Beam Dynamics in Linacs I

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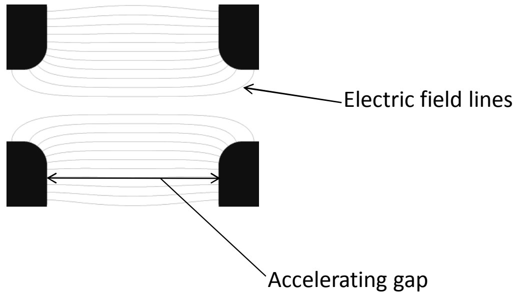
Lecture I: Linac Transverse Dynamics

- Cavities and gaps
- RF defocusing in a gap
- Quadrupole focusing
- Periodic solutions
- Ellipse transformation
- FODO lattice
- Smooth approximation

Cavities and Gaps

A proton linac typically consists of a series of RF cavities with gaps designed to efficiently accelerate the particles.



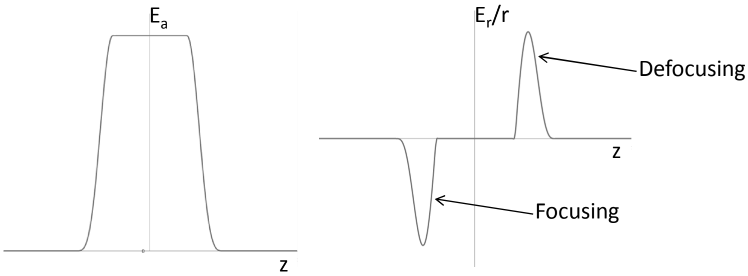
An off-axis particle in a gap will experience a radial force.

Radial field components

From Gauss's Law,

$$\nabla \cdot \mathbf{E} = 0$$

any change in longitudinal field results in a radial field.



Because the field is usually rising in time in a proton linac, net de-focussing can occur.

RF De-focusing

A typical linac cavity only has three non-zero field components:

E_z, E_r and B_θ

The radial momentum impulse on a charge q with velocity βc is

$$\Delta p_r = q \int_{-L/2}^{L/2} (E_r - \beta c B_\theta) \frac{dz}{\beta c}$$

If E_z is independent of r near the axis, Maxwell's equations lead to

$$E_r = -\frac{r}{2}\frac{\partial E_z}{\partial z}, B_\theta = \frac{r}{2c^2}\frac{\partial E_z}{\partial t}$$

giving

$$\Delta p_r = -\frac{q}{2}\int_{-L/2}^{L/2} r\left(\frac{\partial E_z}{\partial z} + \frac{\beta}{c}\frac{\partial E_z}{\partial t}\right)\frac{dz}{\beta c}$$

Radial RF Impulse

The total rate of change of field is the sum of changes due to time and position

$$\frac{dE_z}{dz} = \frac{\partial E_z}{\partial z} + \frac{1}{\beta c}\frac{\partial E_z}{\partial t}$$

leading to

$$\Delta p_r = -\frac{qr}{2\beta c}\int_{-L/2}^{L/2}\left(\frac{dE_z}{dz} - \frac{1}{\gamma^2 \beta c}\frac{\partial E_z}{\partial t}\right)dz$$

If

$$E_z = E_a(z)\cos(\omega t + \phi)$$

then

$$\Delta p_r = -\frac{qr\omega}{2\gamma^2\beta^2 c^2}\sin\phi \int_{-L/2}^{L/2} E_a(z)\cos(kz)dz$$

$$k = 2\pi / \beta\lambda, \omega t = kz$$

Effective voltage $E_0 TL$

Deflection Due to RF De-focusing

Radial momentum is given by

$$p_r = mc\beta\gamma r'$$
$$r' = \frac{dr}{dz}$$

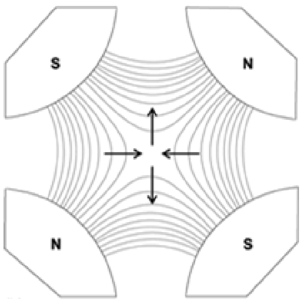
leading to the thin lens radial deflection in a cavity

$$\Delta(\beta\gamma r') = -\frac{\pi q E_0 T L r \sin(\phi)}{mc^2 \beta^2 \gamma^2 \lambda}$$

For phase stable acceleration $\phi < 0$ giving a +ve deflection ie RF de-focussing.

Quadrupole Focusing

It is usually necessary to compensate for the radial RF defocusing in linacs by the use of magnetic lenses. The quadrupole lens is by far the most common.



Quadrupole Fields

An ideal quadrupole lens has constant transverse magnetic gradient

$$G = \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \frac{B_0}{a}$$

Pole tip field

Pole tip radius

The Lorentz force components on a particle of velocity v are

$$F_x = -qvGx$$
$$F_y = qvGy$$

Force is linear in transverse position, focusing in one plane defocusing in the other. +ve G gives focusing in x.
Lenses are arranged with alternating polarities to give overall focusing.

Particle Motion in a Quadrupole

The equations of motion in a perfect quadrupole are

$$\frac{d^2x}{ds^2} + \kappa^2(s)x = 0$$

Where s is the axial position

$$\frac{d^2y}{ds^2} - \kappa^2(s)y = 0$$

$$\kappa^2(s) = \frac{|qG(s)|}{m\beta\gamma c}$$

In a hard edged quadrupole, $G(s) = G_0$ and the particles perform simple harmonic motion with angular frequency κ.

Transfer Matrix Representation

The solution to the equation of motion (in in one plane)

$$x'' + K(s)x = 0$$
$$|K(s)| = \kappa(s)^2$$

can be expressed in matrix form as

$$\begin{bmatrix} x_1 \\ x_1' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

$$x' = \frac{dx}{ds}, x'' = \frac{d^2x}{ds^2}$$

where subscript *0* indicates initial conditions, *1* indicates final conditions.

Drift Space

Denoting the 2x2 transfer matrix as **R**

For a drift of length *l*

$$\mathbf{R} = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}$$

Focusing Quadrupole

For a focusing quadrupole of length l and gradient G

$$\mathbf{R} = \begin{bmatrix} \cos\sqrt{K}l & \dfrac{\sin\sqrt{K}l}{\sqrt{K}} \\ -\sqrt{K}\sin\sqrt{K}l & \cos\sqrt{K}l \end{bmatrix}$$

$$K = \frac{qG}{mc\beta\gamma} > 0$$

Thin lens approximation

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$$

Focal length f given by $\dfrac{1}{f} = |Kl| = \left|\dfrac{qGl}{mc\beta\gamma}\right|$

De-focusing Quadrupole

For a de-focusing quadrupole of length l and gradient G

$$\mathbf{R} = \begin{bmatrix} \cosh\sqrt{|K|}l & \dfrac{\sinh\sqrt{|K|}l}{\sqrt{K}} \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{bmatrix}$$

$$K = \frac{qG}{mc\beta\gamma} < 0$$

Thin lens approximation

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{f} & 1 \end{bmatrix}$$

$$\frac{1}{f} = |Kl| = \left|\frac{qGl}{mc\beta\gamma}\right|$$

Periodic solutions

For the equation of motion

$$x'' + K(s)x = 0$$

with periodic function *K(s)* the general solution has the amplitude-phase form

$$x(s) = \sqrt{\varepsilon_1 \tilde{\beta}(s)}\cos[\tilde{\phi}_1 + \tilde{\phi}(s)]$$

Phase function $\tilde{\phi}(s) = \int \frac{ds}{\tilde{\beta}(s)}$

Amplitude function $\tilde{\beta}(s)$

ε_1 & $\tilde{\phi}_1$ are determined by the initial conditions.

Twiss Parameters

Defining the two additional functions

$$\tilde{\alpha}(s) = -\frac{1}{2}\frac{d\tilde{\beta}(s)}{ds}$$

$$\tilde{\gamma}(s) = \frac{1+\tilde{\alpha}(s)^2}{\tilde{\beta}(s)}$$

gives the 3 *Courant-Snyder* or *Twiss parameters*

$$\tilde{\alpha}(s), \tilde{\beta}(s) \,\&\, \tilde{\gamma}(s)$$

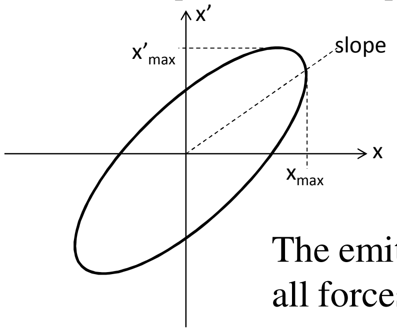
which are periodic with the same period as *K(s)*

Ellipse Representation

From the amplitude-phase form of the periodic solution and the Twiss parameters it can be shown the coordinates *x* & *x'* satisfy the relation

$$\tilde{\gamma}(s)x^2 + 2\tilde{\alpha}(s)xx' + \tilde{\beta}(s)x'^2 = \varepsilon$$

Which is the equation of an ellipse with area $\pi\varepsilon$



$$x_{\max} = \sqrt{\varepsilon\tilde{\beta}}$$

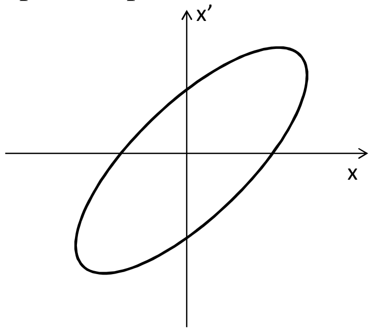
$$x'_{\max} = \sqrt{\varepsilon\tilde{\gamma}}$$

$$\text{slope} = \frac{\tilde{\alpha}}{\tilde{\beta}}$$

The emittance ε is constant in *s* if all forces are linear.

Liouville's Theorem

Liouville's Theorem states that the phase space density of particles in a conservative (linear) system is invariant. This means that in the absence of acceleration the area of the phase space ellipse – called the emittance – is constant.



$$\text{Area} = \pi\varepsilon$$

$$\varepsilon = \tilde{\gamma}(s)x^2 + 2\tilde{\alpha}(s)xx' + \tilde{\beta}(s)x'^2$$

Under acceleration the conserved quantity is the normalised emittance

$$\varepsilon_n = \beta\gamma\varepsilon$$

Ellipse Transformations

The ellipse equation can be written in matrix form as

$$\mathbf{X}^T \boldsymbol{\sigma}^{-1} \mathbf{X} = \varepsilon$$

where $$\mathbf{X} = \begin{bmatrix} x \\ x' \end{bmatrix}$$

$$\boldsymbol{\sigma}^{-1} = \begin{bmatrix} \tilde{\gamma} & \tilde{\alpha} \\ \tilde{\alpha} & \tilde{\beta} \end{bmatrix}$$

and $$\boldsymbol{\sigma} = \begin{bmatrix} \tilde{\beta} & -\tilde{\alpha} \\ -\tilde{\alpha} & \tilde{\gamma} \end{bmatrix}$$

Ellipse Transformations

Knowing that $$\mathbf{X}_1^T \boldsymbol{\sigma}_1^{-1} \mathbf{X}_1 = \varepsilon$$

$$\mathbf{X}_2^T \boldsymbol{\sigma}_2^{-1} \mathbf{X}_2 = \varepsilon$$

and $$\mathbf{X}_2 = \mathbf{R}\mathbf{X}_1$$

it can be shown that $$\boldsymbol{\sigma}_2 = \mathbf{R}\boldsymbol{\sigma}_1\mathbf{R}^T$$

or $$\begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} R_{11}^2 & -2R_{11}R_{12} & R_{12}^2 \\ -R_{11}R_{21} & 1+R_{12}R_{21} & -R_{12}R_{22} \\ R_{21}^2 & -2R_{21}R_{22} & R_{21}^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}$$

For the matched periodic beam $\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1$

Periodic Matrix

For a periodic channel of length L the transfer matrix is

$$\mathbf{P} = \mathbf{R}(s \to s+L) = \begin{bmatrix} \cos\sigma + \tilde{\alpha}\sin\sigma & \tilde{\beta}\sin\sigma \\ -\tilde{\gamma}\sin\sigma & \cos\sigma - \tilde{\alpha}\sin\sigma \end{bmatrix}$$

where

$$\sigma = \Delta\tilde{\phi} = \int^{L} \frac{ds}{\tilde{\beta}(s)}$$

is the phase advance per period.

Periodic Matrix

By constructing the total transfer matrix or transporting two orthogonal particles $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ x_1' \end{bmatrix}$ through the lattice the elements of **P** and hence the phase advance can be calculated

$$\begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = \mathbf{P}\begin{bmatrix} x_1 \\ 0 \end{bmatrix} \text{and} \begin{bmatrix} x_3 \\ x_3' \end{bmatrix} = \mathbf{P}\begin{bmatrix} 0 \\ x_1' \end{bmatrix}$$

$$x_2 = \mathbf{P}_{11}x_1 = (\cos\sigma + \tilde{\alpha}\sin\sigma)x_1$$

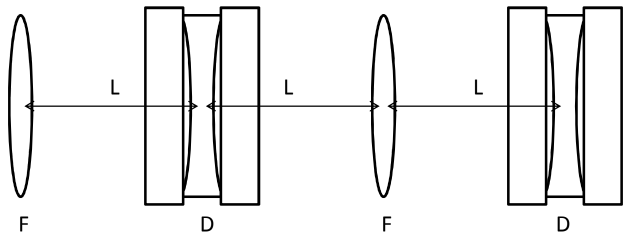
$$x_3' = \mathbf{P}_{22}x_1' = (\cos\sigma - \tilde{\alpha}\sin\sigma)x_1'$$

$$2\cos\sigma = \mathbf{P}_{11} + \mathbf{P}_{22} = \frac{x_2}{x_1} + \frac{x_3'}{x_1'}$$

and the Twiss parameters follow directly from the other elements of **P**.

FODO lattice

The FODO lattice is the most common structure used in linacs.



Assuming thin lenses separated by distance L the period length is $2L$.

FODO Twiss Parameters

Neglecting any acceleration the transfer matrix **P** is the product of the individual transfer matrices.
Starting in the centre of the F quad

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ -(2f)^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ f^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(2f)^{-1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\sigma + \tilde{\alpha}\sin\sigma & \tilde{\beta}\sin\sigma \\ -\tilde{\gamma}\sin\sigma & \cos\sigma - \tilde{\alpha}\sin\sigma \end{bmatrix}$$

Evaluating the terms gives

$$\cos\sigma = 1 - \frac{L^2}{2f^2} \qquad \tilde{\beta} = 2L\frac{1+\sin(\sigma/2)}{\sin\sigma} \qquad \sin\frac{\sigma}{2} = \frac{L}{2f} \qquad \tilde{\alpha} = 0$$

FODO Twiss Parameters

Starting in the centre of the drift after the F quad

$$\cos\sigma = 1 - \frac{L^2}{2f^2}$$

$$\tilde{\beta} = \frac{L}{\sin\sigma}\left(2 - \sin^2\frac{\sigma}{2}\right)$$

$$\tilde{\alpha} = \frac{2\sin(\sigma/2)}{\sin\sigma}$$

$$\sin\frac{\sigma}{2} = \frac{L}{2f}$$

FODO Twiss Parameters

Starting in the centre of the D quad

$$\cos\sigma = 1 - \frac{L^2}{2f^2}$$

$$\tilde{\beta} = 2L\frac{1-\sin(\sigma/2)}{\sin\sigma}$$

$$\tilde{\alpha} = 0$$

$$\sin\frac{\sigma}{2} = \frac{L}{2f}$$

FODO Twiss Parameters

Starting in the centre of the drift after the D quad

$$\cos\sigma = 1 - \frac{L^2}{2f^2}$$

$$\tilde{\beta} = \frac{L}{\sin\sigma}\left(2 - \sin^2\frac{\sigma}{2}\right)$$

$$\tilde{\alpha} = \frac{-2\sin(\sigma/2)}{\sin\sigma}$$

$$\sin\frac{\sigma}{2} = \frac{L}{2f}$$

FODO Beam Envelope

The envelope is maximum in the F quad and minimum in the D quad. The ratio between maximum and minimum beta functions is

$$\frac{\tilde{\beta}_{\max}}{\tilde{\beta}_{\min}} = \frac{1+\sin(\sigma/2)}{1-\sin(\sigma/2)}$$

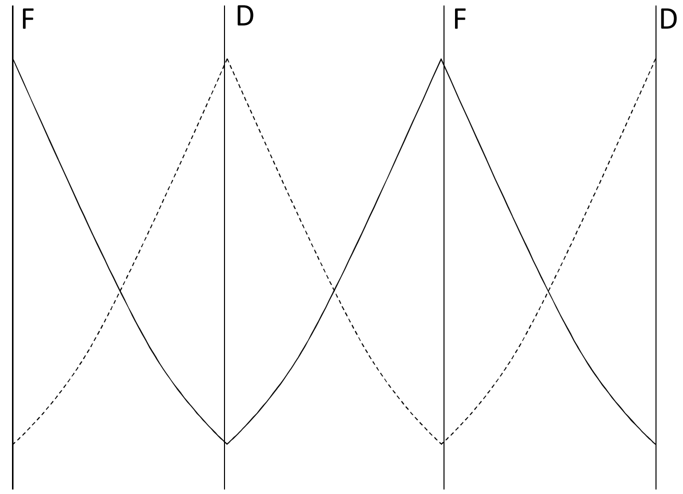
In the centre of the drifts the beam is symmetrical

$$\tilde{\beta}_x = \tilde{\beta}_y$$

$$\tilde{\alpha}_x = -\tilde{\alpha}_y$$

The solution is stable for $\sigma < \pi$ which means $f \geq L/2$.

FODO Beam Envelope



Smooth Approximation

A similar treatment can be carried out including thin lens RF de-focusing in the drifts. The matrix terms quickly become unmanageable so an approximation is necessary. In the smooth approximation the trajectories become sinusoidal.

$$\sigma^2 \approx \left(\frac{L}{f_q}\right)^2 - \left(\frac{4L}{f_g}\right)$$

$$\frac{1}{f_q} = \frac{qGl}{mc\beta\gamma}$$

$$\frac{1}{f_g} = \frac{\pi q E_0 T L \sin(-\phi)}{mc^2 (\beta\gamma)^3 \lambda}$$

Smooth Approximation

$$\sigma^2 \approx \left(\frac{qGlL}{mc\beta\gamma}\right)^2 - \frac{\pi q E_0 T \sin(-\phi)(2L)^2}{mc^2(\beta\gamma)^3\lambda}$$

For a stable solution the quadrupole term must be greater than the RF de-focusing term. The $(\beta\gamma)^3$ in the RF de-focus term means it is more important at low velocities.
The phase advance per unit length

$$\left(\frac{\sigma}{2L}\right)^2$$

is independent of the period length. The length is limited by the stability requirement $\sigma < \pi$.