Metaplectic Flavor Symmetries from Magnetized Torus

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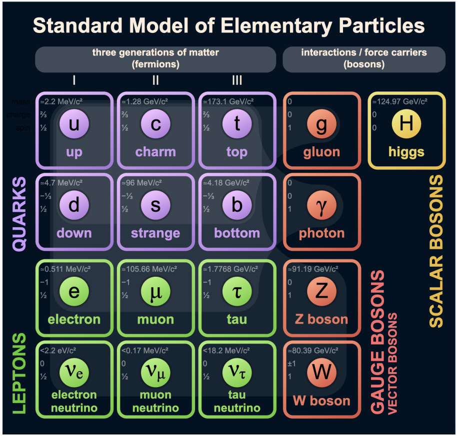
Based in arXiv:2102.11286 with Yahya Almumin, Mu-Chun Chen, Saúl Ramos-Sánchez, Michael Ratz and Shreya Shukla

Standard Model and its unpredicted parameters



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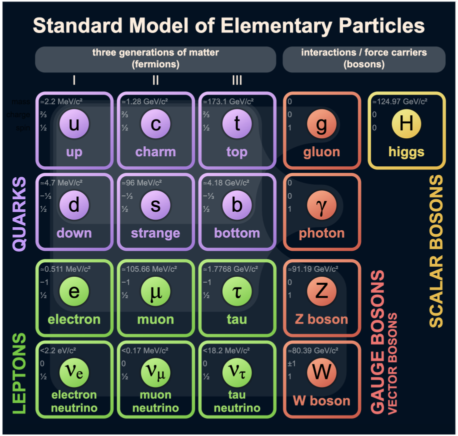

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Unpredicted parameters:

- $m_u, m_d, m_c, m_s, m_t, m_b$
- m_e, m_μ, m_τ
- $m_{\nu_1}, m_{\nu_2}, m_{\nu_3}$

CKM:

- δ^q_{CP}
- $\theta^q_{12}, \theta^q_{13}, \theta^q_{23}$

MNS matrix:

- δ^l_{CP}
- ϕ_1, ϕ_2
- $\theta^l_{12}, \theta^l_{13}, \theta^l_{23}$

Modular flavor symmetries

Consider an $N = 1$ supersymmetric theory with superpotential given by

$$\mathcal{W} = \sum_n \underbrace{Y_{I_1 \cdots I_n}(\tau)}_{\text{Modular forms}} \varphi^{I_1} \cdots \varphi^{I_n} \tag{1}$$

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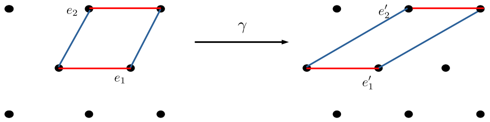
What are modular forms?

Torus and Modular Forms



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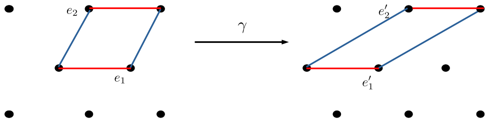

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$$\tau = \frac{e_2}{e_1} \qquad \text{where} \qquad \tau \xrightarrow{\gamma} \frac{a\tau + b}{c\tau + d}$$

with $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$

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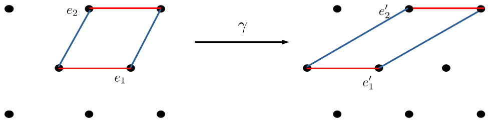

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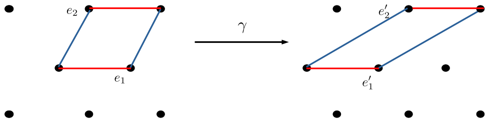

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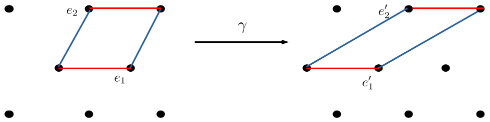

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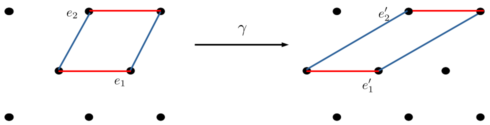

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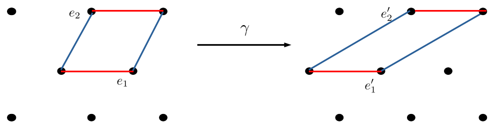

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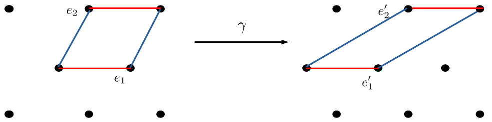

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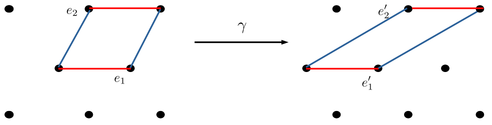

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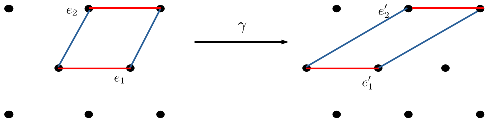

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- Examples: $\Gamma_2 \cong S_3$(Triangle), $\Gamma_3 \cong A_4$(tetrahedron)

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Two approaches for modular symmetries:

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  - We will use this one

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Field	# of generations
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$$M_1+M_2+M_3=0$$

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4. It turns out [Y. Almumin, M. C. Chen, V. Knapp-Perez, S. Ramos-Sanchez, M. Ratz and S. Shukla(2102.11286)]

$$Y = Y(\lambda) \qquad \text{where} \qquad \lambda = \text{l.c.m}(M_1, M_2, M_3)$$

and

$$Y(\tau) \xrightarrow{\gamma} Y(\gamma\tau) =: \pm(c\tau + d)^{\frac{1}{2}}[\rho(\gamma)]f(\tau) \tag{3}$$

Metaplectic Modular Groups

- For $k = \frac{\mathbb{N}}{2}$ we must consider the double cover of $SL(2,\mathbb{Z})$, **the metaplectic group**

$$\mathrm{Mp}(2,\mathbb{Z}) = \{\tilde{\gamma} = (\gamma, \varphi(\gamma,\tau)) | \gamma \in \Gamma, \varphi(\gamma,\tau) = \pm(c\tau + d)^{\frac{1}{2}}\} \quad (4)$$

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$M_1 = 3$ $M_2 = 3$ $M_3 = -6$	$\tilde{\Gamma}_{6\cdot 2} = \tilde{\Gamma}_{12}$
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Conjecture:

Magnetized torus exhibit a $\tilde{\Gamma}_{2\lambda}$ modular flavor symmetry.

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- Wilson lines = Scalar particles with mass corrections under control [W. Buchmuller, M. Dierigl, E. Dudas and J. Schweizer(Arxiv:1611.03798)]

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- Wilson lines = Scalar particles with mass corrections under control [W. Buchmuller, M. Dierigl, E. Dudas and J. Schweizer(Arxiv:1611.03798)]
- **Is SUSY needed?**

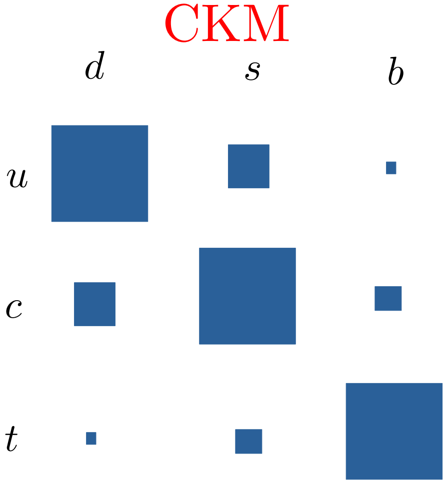
Summary and Outlook

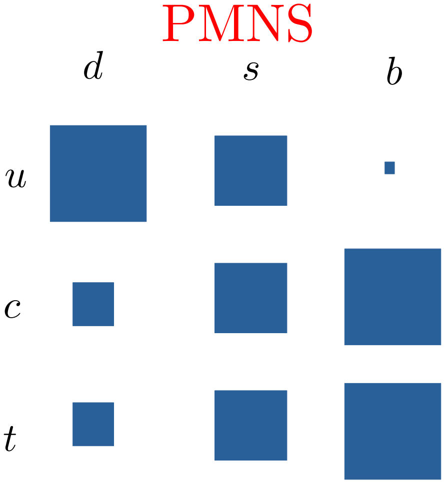
- Modular flavor symmetries are derived $\implies$ No loss of predictivity.
- Contact with bottom-up constructions ($\tilde{\Gamma}_4$) [X.-G. Liu, C.-Y. Yao, B.-Y. Qu, and G.-J. Ding(2007.13706)]
- There are other symmetries which need to be studied carefully
- Not realistic models, but they can give rise to realistic models through $D-$brane and heterotic string constructions.
- Wilson lines $=$ Scalar particles with mass corrections under control [W. Buchmuller, M. Dierigl, E. Dudas and J. Schweizer(Arxiv:1611.03798)]
- **Is SUSY needed?**

The End

Extra slides

Motivation





Motivation

Let the Yukawa quark Lagrangian be

$$\mathcal{L}_{Yukawa,q} = \bar{\psi}^f_R Y_{fg} \psi^g_L + \text{h.c.} \tag{1}$$

Then, we can diagonalize $Y = Y_{fg}$ (we have two Y_u and Y_d) with $U_R^\dagger Y U_L = \text{diag}(y_1, y_2, y_3)$ where

$$U_L^\dagger Y^\dagger Y U_L = \text{diag}(|y_1|^2, |y_2|^2, |y_3|^2)$$
$$U_R^\dagger Y Y^\dagger U_R = \text{diag}(|y_1|^2, |y_2|^2, |y_3|^2).$$

Then, to calculate CKM matrix we

1. Switch to the basis where Y_u is diagonal

$$Y_u \to (U_R^u)^\dagger Y_u U_L^{(u)} = \text{diag}(y_u, y_c, y_t)$$
$$Y_d \to (U_R^u)^\dagger Y_d U_L^{(u)} = Y_d'$$

2. Finally, $U_L^{(d)} = U_{CKM}$

Motivation

For the MNS matrix, we switch to the basis where

- $Y_e \to U_R^\dagger Y_e U_L = \mathrm{diag}(y_e, y_\mu, y_\tau)$
- $m_\nu \to U_L^T m_\nu U_L = m'_\nu$.

Then, U_{MNS} fulfills

$$U_{MNS}^T m'_\nu U_{MNS} = \mathrm{diag}(m_1, m_2, m_3)$$

Traditional flavor symmetry.

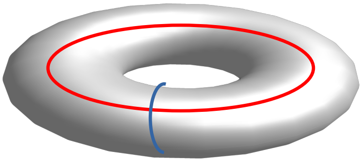
Additional **flavor symmetry** G_{fl}, e.g. $G_{fl} = A4$.

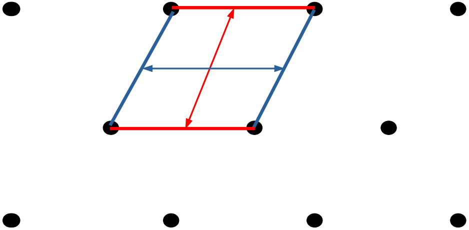
1. Assign matter fields to representations of G_{fl}, e.g. (L_e, L_μ, L_τ) as **3**
2. Introduce **flavons** to break G_{fl}
3. Extract mass matrices.

Some **drawbacks**

- Additional symmetries and fields are needed to align the flavon vevs.
- Hard to fit with current data

Tori





Tori

$$\begin{pmatrix} e_2 \\ e_1 \end{pmatrix} \xrightarrow{\gamma} \begin{pmatrix} e_2' \\ e_1' \end{pmatrix} = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{a,b,c,d\in\mathbb{Z}} \begin{pmatrix} e_2 \\ e_1 \end{pmatrix} =: \gamma \begin{pmatrix} e_2 \\ e_1 \end{pmatrix}$$

$\det\gamma=1 \leftrightarrow$ The volume of fundamental domain is invariant

where $\gamma \in \mathrm{SL}(2,\mathbb{Z})$

Modular Groups and Forms

Generators of SL(2, $\mathbb{Z}$):

$$T\colon\ e_2 \to e_2' = e_1 + e_2 \text{ and } \ e_1 \to e_1' = e_1 \qquad T =: \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S\colon\ e_2 \to e_2' = e_1 \ \text{ and } \ e_1 \to e_1' = -e_2 \qquad S =: \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\boxed{\text{where } S^4 = (ST)^3 = \mathbb{I} \textit{ and } S^2 T = TS^2}$$

Modular Groups and Forms

Finite subgroups for $N < 5$:

$$\boxed{\Gamma_N = \langle S, T | S^4 = (ST)^3 = \mathbb{I}, T^N = \mathbb{I} \rangle}$$

$$\boxed{\Gamma'_N = \langle S, T | S^4 = (ST)^3 = \mathbb{I}, S^2 T = TS^2, T^N = \mathbb{I} \rangle}$$

- $\Gamma'_N \cong \mathrm{SL}(2, \mathbb{Z}_N)$
- Complex coordinate z: $\mathbb{R}^2 \cong \mathbb{C}$ where $z \xrightarrow{\gamma} \frac{z}{c\tau+d}$
- Modulus τ: $\tau = \frac{e_2}{e_1}$ where $\tau \xrightarrow{\gamma} \frac{a\tau+b}{c\tau+d}$
- T: $z \to z$ and S: $z \to \frac{-z}{\tau}$
- T: $\tau \to \tau + 1$ and S: $\tau \to \frac{-1}{\tau}$

Modular Groups and Forms

Modular forms:

$$f_\alpha(\tau) \xrightarrow{\gamma} f_\alpha(\gamma\tau) =: (c\tau + d)^k [\rho(\gamma)_{\alpha\beta}] f_\beta(\tau)$$

- $(c\tau + d)$ is the automorphy factor
- $k \in \mathbb{N}$ is the modular weight
- ρ is the irrep. matrix of γ under Γ'_N
- Example: $\Gamma_2 \cong S_3$(Triangle) and $\Gamma_3 \cong A_4$(Tetrahedron)

Zero-Modes Wavefunction

Starting with a gauge theory in 6D$= \overbrace{4D}^{\text{Minkowski}} + \overbrace{2D}^{\text{Compactified Torus}}$

The torus $\mathbb{T}^2 \in \mathbb{C}$ with Magnetic Flux $M(B) \to M(B)$-Chiral zero-modes with charge q

Flux potential:

$$A(z) = \frac{b}{2\text{Im}\tau}\text{Im}(\bar{z}dz)$$

which transforms under lattice transformations as

$$A(z+1) = A(z) + \frac{b}{2\text{Im}\tau}\text{Im}dz$$
$$A(z+\tau) = A(z) + \frac{b}{2\text{Im}\tau}\text{Im}\bar{\tau}dz$$

Zero-Modes Wavefunction

Then, the wavefunctions must satisfy the boundary conditions

$$\psi(z+1,\tau) = \exp\left(\mathrm{i}\frac{qB}{2\mathrm{Im}\tau}\mathrm{Im}z\right)\psi(z,\tau)$$
$$\psi(z+\tau,\tau) = \exp\left(\mathrm{i}\frac{qB}{2\mathrm{Im}\tau}\mathrm{Im}\bar{\tau}z\right)\psi(z,\tau)$$

with the quantization condition

$$\frac{qB}{2\pi} = M \in \mathbb{Z}.$$

We can add a Wilson line which is a translation $z \to z + \zeta$ for $\zeta \in \mathbb{C}$.

Zero-Modes Wavefunction

The Dirac operator is

$$[\bar{\partial} + q\frac{\pi M}{2\text{Im}\tau}(z+\zeta)]\psi_{+} = 0$$
$$[\partial - q\frac{\pi M}{2\text{Im}\tau}(\bar{z}+\bar{\zeta})]\psi_{-} = 0$$

Wavefunctions of the zero modes of the Dirac operator:

$$\boxed{\psi^{j,M}(z,\tau,\zeta) = N e^{i\pi M(z+\zeta)\frac{Im(z+\zeta)}{Im\tau}}\theta\begin{bmatrix} j/M \\ 0 \end{bmatrix}(M(z+\zeta), M\tau)}$$

$$\boxed{\theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(z,\tau) = \sum_{l=-\infty}^{\infty} e^{\pi i(\alpha+l)^2\tau} e^{2\pi i(\alpha+l)(z+\beta)}}$$

Zero-Modes Wavefunction

$$\psi^{j,M}(z,\tau,\zeta) = N e^{i\pi M(z+\zeta)\frac{Im(z+\zeta)}{Im\tau}} \theta \begin{bmatrix} j/M \\ 0 \end{bmatrix} (M(z+\zeta), M\tau)$$

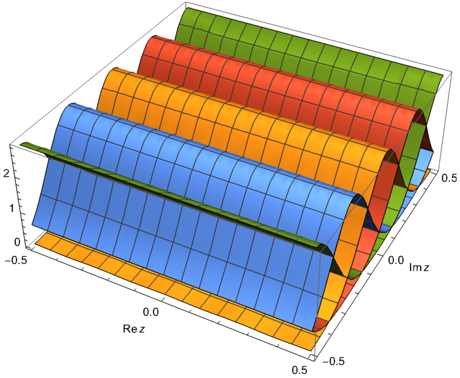


Figure: : Squares of the absolute values of the wave functions on a quadratic torus for M = 4.

Zero-Modes Wavefunction

$$\boxed{\psi^{j,M}(z+1+\tau,\tau,\zeta)=\psi^{j,M}(z+\tau+1,\tau,\zeta)}$$

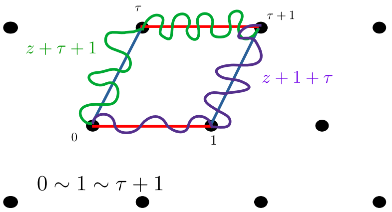


Figure: Quantization condition by a closed loop

Zero-Modes Wavefunction

$$\boxed{\psi^{j,M}(z+1+\tau,\tau,\zeta)=\psi^{j,M}(z+\tau+1,\tau,\zeta)}$$

This relation gives us the flux quantization condition See [Cremades, Ibañez, Marcheano (0404229)]:

$$\boxed{M=\frac{qB}{2\pi}}$$

- For $M\in\mathbb{Z}^+,\psi^{j,M}=f(z)\xrightarrow{holomorphic}$ Left handed particles (no right handed particles)
- For $M\in\mathbb{Z}^-,\psi^{j,M}=f(\bar{z})\xrightarrow{anti-holomorphic}$ right handed particles (no left handed particles)

Yukawa Couplings

The introduction of the magnetic flux in the compact dimension:

$$F_{z\bar{z}} = \frac{\pi i}{Im\tau} = \begin{pmatrix} \frac{m_a}{N_a}\mathbb{I}_{N_a x N_a} & 0 & 0 \\ 0 & \frac{m_b}{N_b}\mathbb{I}_{N_b x N_b} & 0 \\ 0 & 0 & \frac{m_c}{N_c}\mathbb{I}_{N_c x N_c} \end{pmatrix}$$

breaks our gauge groups as following:

$$U(N) \to U(N_a) x U(N_b) x U(N_c)$$

$$\text{where } s_\alpha = \frac{m_\alpha}{N_\alpha} \in \mathbb{Z};\ \alpha \in \{a, b, c\}$$

Yukawa Couplings

The Yukawa coupling is given by:

$$\boxed{Y_{ijk}(\tilde{\zeta},\tau) = -g\int_{\mathbb{T}^2} d^2z\,\psi^{i,\mathcal{I}_{ab}}(z,\tau,\zeta_{ab})\psi^{j,\mathcal{I}_{ca}}(z,\tau,\zeta_{ca})(\psi^{k,\mathcal{I}_{bc}}(z,\tau,\zeta_{bc}))^*}$$

- $\psi^{i,\mathcal{I}_{ab}}$ are the chiral fermions bifundementals transforming as $(N_a,\bar{N}_b)$ under $U(N_a)xU(N_b)$
- $\mathcal{I}_{\alpha\beta} =: s_\alpha - s_\beta$ which implies that $\mathcal{I}_{ab}+\mathcal{I}_{ca}+\mathcal{I}_{bc}=0$
- Wilson lines: $\zeta_{\alpha\beta} = \frac{s_\alpha\zeta_\alpha - s_\beta\zeta_\beta}{\mathcal{I}_{\alpha\beta}}$

Yukawa Couplings

The key to simplify the Yukawa is the following:

- Products of θ–functions can be expanded in terms of θ–functions.
- The θ–functions fulfill certain orthogonality and completeness relations.

$$\Longrightarrow \mathcal{I}_{ab} m = k - i - j \mod \mathcal{I}_{bc}$$

$$\Longrightarrow \text{ We need } \mathcal{I}_{ab}^{-1} \mod \mathcal{I}_{bc}$$

Yukawa Couplings

In order to solve this modular linear equation for m we use

- Euler's Theorem: $\underbrace{a^{\phi(n)} = 1 \mod n}_{n \text{ and } a \text{ are coprime}}$,

  where $\phi(n)$ is the Euler's function given by

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

  for all p prime numbers that divide n.

$$\implies a^{-1} = a^{\phi(n)-1} \mod n$$

- Solution to modular equation:

$$m = m_0 - \left|\frac{\mathcal{I}_{bc}}{d}\right| t \quad \text{for} \quad t = 0, 1, \cdots (d-1)$$

- Then the solution:

$$m_0 = \left(\frac{\mathcal{I}_{ab}}{d}\right)^{\phi(\left|\frac{\mathcal{I}_{bc}}{d}\right|)} \left(\frac{k-i-j}{d}\right) \mod \left|\frac{\mathcal{I}_{bc}}{d}\right|$$

Yukawa Couplings

These give us:

$$\boxed{Y_{ijk}(\tilde{\zeta},\tau) = A_{abc}\,\theta\begin{bmatrix}\frac{\hat{\alpha}_{ijk}}{\lambda}\\ 0\end{bmatrix}(\frac{\tilde{\zeta}}{d},\lambda\tau)},$$

if $i+j-k=0 \mod d$ and

- $\lambda = \text{l.c.m}(\#\text{of flavors}) = \text{l.c.m}(\mathcal{I}_{ab},\mathcal{I}_{cb},\mathcal{I}_{ca})$
- $d = \text{g.c.d}(\#\text{of flavors}) = \text{g.c.d}(\mathcal{I}_{ab},\mathcal{I}_{cb},\mathcal{I}_{ca})$
- $\hat{\alpha}_{ijk} = \frac{\mathcal{I}'_{ca}i-\mathcal{I}'_{ab}j+\mathcal{I}'_{ca}(\mathcal{I}'_{ab})^{\phi(|\mathcal{I}'_{bc}|)}(k-i-j)}{\lambda}$
- $\mathcal{I}'_{\alpha\beta} = \frac{\mathcal{I}_{\alpha\beta}}{\lambda}$
- $\tilde{\zeta} = -\mathcal{I}_{ab}\mathcal{I}_{ca}(\zeta_{ca}-\zeta_{ab})$

Since the Yukawa can be written as a single theta function, it transforms similarly to the wavefunctions we introduced earlier!

Yukawa Couplings

$$\boxed{Y_{ijk}(\tilde{\zeta},\tau) = A_{abc}\theta \begin{bmatrix} \frac{\hat{\alpha}_{ijk}}{\lambda} \\ 0 \end{bmatrix} (\frac{\tilde{\zeta}}{d},\lambda\tau)}$$

Theta functions have more identities which reduce the number of independent Yukawas as following:

- $\theta \begin{bmatrix} a+1 \\ b \end{bmatrix} (z,\tau) = \theta \begin{bmatrix} a \\ b \end{bmatrix} (z,\tau) \rightarrow$ at most λ independent Yukawas
- $\theta \begin{bmatrix} a \\ b \end{bmatrix} (z,\tau) = \theta \begin{bmatrix} -a \\ b \end{bmatrix} (z,\tau) \rightarrow$ at most $\frac{\lambda}{2}+1$ independent Yukawas

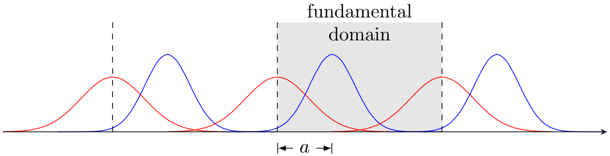
Geometrical interpretation.

The Yukawas go like

$$\mathcal{Y}_{\hat{\beta}} \propto (\mathrm{Im}\tau)^{-\frac{1}{4}} \vartheta \begin{bmatrix} \frac{\beta}{\lambda} \\ 0 \end{bmatrix} (0, \lambda\tau) = (\mathrm{Im}\tau)^{-\frac{1}{4}} \sum_{l=-\infty}^{\infty} e^{-\pi\lambda \left(\underbrace{\mathrm{Im}\tau}_{\text{exp suppression}} - \mathrm{iRe}\tau \right) \left(\frac{\beta}{\lambda} + l \right)^2} .$$

On the other hand, the overlap of two gaussians is

$$\int_{-\infty}^{\infty} dx \frac{e^{-x^2/b_1}}{\sqrt{\pi b_1}} \frac{e^{-(x-a)^2/b_2}}{\sqrt{\pi b_2}} = \frac{e^{-a^2/(b_1+b_2)}}{\sqrt{\pi}\sqrt{b_1 + b_2}}$$



Geometrical Interpretation.

In our scenario $a = \min\left\{\left|\frac{\hat{\beta}}{\lambda}\right|, \left|1 - \frac{\hat{\beta}}{\lambda}\right|\right\}$

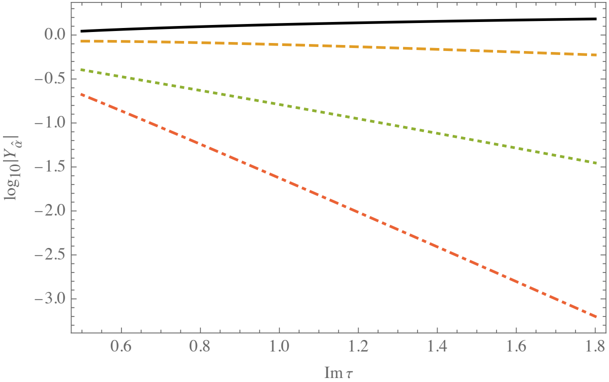


Figure: Dependence of the magnitude of the Yukawa couplings $Y_{\hat{\alpha}}$ for Re$\tau = 0.1$. The black solid, orange dashed, green dotted and red dash–dotted curves represent $\hat{\alpha} = 0, 1, 2$ and 3, respectively.

Boundary Conditions

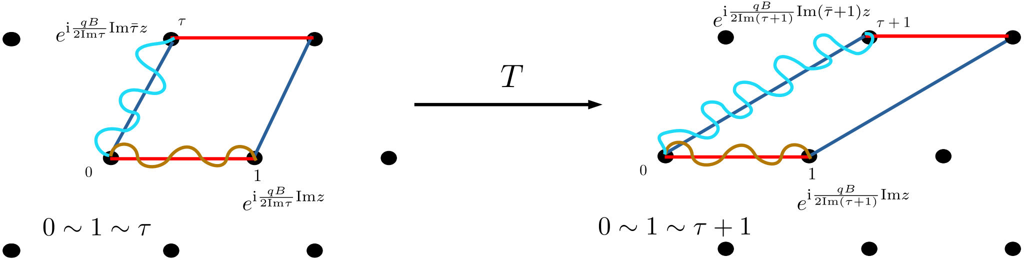


Figure: (Left) Boundary conditions for τ. (Right) Boundary conditions for $\tau + 1$.

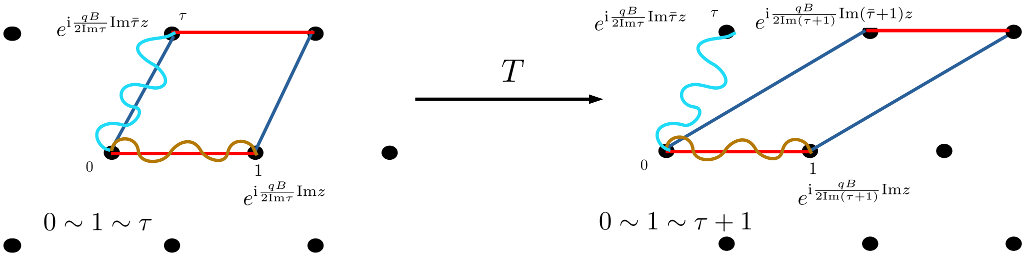
Are boundary conditions still satisfied for modular transformed wavefunctions? **They are!**

Boundary Conditions

Do modular transformed wavefunctions

$$\psi(z,\tau,\underbrace{0}_{\text{Assume }\zeta=0}) \xrightarrow{S} \psi\left(-\frac{z}{\tau}, -\frac{1}{\tau}, 0\right) \quad \text{and} \quad \psi(z,\tau,0) \xrightarrow{T} \psi(z,\tau+1,0)$$

follow the old boundary contitions? **Well, they don't have to**



Boundary Conditions

We have for the new torus

$$\psi^{j,M}(\underbrace{z+\tau}_{\substack{\text{old}\\\text{transformation}}}, \overbrace{\tau+1}^{\text{new torus}}, 0) = \underbrace{e^{-\pi \mathrm{i} M}}_{\substack{\text{new}\\\text{phase}}} e^{\mathrm{i}\frac{\pi M}{\mathrm{Im}\tau}\mathrm{Im}z}\psi^{j,M}(z,\tau+1,0)$$

(a) For even $M \implies e^{-\mathrm{i}\pi M} = 1$. **Possible to find $\rho(T)$!**

(b) For odd $M \implies e^{-\mathrm{i}\pi M} = -1 \implies$ $\boxed{\psi^{j,M}(z,\tau+1,0) \neq \sum_{j=0}^{M-1} \rho(T)_{jj'}\psi^{j,M}(z,\tau,0)}$

$\implies$ Not possible to find $\rho(T)$

Modular Transformations on Torus Wavefunctions.

Recall that

$$\psi^{j,M}(z,\tau,\zeta) = \mathcal{N} e^{i\pi M(z+\zeta)\frac{\mathrm{Im}(z+\zeta)}{\mathrm{Im}\tau}} \vartheta \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (M(z+\zeta), M\tau).$$

The S transformation for $M \in \mathbb{Z}$ is

$$\psi^{j,M}(z,\tau,\overbrace{0}^{\text{Assumption } \zeta=0}) \xrightarrow{S} \psi^{j,M}(-\frac{z}{\tau},-\frac{1}{\tau},0)$$

$$= -\underbrace{\left(-\frac{\tau}{|\tau|}\right)}_{\text{automorphy factor}}^{\overbrace{\frac{1}{2}}^{\text{modular weight}}} \sum_{k=0}^{M-1} \underbrace{-\frac{e^{i\frac{\pi}{4}} e^{\frac{2\pi ijk}{M}}}{\sqrt{M}}}_{\text{rep. matrix}} \psi^{k,M}(z,\tau,0) \tag{2}$$

$$\implies \rho^{\psi}_{M}(\tilde{S}) = -\frac{e^{i\frac{\pi}{4}} e^{\frac{2\pi ijk}{M}}}{\sqrt{M}} \tag{3}$$

Modular Transformations on Torus Wavefunctions.

For the T transformation, $\cdots$ it is not so simple. We have

$$\psi^{j,M}(z,\tau,0) \xrightarrow{T} \psi^{j,M}(z,\tau+1,0) = \mathcal{N} e^{-\mathrm{i}\pi j\left(1-\frac{j}{M}\right)} e^{\mathrm{i}\pi M z \frac{\mathrm{Im} z}{\mathrm{Im}\tau}} \begin{bmatrix} \frac{j}{M} \\ \frac{M}{2} \end{bmatrix} (Mz, M\tau) \tag{4}$$

(a) If M is even, then we can use

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z,\tau) = e^{2\pi\mathrm{i}\alpha\beta} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} (z,\tau), \quad \text{for } \beta \in \mathbb{Z}$$

to get

$$\psi^{j,M}(z,\tau+1,0) = \underbrace{e^{\frac{\mathrm{i}\pi j^2}{M}}}_{\text{rep. matrix}} \psi^{j,M}(z,\tau,0) \tag{5}$$

(b) If M is odd, then $\frac{M}{2} \notin \mathbb{Z}$ and we cannot do the same. It was argued in See [Ohki, Uemura, Watanabe (2003.04174)] that odd M was not possible because of this. What about $M=3$?

Modular Tranformations on Torus Wavefunctions

To fix this, we assume that under T

$$z \xrightarrow{T} z + \underbrace{\Delta z}_{\substack{\text{torus}\\ \text{origin}\\ \text{translation}}} \qquad \text{and} \qquad \tau \xrightarrow{T} \tau + 1$$

$$\implies \psi^{j,M}(z+\Delta z, \tau+1, 0) = \mathcal{N} e^{-\mathrm{i}\pi j\left(1-\frac{j}{M}\right)} e^{\mathrm{i}\pi M \Delta z \frac{\mathrm{Im} z}{\mathrm{Im}\tau}} \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (M(z + \underbrace{\Delta z + \frac{1}{2}}_{N}), M\tau) \tag{6}$$

If $N \in \mathbb{Z}$, we may use $\begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (Mz + N, M\tau) = e^{2\pi \mathrm{i} N\alpha} \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (Mz, M\tau)$

$$\implies \Delta z = \begin{cases} \frac{k}{2} \text{with odd } k, & \text{for odd } M \\ \frac{k}{2} \text{with } k \in \mathbb{Z}, & \text{for even } M \end{cases}$$

Modular Transformations on Torus Wavefunctions.

Then,

$$\psi^{j,M}(z,\tau,0) \xrightarrow{T} \psi^{j,M}(z,\tau+1,0) = \underbrace{e^{\mathrm{i}\pi M \frac{\mathrm{Im} z}{\mathrm{Im} \tau}}}_{\text{ugly phase}} \underbrace{e^{\mathrm{i}\pi j\left(\frac{j}{M}+1\right)}}_{\text{rep. matrix}} \psi^{j,M}(z - \underbrace{\frac{1}{2}}_{\text{translation}}, \tau, 0),$$

where we have chosen $\Delta z = \frac{1}{2}$ for $M \in \mathbb{Z}$.

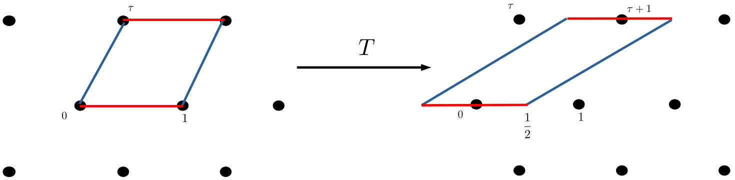


Figure: T shifts $\tau \to \tau + 1$ and translates torus origin by $\frac{1}{2}$.

Modular transformations of the wavefunctions.

What about the ugly phase? It doesn't affect the 4D physics! Since the Yukawas transform as

$$Y_{ijk} \xrightarrow{T} e^{\frac{\mathrm{i}\pi \mathrm{Im} z}{2 \mathrm{Im}\tau}\left(\underbrace{\mathcal{I}_{ab}+\mathcal{I}_{bc}+\mathcal{I}_{ca}}_{0}\right)} [\rho(T)]_{ii'} [\rho(T)]_{jj'} [\rho(T)]_{kk'} Y_{i'j'k'}$$

For S, the transformation was already known, but not for T and odd M [Ohki, Uemura, Watanabe (2003.04174)]

$$\psi^{j,M}(z,\tau,0) \xrightarrow{S} (-\tau)^{1/2} \rho(S) \psi(z,\tau,0)$$

Modular transformations of the wavefunctions

For half-integer weight we must consider **the metaplectic group** which is generated by $\tilde{S} = (S, -\sqrt{\tau})$ and $\tilde{T} = (T, +1)$. Here, the modular forms of weight $\frac{k}{2}$ and level $4N$ transform as

$$f_{\hat{\alpha}}(\tau) \xrightarrow{\tilde{\gamma}} f_{\hat{\alpha}}(\tilde{\gamma}) = \phi(\gamma, \tau)^k \rho_r(\tilde{\gamma})_{\hat{\alpha}\hat{\beta}} f_{\hat{\beta}}(\tau)$$

$\rho(\tilde{\gamma})_r$ are irreps. of $\tilde{\Gamma}_{4N}$ which satisfy the relations (8) and more relations for $N > 1$.
The generators satisfy

$$\tilde{S}^8 = (\tilde{S}\tilde{T})^3 = 1 \quad \tilde{S}^2\tilde{T} = \tilde{T}\tilde{S}^2 \quad \tilde{T}^{4N} = 1.$$

And for finiteness we choose the relations

$$\tilde{S}^5\tilde{T}^6\tilde{S}\tilde{T}^4\tilde{S}\tilde{T}^2\tilde{S}\tilde{T}^4 = 1 \text{ for } N = 2$$
$$\tilde{S}\tilde{T}^3\tilde{S}\tilde{T}^{-2}\tilde{S}^{-1}\tilde{T}\tilde{S}\tilde{T}^{-3}\tilde{S}^{-1}\tilde{T}^2\tilde{S}^{-1}\tilde{T}^{-1} = 1$$

Modular transformations of the $4D$ theory.

The superpotential of the 4D theory will be

$$W \supset Y_{ijk}\phi^{i,\mathcal{I}_{ab}}\phi^{j,\mathcal{I}_{ca}}\phi^{k,\mathcal{I}_{bc}}.$$

- For Y_{ijk} the modular transformations are derived $\rightarrow$ They transform as a $\lambda-$plet.
- For $\phi^{i,M}$ the modular transformations are proposed$\rightarrow$ They transform as a $M-$plet.

The modular weights are given by

Field	$\psi^{j,M}$	$\phi^{j,M}$	Y_{ijk}	W
Modular weight	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1

The Kähler potential (in the absence of Wilson lines) is given by

$$K = -\ln(\underbrace{S}_{\substack{\text{Axio}\\ \text{Dilaton}}}+\bar{S}) - \ln(\underbrace{T}_{\substack{\text{Kähler}\\ \text{Modulus}}}+\bar{T}) - \ln(\underbrace{U}_{\substack{\text{Complex}\\ \text{Structure}\\ \text{Moulus}}}+\bar{U})$$

$$\text{Re}S \propto \frac{1}{g^2} \quad \text{Re}T \propto \mathcal{A} \quad \text{Re}U = \text{Im}\tau$$

Modular Transformations of the 4D theory

How can we construct a modular symmetric theory? First, we will distinguish two Yukawas

$$\underbrace{Y_{ijk}(\tau)}_{\substack{\text{Overlap integral}\\ \text{Non-holomorphic}\\ \text{Normalized and physical}}} = \mathcal{N}_{abc} \underbrace{\mathcal{Y}_{ijk}(\tau)}_{\substack{\text{Holomorphic}\\ \text{unnormalized}\\ \text{non-physical}}}$$

$$\implies \mathcal{Y}_{ijk}(\tau) = \vartheta\begin{bmatrix}\hat{\alpha}_{ijk}\\ 0\end{bmatrix}(0, \lambda\tau) \quad \text{with} \quad \hat{\alpha}_{ijk} = \mathcal{I}'_{ca} i - \mathcal{I}'_{ab} j + \mathcal{I}_{ca} (\mathcal{I}'_{ab})^{\phi(\mathcal{I}'_{cb})} (k - i - j)$$

The superpotential is then

$$W \supset \mathcal{Y}_{ijk} \phi^{i,\mathcal{I}_{ab}} \phi^{j,\mathcal{I}_{ca}} \phi^{k,\mathcal{I}_{cb}}$$

(a) For $\mathcal{Y}_{ijk}$ we **derive** the modular transformations.

(b) For $\phi^{i,\mathcal{I}_{ab}}$ we **propose** them.

Modular Transformations of the 4D theory.

(a) For $\mathcal{Y}_{ijk}$ we arrange them in a $\lambda-$plet $\mathcal{Y}_\lambda(\tau)$.

$$\mathcal{Y}_\alpha(\tau) \xrightarrow{\tilde{\gamma}} \mathcal{Y}_\alpha(\tilde{\gamma}\tau) = \pm(c\tau+d)^{\overbrace{1/2}^{\text{modular weight}}} \rho_\lambda(\gamma)_{\hat{\alpha}\hat{\beta}}\mathcal{Y}_{\hat{\beta}}(\tau)$$

with

$$\rho_\lambda(\tilde{S})_{\hat{\alpha}\hat{\beta}} = -\frac{e^{\mathrm{i}\frac{\pi}{4}}\, e^{2\pi\mathrm{i}\frac{\hat{\alpha}\hat{\beta}}{\lambda}}}{\sqrt{\lambda}} \quad \text{and} \quad \rho(\tilde{T})_{\hat{\alpha}\hat{\beta}} = e^{\frac{\mathrm{i}\pi\hat{\alpha}^2}{\lambda}}\delta_{\hat{\alpha}\hat{\beta}}$$

(b) For $\phi^{i,\mathcal{I}_{ab}}$ we assume

$$\phi^{j,M} \xrightarrow{\tilde{\gamma}} \pm(c\tau+d)^{\overbrace{-1/2}^{\text{modular weight}}} \underbrace{\rho^{\phi}_M(\tilde{\gamma})^{-1}_{jk}}_{\text{rep. matrices}}\phi^{k,M}$$

Modular Transformations of the 4D theory.

Thus, we look for $\rho_M^\phi(\tilde{\gamma})^{-1}$ such that

$$\boxed{\mathcal{Y}_{ijk}(\tilde{\gamma})\left[\rho^*_{\mathcal{I}_{ab}}(\tilde{\gamma})\right]^{-1}_{ii'}\phi^{i'\mathcal{I}_{ab}}\left[\rho^*_{\mathcal{I}_{ca}}(\tilde{\gamma})\right]^{-1}_{jj'}\phi^{j'\mathcal{I}_{ca}}\left[\rho^*_{\bar{\mathcal{I}}_{cb}}(\tilde{\gamma})\right]^{-1}_{kk'}\phi^{k'\mathcal{I}_{cb}}\stackrel{!}{=}\mathcal{Y}_{ijk}(\tau)\phi^{i,\mathcal{I}_{ab}}\phi^{j,\mathcal{I}_{ca}}\phi^{k,\mathcal{I}_{cb}}} \tag{7}$$

One way to solve this is using the fact that

$$\mathcal{Y}_{ijk}\xrightarrow{\tilde{\gamma}}\left[\rho^\psi_{\mathcal{I}_{ab}}(\tilde{\gamma})\right]_{ii'}\left[\rho^\psi_{\mathcal{I}_{ca}}(\tilde{\gamma})\right]_{jj'}\left[\rho^\psi_{\mathcal{I}_{cb}}(\tilde{\gamma})\right]^*_{kk'}\mathcal{Y}_{i'j'k'} \tag{8}$$

to take

$$\rho^\phi_{\mathcal{I}_{ab}}(\tilde{\gamma})=\rho^\psi_{\mathcal{I}_{ab}}(\tilde{\gamma}), \tag{9}$$

which leaves W invariant.

Modular Transformations of the 4D theory.

However, this choice do not follow the $\tilde{\Gamma}_N$ conditions. We may use a $U(1)$ symmetry of W given by $\phi^{j,\mathcal{I}_{ab}} \xrightarrow{U(1)} e^{\mathrm{i}q\theta\mathcal{I}_{ab}}\phi^{j,\mathcal{I}_{ab}}$ with charges

	$\phi^{\mathcal{I}_{ab}}$	$\phi^{\mathcal{I}_{ca}}$	$\phi^{\mathcal{I}_{cb}}$
q	$+1$	$+1$	-1

Choosing $\theta = \frac{3}{4}$ we define

$$\rho_M^{\phi}(\tilde{S})_{jk} = \underbrace{e^{\mathrm{i}\pi\frac{3M}{4}}}_{U(1)\ \text{phase}}\ \rho_M^{\psi}(\tilde{S})_{jk} = -\frac{e^{\mathrm{i}\pi\frac{(3M+1)}{4}}\,e^{\frac{2\pi\mathrm{i}jk}{M}}}{\sqrt{M}}$$

$$\rho_M^{\phi}(\tilde{T})_{jk} = \rho_M^{\psi}(\tilde{T})_{jk} = e^{\mathrm{i}\pi j\left(\frac{j}{M}+1\right)}\delta_{jk}, \qquad (10)$$

which fulfill the $\tilde{\Gamma}_{2\lambda}$ conditions. Thus, we conjecture

Magnetized torus with $\lambda = $ l.c.m(#of flavors) exhibit a $\tilde{\Gamma}_{2\lambda}$ modular flavor symmetry.

Model 1: $\mathcal{I}_{ab} = \mathcal{I}_{ca} = 1$ and $\mathcal{I}_{bc} = -2$.

(a) **Gauge breaking:** $F = \frac{\pi i}{\text{Im}\tau}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $U(3) \xrightarrow{U} (1)_a \times U(1)_b \times U(1)_c$

(b) **Yukawas**: $\mathcal{Y}_{ijk}(\tau) = \vartheta\begin{bmatrix} \frac{k}{2} \\ 0 \end{bmatrix}(0, 2\tau)$ is a doublet $\hat{2}$ which transforms as

$$\mathcal{Y}_{\hat{2}} = \begin{bmatrix} \mathcal{Y}_0 \\ \mathcal{Y}_1 \end{bmatrix} := \begin{bmatrix} \mathcal{Y}_{000} \\ \mathcal{Y}_{001} \end{bmatrix} \tag{11}$$

(c) **Matter fields**:

	$\phi^{\mathcal{I}_{ab}}$	$\phi^{\mathcal{I}_{ca}}$	$\phi^{\mathcal{I}_{cb}}$
$\tilde{\Gamma}_4$	$\mathbf{1}$	$\mathbf{1}$	$\bar{\hat{\mathbf{2}}}$

(d) **Superpotential**: $W \supset \phi_{ab}\phi_{ca}\left(\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, 2\tau)\phi^0_{cb} + \vartheta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(0, 2\tau)\right)\phi^1_{cb}$

(e) **Contact with bottom-up:** $\tilde{\Gamma}_4$ [X.-G. Liu, C.-Y. Yao, B.-Y. Qu, and G.-J. Ding(2007.13706)].

Model 2: $\mathcal{I}_{ab} = \mathcal{I}_{ca} = 3$ and $\mathcal{I}_{bc} = -6$.

(a) **Gauge breaking:** $F = \frac{\pi \mathrm{i}}{\mathrm{Im}\tau} \begin{pmatrix} \mathbb{I}_{2\times 2} & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ with

$$U(3) \xrightarrow{S} U(2) \times U(1)_a \times U(1)_b \times U(1)_c$$

(b) **Matter fields**:

Field	Quantum number	# of copies	$\tilde{\Gamma}_{12}$
$\phi^{\mathcal{I}_{ab}} = L$	$\mathbf{2}_{(1,-1,0)}$	$\mathcal{I}_{ab} = 3$	$\mathbf{3} = \mathbf{2}'' + \mathbf{1}'$
$\phi^{\mathcal{I}_{cb}} = R$	$\mathbf{1}_{(0,1,-1)}$	$\mathcal{I}_{bc} = -6$	$\mathbf{6}' = \mathbf{4}' + \mathbf{2}'$
$\phi^{\mathcal{I}_{ca}} = H$	$\mathbf{2}_{(-1,0,1)}$	$\mathcal{I}_{ca} = 3$	$\mathbf{3} = \mathbf{2}'' + \mathbf{1}'$

Model 2: $\mathcal{I}_{ab} = \mathcal{I}_{ca} = 3$ and $\mathcal{I}_{bc} = -6$.

(a) **Superpotential**: $W \supset \mathcal{Y}_{ijk} L^i H^j R^k$

(b) **Yukawas**: Although $\lambda = 6$, the number of independent yukawas is $\frac{\lambda}{2} + 1 = 4$ and satisfy

$$Y_0 := Y_{i=j,j,k=2j} \ , \qquad Y_1 := Y_{i=j+1,j,k=2j+1} = Y_5 := Y_{i=j+2,j,k=2j+5} \ , \tag{12a}$$

$$Y_3 := Y_{i=j,j,k=2j+3} \ , \qquad Y_2 := Y_{i=j+2,j,k=2j+2} = Y_4 := Y_{i=j+1,j,k=2j+4} \ , \tag{12b}$$

We form an irreducible 4−plet ($\mathbf{6} = \mathbf{4} + \mathbf{2}$) with the projection matrix

$$P_{6\to4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \tag{13}$$

with $\mathcal{Y}_4 = P_{6\to4}\mathcal{Y}_6$ and $\rho_4(\tilde{\gamma}) = P^T_{6\to4}\rho_6(\tilde{\gamma})P_{6\to4}$ where

$$\mathcal{Y}_6 = (\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_2, \mathcal{Y}_1)^T \tag{14}$$

Model 3: $\mathcal{I}_{ab} = \mathcal{I}_{ca} = 2$ and $\mathcal{I}_{bc} = -4$.

(a) **Gauge breaking:** $F = \frac{\pi \mathrm{i}}{\mathrm{Im}\tau} \begin{pmatrix} \mathbb{I}_{2\times 2} & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(b) **Matter fields**:

Field	# of copies	$\tilde{\Gamma}_8$
$\phi^{\mathcal{I}_{ab}}$	$\mathcal{I}_{ab} = 2$	$\mathbf{2}$ (irreducible)
$\phi^{\mathcal{I}_{cb}}$	$\mathcal{I}_{bc} = -4$	$\mathbf{4}' = \mathbf{3}' + \mathbf{1}''$
$\phi^{\mathcal{I}_{ca}}$	$\mathcal{I}_{ca} = 2$	$\mathbf{2}$ (irreducible)

Model 3: $\mathcal{I}_{ab} = \mathcal{I}_{ca} = 2$ and $\mathcal{I}_{bc} = -4$.

(a) **Superpotential**: $W \supset \mathcal{Y}_{ijk} \phi^{i,\mathcal{I}_{ab}} \phi^{j,\mathcal{I}_{ca}} \phi^{k,\mathcal{I}_{bc}}$

(b) **Yukawas**: Although $\lambda = 4$, the number of independent yukawas is $\frac{\lambda}{2} + 1 = 3$ and satisfy

$$Y_0 := Y_{i=j,j,k=2j} \ , \qquad Y_1 := Y_{i=j+1,j,k=2j+1} = Y_3 := Y_{i=j+1,j,k=2j+3} \ , \tag{15a}$$

$$Y_2 := Y_{i=j,j,k=2j+2} \ , \tag{15b}$$

We form an irreducible 3−plet ($\mathbf{4} = \mathbf{3} + \mathbf{1}'$) with the projection matrix

$$P_{4\to 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{16}$$

with $\mathcal{Y}_3 = P_{4\to 3}\mathcal{Y}_4$ and $\rho_3(\tilde{\gamma}) = P^T_{4\to 3}\rho_4(\tilde{\gamma})P_{4\to 3}$ where

$$\mathcal{Y}_6 = (\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3)^T \tag{17}$$

Model 4: $\mathcal{I}_{ab} = 1$, $\mathcal{I}_{ca} = 2$ and $\mathcal{I}_{bc} = -3$.

(a) **Gauge breaking:** $F = \frac{\pi \mathrm{i}}{\mathrm{Im}\tau} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $U(3) \to U(1)_a \times U(1)_b \times U(1)_c$

(b) **Matter fields**:

Field	# of copies	$\tilde{\Gamma}_{12}$
$\phi^{\mathcal{I}_{ab}}$	$\mathcal{I}_{ab} = 1$	**1**
$\phi^{\mathcal{I}_{cb}}$	$\mathcal{I}_{bc} = -3$	**2**
$\phi^{\mathcal{I}_{ca}}$	$\mathcal{I}_{ca} = 2$	**3**

Model 4: $\mathcal{I}_{ab} = 1$, $\mathcal{I}_{ca} = 2$ and $\mathcal{I}_{bc} = -3$.

(a) **Superpotential**: $W \supset \mathcal{Y}_{ijk} \phi^{i,\mathcal{I}_{ab}} \phi^{j,\mathcal{I}_{ca}} \phi^{k,\mathcal{I}_{bc}}$

(b) **Yukawas** Although $\lambda = 6$, the number of independent yukawas is $\frac{\lambda}{2} + 1 = 4$ and satisfy

$$Y_0 := Y_{000} \ , \qquad Y_1 := Y_{011} = Y_5 := Y_{012} \ , \tag{18a}$$

$$Y_3 := Y_{010} \ , \qquad Y_2 := Y_{001} = Y_4 := Y_{002} \ , \tag{18b}$$

We form an irreducible 4−plet ($\mathbf{6} = \mathbf{4} + \mathbf{2}$) with the projection matrix

$$P_{6\to4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \tag{19}$$

with $\mathcal{Y}_4 = P_{6\to4}\mathcal{Y}_6$ and $\rho_4(\tilde{\gamma}) = P^T_{6\to4}\rho_6(\tilde{\gamma})P_{6\to4}$ where

$$\mathcal{Y}_6 = (\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_2, \mathcal{Y}_1)^T \tag{20}$$

Kähler parameters loss of predictivity

In [ArXiv: F. Feruglio(1706.08749)]the Káhler potential was

$$K = \alpha_0(-\mathrm{i}\tau + \mathrm{i}\bar{\tau})^{-1}(\bar{L}L)_{\mathbf{1}}.$$

However, there are actually more terms [M. C. Chen, S. Ramos-Sánchez and M. Ratz (1909.06910)]

$$\Delta K = \sum_{k=1}^{7} \alpha_k(-\mathrm{i}\tau + \mathrm{i}\bar{\tau})^{-1}(YL\bar{Y}\bar{L})_{\mathbf{1},k}.$$

Then, the metric needs to be diagonalized

$$K_L = U_L^\dagger D^2 U_L.$$

Then, to extract the MNS matrix we now have

$$U_\nu^T D^{-1} U_L^* m_\nu U_L^\dagger D^{-1} \hat{U}_\nu = \mathrm{diag}(m_1, m_2, m_3)$$
$$U_e^\dagger D^{-1} U_L^* Y_e Y_e^\dagger U_L^T D^{-1} \hat{U}_e = \mathrm{diag}(y_e^2, y_\mu^2, y_\tau^2).$$

Then, if D is proportional to the matrix, there is no change in the MNS parametes, otherwise there is.