)}=\underline{\underline{0}}$$

Can we do it a little bit easier ?
in thin lens approximation

Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K}*l) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}*l) \\ -\sqrt{K}\sin(\sqrt{K}*l) & \cos(\sqrt{K}*l) \end{pmatrix}$$

If the focal length f is much larger than the length of the quadrupole magnet,

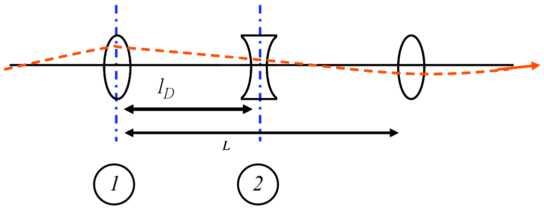
$$f = 1/kl_Q >> l_Q$$

the transfer matrix can be approximated by

$$kl_q = const, \; l_q \to 0$$

$$M = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

FoDo in thin lens approximation



$l_D = L/2,$
$\tilde{f} = 2f$

Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$M_{half\ Cell} = M_{QD/2} * M_{lD} * M_{QF/2}$$

$$M_{half\ Cell} = \begin{pmatrix} 1 & 0 \\ 1/\tilde{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix}$$

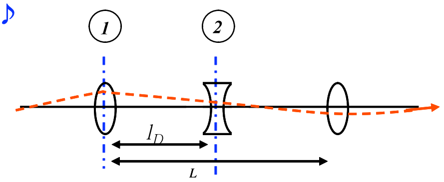
note: $\tilde{f}$ denotes the focusing strength of half a quadrupole, so $\tilde{f} = 2f$

$$M_{half\ Cell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

for the second half cell set f → -f

Transfer matrix for half a FoDo cell:

$$M_{half\ cell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$



Compare to the twiss parameter form of M

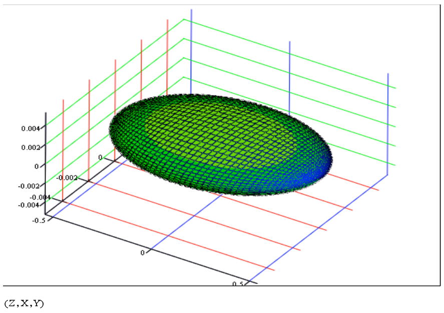
$$M_{1\rightarrow 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1+\alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2\sin\psi_{12}) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we allways have $\alpha = 0$, and the half cell will lead us from β_{max} to β_{min}

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}}\cos\frac{\mu}{2} & \sqrt{\check{\beta}\hat{\beta}}\sin\frac{\mu}{2} \\ \frac{-1}{\sqrt{\hat{\beta}\check{\beta}}}\sin\frac{\mu}{2} & \sqrt{\frac{\hat{\beta}}{\check{\beta}}}\cos\frac{\mu}{2} \end{pmatrix}$$

Solving for β_{max} and β_{min} and remembering that …. $\sin\frac{\mu}{2} = \frac{l_D}{\tilde{f}} = \frac{L}{4f}$

$$\left.\begin{array}{l} \frac{S'}{C} = \frac{\hat{\beta}}{\check{\beta}} = \frac{1 + l_D/\tilde{f}}{1 - l_D/\tilde{f}} = \frac{1+\sin\frac{\mu}{2}}{1-\sin\frac{\mu}{2}} \\ \\ \frac{S}{C'} = \hat{\beta}\check{\beta} = \tilde{f}^2 = \frac{l_D^2}{\sin^2\frac{\mu}{2}} \end{array}\right\} \rightarrow \begin{array}{l} \boxed{\hat{\beta} = \frac{(1+\sin\frac{\mu}{2})L}{\sin\mu}} \\ \boxed{\check{\beta} = \frac{(1-\sin\frac{\mu}{2})L}{\sin\mu}} \end{array}$$



The maximum and minimum values of the β-function are solely determined by the phase advance and the length of the cell.

Longer cells lead to larger β

typical shape of a proton bunch in the HERA FoDo Cell

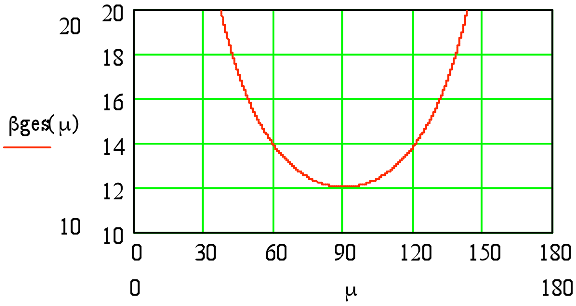
Optimising the FoDo phase advance

$$r^2 = \varepsilon_x \beta_x + \varepsilon_y \beta_y$$

search for the phase advance μ that results in a minimum of the sum of the beta's

$$\hat{\beta} + \check{\beta} = \frac{(1+\sin\frac{\mu}{2})*L}{\sin\mu} + \frac{(1-\sin\frac{\mu}{2})*L}{\sin\mu}$$

$$\hat{\beta} + \check{\beta} = \frac{2L}{\sin\mu} \qquad \frac{d}{d\mu}\left(2L/\sin\mu\right) = 0$$



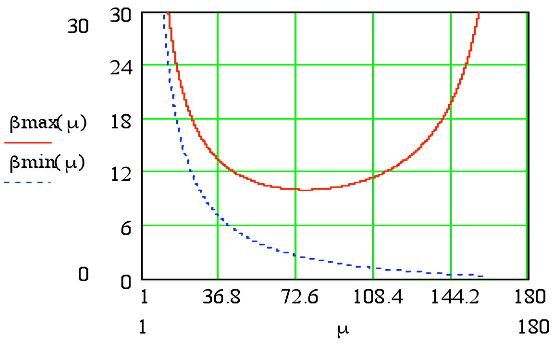
$$\frac{L}{\sin^2\mu} * \cos\mu = 0 \rightarrow \underline{\underline{\mu = 90°}}$$

Electrons are different

electron beams are usually flat, $\varepsilon_y \approx 2 - 10\ \%\ \varepsilon_x$

→ optimise only β_{hor}

$$\frac{d}{d\mu}(\hat{\beta}) = \frac{d}{d\mu}\frac{L(1+\sin\frac{\mu}{2})}{\sin\mu} = 0 \rightarrow \mu \approx 76°$$



red curve: β_{max}
blue curve: β_{min}
as a function of the phase advance μ