On a rigorous Framework for a Quantum N-Body Theory on curved Spacetime

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Motivation

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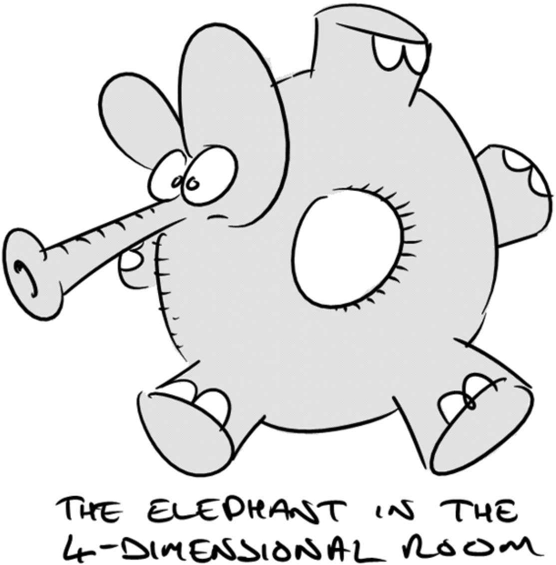

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"These equations lead to infinities when one tries to solve them; these infinities ought not to be there. They remove them artificially. [...] Indeed, there is some justification for that because rules can be set up to remove the infinities. This is the renormalization process. It turns out that, sometimes, one gets very good agreement with experiments working with these rules. [...] Most physicists say that these working rules are, therefore, correct. I feel that is not an adequate reason. Just because the results happen to be in agreement with observation does not prove that one's theory is correct. After all, the Bohr theory was correct in simple cases. It gave very good answers, but still the Bohr theory had ***the wrong concepts*** *[emphasis added]. Correspondingly, the renormalized kind of quantum theory with which physicists are working nowadays is not justifiable by agreement with experiments under certain conditions."*

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— P. A. M. Dirac*

*Dirac, *The inadequacies of quantum field theory* in 'Paul Adrien Maurice Dirac: Reminiscences about a great Physicist' (eds. Kursunoglu & Wigner) 194-198 (Cambridge University Press, 1987).

Motivation

"There remains the task to create a mathematically rigorous theory which explains the great success of formal perturbation theory."

— E. Zeidler*

 †

*Zeidler, *Quantum Field Theory*. vol. II (Springer, 2009), p. 109.

Motivation

- further reading: Fraser, *Brit. J. Philos. Sci.* **71**, 391 (2018)

*Haag, *Local Quantum Physics* (Springer, 1996)
†Hollands & Wald, *Commun. Math. Phys.* **293**, 85 (2009)
‡Bär & Fredenhagen, *Quantum Field Theory on Curved Spacetimes* (Springer, 2009)

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- further reading: Fraser, *Brit. J. Philos. Sci.* **71**, 391 (2018)
- mathematically rigorous approaches to relativistic quantum theory exist,[*][†][‡] but question remains open

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  - QT is a statistical theory

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  - relativistic QT should not depend on the symmetries of Minkowski spacetime[†]

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- mathematically rigorous approaches to relativistic quantum theory exist,[*][†][‡] but question remains open
- our approach:
  - QT is a statistical theory
  - relativistic QT should not depend on the symmetries of Minkowski spacetime[†]
  - therefore, study conservation of particle detection probability on curved spacetime to understand structure of relativistic QT

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Overview

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1 Quantum one-body theory

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Quantum one-body theory

Reddiger & Poirier, arXiv:2012.05212 [math-ph] (2020)

Quantum one-body theory*

seek: generalization of non-relativistic, one-body Born rule

$$\mathbb{P}_t\,(U) = \int_U \rho\,(t, \vec{x})\;\mathrm{d}^3x \quad \text{with} \quad \mathbb{P}_t(\mathbb{R}^3) = 1$$

and conserved ρ:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\,\vec{v}) = 0$$

*Reddiger & Poirier, arXiv:2012.05212 [math-ph] (2020)

Quantum one-body theory: Static case

- setting: 4-spacetime $\mathcal{Q}$, local coordinates κ
- for 'fixed time', choose 'suitable' hypersurface $\mathcal{S}_0$
- inclusion map in coordinates:

$$\xi = (\xi^1, \xi^2, \xi^3) \mapsto \kappa(\xi) = \Big(\kappa^0(\xi), \kappa^1(\xi), \kappa^2(\xi), \kappa^3(\xi)\Big)$$

- current density vector field J *over* $\mathcal{S}_0$ given:

$$\xi \mapsto J^i(\xi) \quad , \qquad i \in \{0, 1, 2, 3\}$$

- J orients $\mathcal{S}_0$, i.e. for κ right-handed on $\mathcal{Q}$, we require of ξ that

$$\det\left(J, \frac{\partial \kappa}{\partial \xi^1}, \frac{\partial \kappa}{\partial \xi^2}, \frac{\partial \kappa}{\partial \xi^3}\right) > 0$$

- This works iff J is non-tangent to $\mathcal{S}_0$

Quantum one-body theory: Static case

- if ξ are global coordinates on $\mathcal{S}_0$, then set

$$\mathbb{P}_{\mathcal{S}_0}(U) = \int_U h(\xi)\,\sqrt{|g|}(\kappa(\xi))\;\mathrm{d}^3\xi$$

  with

$$h(\xi) = \varepsilon_{i_0 i_1 i_2 i_3}\, J^{i_0}\, \frac{\partial\kappa^{i_1}}{\partial\xi^1}\,\frac{\partial\kappa^{i_2}}{\partial\xi^2}\,\frac{\partial\kappa^{i_3}}{\partial\xi^3}\,,$$

  assuming $\mathbb{P}_{\mathcal{S}_0}(\mathcal{S}_0) = 1$ (normalization of J)

- $\mathbb{P}_{\mathcal{S}_0}(U)$ is the *probability to find the particle in* U

  $\rightsquigarrow$ relativistic one-body Born rule

- Ehlers first suggested above integral in context of relativistic fluid dynamics (mass/charge/entropy)*

*Ehlers, *General relativity and kinetic theory* in *Proceedings of the International School of Physics "Enrico Fermi", Course XLVII* (ed. Sachs, Academic Press, 1971)

Quantum one-body theory: Lagrangian picture

- as in nonrelativistic continuum mechanics, $\exists$ *Lagrangian picture* and *Eulerian picture*
- *Lagrangian picture ingredients:*
  - 'flow domain' $\mathcal{S} \subseteq \mathbb{R} \times \mathcal{S}_0$
  - inclusion $\mathcal{S} \to \mathcal{Q}$: $(\tau, \xi) \mapsto \kappa\,(\tau, \xi)$
  - $X = \partial\kappa/\partial\tau$ future-directed timelike
  - scalar field ρ on $\mathcal{S}$ is *invariant probability density**
  - use

    $$J^i(\tau,\xi) = \rho(\tau,\xi)\,\frac{\partial\kappa^i}{\partial\tau}(\tau,\xi)$$

    for prior Born rule, integrating over $\tau = \mathrm{const.}$ surfaces
- *probability conservation* if

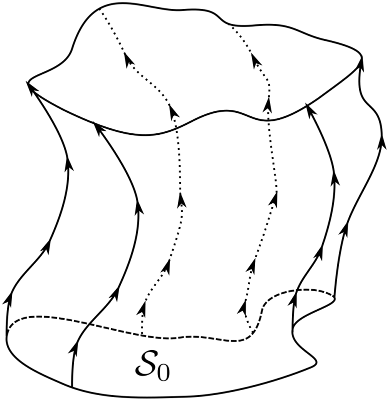
$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial\tau}\left(\rho\sqrt{|g|}\right) = 0$$

* *invariant mass density* first introduced in Eckart, *Phys. Rev.* **58**, 919–924 (1940).

Quantum one-body theory: Lagrangian picture

Theorem (Reddiger & Poirier 2021, in preparation)

The relativistic Born rule in the Lagrangian picture is invariant under reparametrizations of τ.



N-body theory for point masses

N-body theory for point masses

- if N future-directed timelike curves are given, then no need for a different formalism
- for collision-less interactions, however, we will have fields on 'N-body spacetime' (e.g. electromagnetic fields), possibly with singularities
- exclude gravitational interaction here, i.e. 'fixed background'
- define N*-body spacetime* $\mathcal{Q}^N$, open subset of $\bigtimes_{a=1}^{N} \mathcal{Q}$
- locally we have coordinates

$$\kappa = (\kappa_1, \ldots, \kappa_N) = \left(\kappa_1^0, \ldots, \kappa_1^3, \kappa_2^0, \ldots, \kappa_2^3, \ldots, \kappa_N^0, \ldots, \kappa_N^3\right)$$

  and 'metric for ath body' $\kappa \mapsto (g_a)_{ij}(\kappa_a)$
- evolution of N point masses is given by 'future-directed timelike' curve on $\mathcal{Q}^N$

Quantum N-body Theory

Quantum N-body Theory

- Lienert & Tumulka claimed to have found Born rule for N bodies in Minkowski spacetime*
- Miller et al. claimed to have done so in globally hyperbolic spacetimes†
- our construction
  - differs
  - works on any spacetime
  - closely related to the idea of 'multi-time wave functions'‡

*Lienert & Tumulka, *Lett. Math. Phys.* **110**, 753 (2019)
†Miller, Eckstein, Horodecki, & Horodecki, *J. Geom. Phys.* **160**, 103990 (2021)
‡Lienert, Petrat, & Tumulka, *Found. Phys.* **47**, 1582 (2017)

Quantum N-body Theory: Static case

- consider N-body spacetime $\mathcal{Q}^N$
- in analogy to one-body case, consider 'suitable' $3N$-submanifold of $\mathcal{Q}^N$
- let $X = \sum_{a=1}^{N} X_a$ 'future-directed timelike' vector field *over* $\mathcal{S}_0$
- Proposition:
  An $\mathcal{S}_0$, for which each $X_a = X_a^i\,\partial/\partial\kappa_a^i$ is nowhere tangent, admits local coordinates $\xi = (\xi_1^1, \xi_1^2, \xi_1^3, \ldots, \xi_N^1, \xi_N^2, \xi_N^3)$ such that each $\partial/\partial\xi_a^i$ lies in the 'ath body tangent space'
- If $\mathcal{S}_0$ is embedded, we can extend those to 'N-body slice coordinates' κ on $\mathcal{Q}^N$ such that $\mathcal{S}_0$ is the $\kappa_1^0 = \cdots = \kappa_N^0 = 0$ submanifold

Quantum N-body Theory: time-independent case

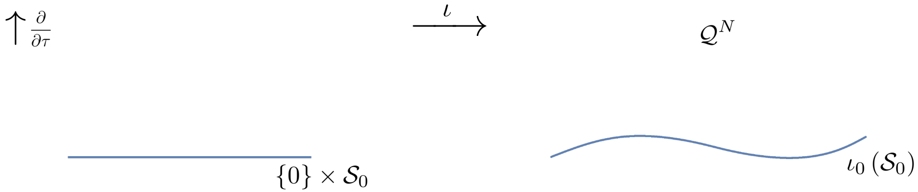
- if ρ_0 is the probability density on $\mathcal{S}_0$ and those coordinates are global, then set

$$\mathbb{P}_{\mathcal{S}_0}(U) := \int_U \rho_0(\xi)\, X_1^0(\xi) \cdots X_N^0(\xi) \sqrt{|g_1|}(0,\xi) \cdots \sqrt{|g_N|}(0,\xi)\ \mathrm{d}^{3N}\xi$$

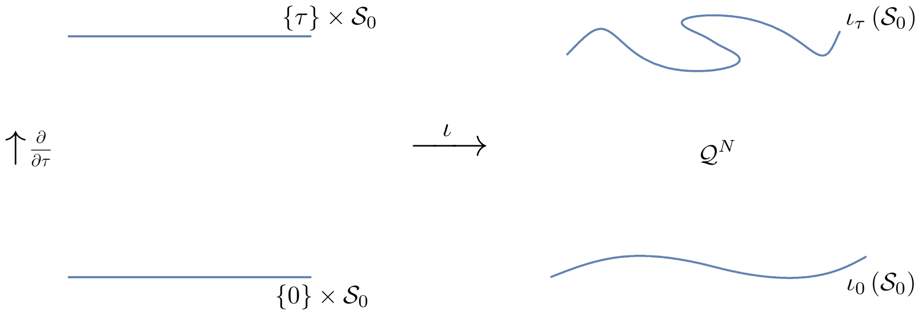
  assuming ρ_0 to be normalized, i.e. $\mathbb{P}_{\mathcal{S}_0}(\mathcal{S}_0) = 1$

- This is the static, relativistic N-body Born rule (in N-body slice coordinates)

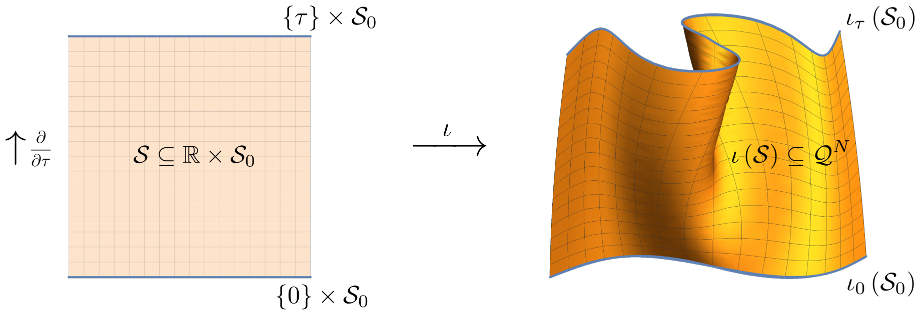
Quantum N-body Theory: Dynamic case



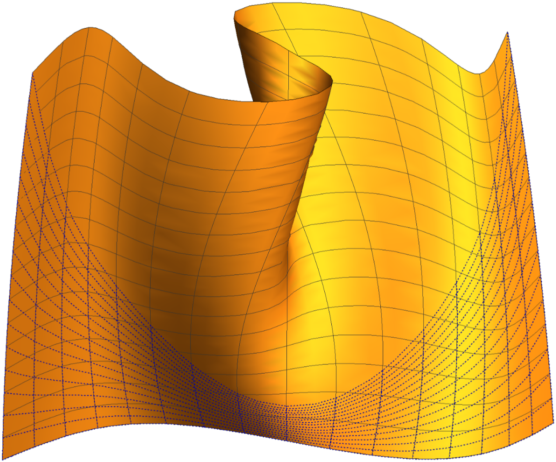
Quantum N-body Theory: Dynamic case



Quantum N-body Theory: Dynamic case



Quantum N-body Theory: Dynamic case



Quantum N-body Theory: time-dependent case

- in N-body slice coordinates with inclusion mapping $(\tau,\xi)\mapsto\kappa(\tau,\xi)$ we have probability conservation for

$$\frac{\partial}{\partial\tau}\left(\rho\sqrt{|g_1|}\cdots\sqrt{|g_N|}\;\tilde{h}\right)=0$$

with

$$\tilde{h}=\sum_{\sigma\in S_{(3N)}}\mathrm{sgn}(\sigma)\Big(X_1^0\,\frac{\partial\kappa_1^1}{\partial\xi^{\sigma(1)}}\frac{\partial\kappa_1^2}{\partial\xi^{\sigma(2)}}\frac{\partial\kappa_1^3}{\partial\xi^{\sigma(3)}}+\dots$$
$$+(-1)^3\frac{\partial\kappa_1^0}{\partial\xi^{\sigma(1)}}\frac{\partial\kappa_1^1}{\partial\xi^{\sigma(2)}}\frac{\partial\kappa_1^2}{\partial\xi^{\sigma(3)}}X_1^3\Big)\cdots$$
$$\Big(X_N^0\,\frac{\partial\kappa_N^1}{\partial\xi^{\sigma(3N-2)}}\frac{\partial\kappa_N^2}{\partial\xi^{\sigma(3N-1)}}\frac{\partial\kappa_N^3}{\partial\xi^{\sigma(3N)}}+\dots$$
$$+(-1)^3\frac{\partial\kappa_N^0}{\partial\xi^{\sigma(3N-2)}}\frac{\partial\kappa_N^1}{\partial\xi^{\sigma(3N-1)}}\frac{\partial\kappa_N^2}{\partial\xi^{\sigma(3N)}}X_N^3\Big)$$

(ξ coordinates relabeled from 1 to $3N$)

Open Questions

- indistinguishable bodies?
- fully correlated example?
- dynamical equations?
- variable number of bodies?

Thank you!

Directly related prior work:

- Reddiger & Poirier, arXiv:2012.05212 [math-ph] (2020)

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www.depts.ttu.edu/chemistry/faculty/poirier/

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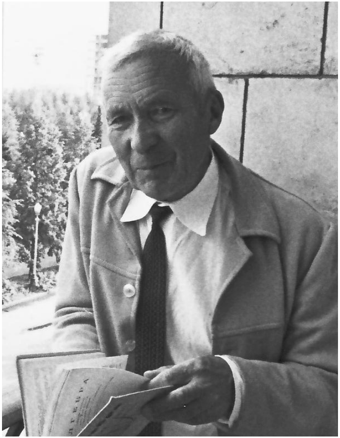
Addendum 1: Foundations of Modern Probability Theory


*

Andrey Kolmogorov (1903-1987)

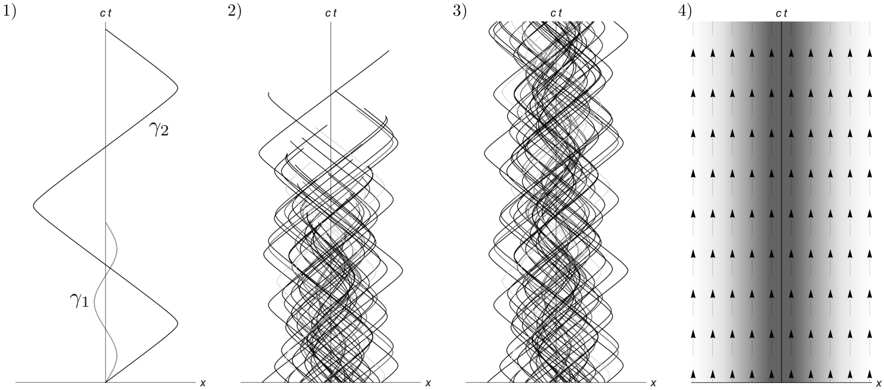
Grundbegriffe der Wahrscheinlichkeitsrechnung (1933)


†

John von Neumann (1903-1957)

Mathematische Grundlagen der Quantenmechanik (1932)

Addendum 2: Stochastic Interpretation



stochastic (3) vs. phenomenological (4) description of an *ensemble* of special-relativistic harmonic oscillators with same (proper) period ω

initial positions and momenta are Gaussian distributed