

Muonium-Antimuonium Oscillations in Effective Field Theory

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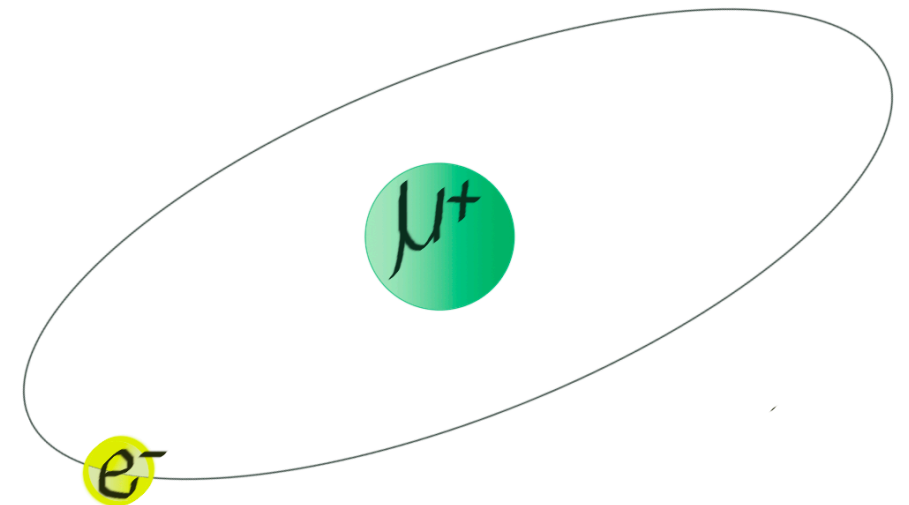
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Outline

- * Motivation
- * Muonium Oscillation Formalism
- * Calculation of mixing parameters Δm and $\Delta \Gamma$
- * Results
- * Summary

Muonium

- Muonium is a non-relativistic QED Coulombic bound state
 - ▶ Comparable to B_c , but with computable non-perturbative matrix elements
- Simpler than atomic hydrogen
 - ▶ Spin-0 (singlet): para-muonium
 - ▶ Spin-1 (triplet): ortho-muonium
 - ▶ Electrically neutral \Rightarrow flavor oscillations are possible
 - ▶ Ideal system for NP searches
- An oscillation is the process $M_\mu(\mu^+e^-) \rightarrow \bar{M}_\mu(\mu^-e^+)$
 - ▶ It violates muon lepton number by two units $\Delta L_\mu = 2$, i.e.: it can probe different types of NP than $\mu \rightarrow e\gamma$ or $\mu + N \rightarrow e + N$
 - ▶ Conversion rate was calculated in several NP models with heavy DOF

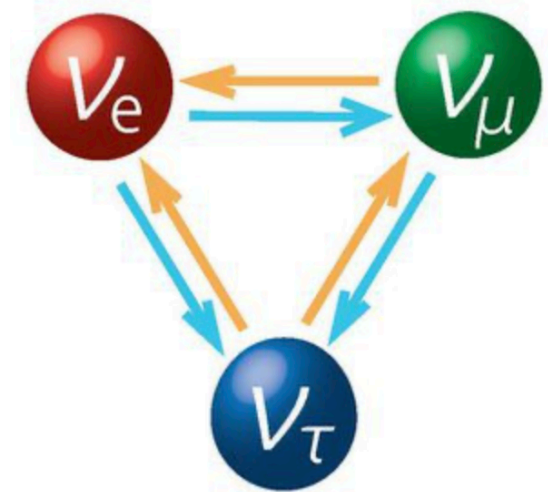


Muonium (μ^+e^-)



New Physics Searches with Muonium

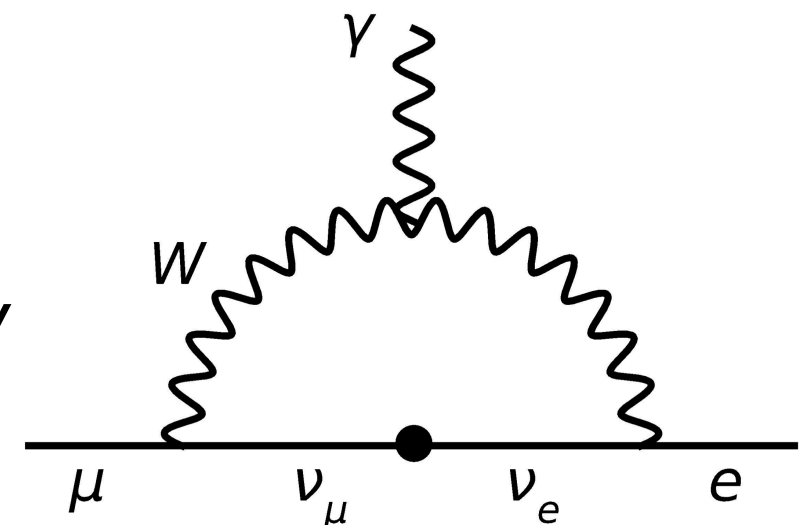
- From experiment neutrino's mix and change flavor
 - Flavor violating processes in the charged lepton sector
 - Highly suppressed in the standard model (SM)
 - $\mu \rightarrow e\gamma$ $\text{Br}(\mu \rightarrow e\gamma)_{SM} \sim 10^{-54}$
 \Rightarrow **no background for NP!**



<https://j-parc.jp/Neutrino/en/intro-t2kexp.html>

- LFV searched for in various processes:

- $\Delta L_\mu = 1$: $\mu \rightarrow e\gamma$, $\mu \rightarrow eee$ and $\mu + N \rightarrow e + N$
- $\Delta L_\mu = 2$: **Muonium anti-muonium oscillations**
 - In some models we can expect $\Delta L_\mu = 2$ contributions to be the dominant ones, e.g. doubly charged Higgs has a tree level contribution to $\Delta L_\mu = 2$, but not to $\mu \rightarrow e\gamma$



Abdallah, W. 2108 arXiv:1105.1047 [hep-ph]



Muonium Oscillation Formalism

- Similar to meson-antimeson oscillations, but unlike $K\bar{K}$ or $B\bar{B}$ oscillations both spin-0 and spin-1 states can oscillate
- Time development of M_μ and \bar{M}_μ is given by a Schrödinger equation

$$i\frac{d}{dt} \begin{pmatrix} |M_\mu(t)\rangle \\ |\bar{M}_\mu(t)\rangle \end{pmatrix} = \begin{pmatrix} m & -i\frac{\Gamma}{2} \\ & \end{pmatrix} \begin{pmatrix} |M_\mu(t)\rangle \\ |\bar{M}_\mu(t)\rangle \end{pmatrix}$$

matrix Hamiltonian

m and Γ are 2×2 Hermitian matrices:
the mass matrix and the decay matrix

- Assume CPT invariance, then the diagonal and off diagonal elements

$$m_{11} = m_{22}, \quad \Gamma_{11} = \Gamma_{22}$$

$$m_{12} = m_{21}^*, \quad \Gamma_{12} = \Gamma_{21}^*$$



Muonium Oscillation Formalism

- Off diagonal element of this matrix

$$\left(m - \frac{i}{2}\Gamma\right)_{12} = \frac{1}{2M_M} \langle \bar{M}_\mu | \mathcal{H}^{\Delta L_\mu=2} | M_\mu \rangle + \frac{1}{2M_M} \sum_n \frac{\langle \bar{M}_\mu | \mathcal{H}^{\Delta L_\mu=1} | n \rangle \langle n | \mathcal{H}^{\Delta L_\mu=1} | M_\mu \rangle}{M_M - E_n + i\epsilon}$$

- M_μ and \bar{M}_μ are not mass eigenstates: diagonalize! $\left(m - i\frac{\Gamma}{2}\right)_{12} \neq 0$
 - ▶ Mass eigenstates, $M_{\mu 1}$ and $M_{\mu 2}$ (assume CP conservation)

$$|M_{\mu 1,2}\rangle = \frac{1}{\sqrt{2}}(|M_\mu\rangle \mp |\bar{M}_\mu\rangle)$$

- $M_{\mu 1}$ and $M_{\mu 2}$ have mass difference and width difference

- ▶ Mass difference (Δm)

$$\Delta m \equiv M_1 - M_2,$$

- ▶ Width difference ($\Delta\Gamma$)

$$\Delta\Gamma \equiv \Gamma_2 - \Gamma_1$$



Muonium Oscillation Time Evolution

- Time development of M and \bar{M}

$$\left| M_\mu(t) \right\rangle = g_+(t) \left| M_\mu \right\rangle + g_-(t) \left| \bar{M}_\mu \right\rangle \quad \left| \bar{M}_\mu(t) \right\rangle = g_-(t) \left| M_\mu \right\rangle + g_+(t) \left| \bar{M}_\mu \right\rangle$$

Where, $g_\pm(t) = \frac{1}{2} e^{-\Gamma_1 t/2} e^{-iM_1 t} \left[1 \pm e^{-\Delta\Gamma t/2} e^{i\Delta m t} \right]$

- Mass eigenstates $M_{\mu 1}$ and $M_{\mu 2}$ **mass (Δm)**, and **width difference ($\Delta\Gamma$)**

$$\Delta m \equiv M_1 - M_2, \quad \Delta\Gamma \equiv \Gamma_2 - \Gamma_1$$

$$x = \frac{\Delta m}{\Gamma} \quad y = \frac{\Delta\Gamma}{2\Gamma} \quad \text{but } x, y \ll 1$$

- Probability of M_μ decaying as \bar{M}_μ at $t > 0$

- Dependence on x and y !

$$P(M_\mu \rightarrow \bar{M}_\mu) = \frac{\Gamma(M_\mu \rightarrow \bar{f})}{\Gamma(M_\mu \rightarrow f)} = R_M(x, y),$$

$$R_M(x, y) = \frac{1}{2}(x^2 + y^2)$$

$$\Gamma(M_\mu \rightarrow \bar{f})(t) = N_f \left| \langle \bar{f} | S | M_\mu(t) \rangle \right|^2,$$

$$\Gamma(M_\mu \rightarrow f)(t) = N_f \left| \langle f | S | M_\mu(t) \rangle \right|^2$$

Need to calculate x and y !



Effective Theory of Oscillations

- An **effective field theory (EFT)** approach: all possible heavy NP models
- Classify effective operators by lepton quantum numbers

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{\Delta L_\mu=0} + \mathcal{L}_{\text{eff}}^{\Delta L_\mu=1} + \mathcal{L}_{\text{eff}}^{\Delta L_\mu=2}$$

Heavy degrees of freedom are “integrated out” only operators local at the scale of muonium mass are contained in \mathcal{L}_{eff}

- Our $\Delta L_\mu = 2$ Lagrangian

$$\mathcal{L}_{\text{eff}}^{\Delta L_\mu=2} = -\frac{1}{\Lambda_1^2} \sum_i C_i(\mu) Q_i(\mu)$$

- Most general set of operators in terms of the electron and muon degrees of freedom, $Q_1 - Q_5$, relevant for x

$$Q_1 = (\bar{\mu}\gamma_\alpha P_L e) (\bar{\mu}\gamma^\alpha P_L e), \quad Q_2 = (\bar{\mu}\gamma_\alpha P_R e) (\bar{\mu}\gamma^\alpha P_R e), \quad Q_3 = (\bar{\mu}\gamma_\alpha P_L e) (\bar{\mu}\gamma^\alpha P_R e),$$

$$Q_4 = (\bar{\mu} P_R e) (\bar{\mu} P_R e), \quad Q_5 = (\bar{\mu} P_L e) (\bar{\mu} P_L e)$$

► Other possible structures can be Fierz'd into the operators above

- $\Delta L_\mu = 2$ operators that will be important, relevant for y

$$Q_6 = (\bar{\mu}_L \gamma_\alpha e_L) (\bar{\nu}_{\mu L} \gamma^\alpha \nu_{eL}), \quad Q_7 = (\bar{\mu}_R \gamma_\alpha e_R) (\bar{\nu}_{\mu L} \gamma^\alpha \nu_{eL})$$



Matrix Elements of Effective Operators

- Rewrite off diagonal element in terms of x

$$\left(m - \frac{i}{2}\Gamma\right)_{12} = \frac{1}{2M_M} \langle \bar{M}_\mu | \mathcal{H}^{\Delta L_\mu=2} | M_\mu \rangle + \frac{1}{2M_M} \sum_n \frac{\langle \bar{M}_\mu | \mathcal{H}^{\Delta L_\mu=1} | n \rangle \langle n | \mathcal{H}^{\Delta L_\mu=1} | M_\mu \rangle}{M_M - E_n + i\epsilon}$$

$$|\Delta m| = 2 \left| \text{Re } m_{12} \right| \longrightarrow x = \frac{\Delta m}{\Gamma}$$

$$\Rightarrow x = \frac{1}{2M_M\Gamma} \text{Re} \left[2 \langle \bar{M}_\mu | \mathcal{H}_{\text{eff}} | M_\mu \rangle + \left\langle \bar{M}_\mu \left| i \int d^4x \text{T} [\mathcal{H}_{\text{eff}}(x) \mathcal{H}_{\text{eff}}(0)] \right| M_\mu \right\rangle \right]$$

- Dominant contribution

$$\langle \bar{M}_\mu | \mathcal{H}_{\text{eff}} | M_\mu \rangle = \langle \bar{M}_\mu | \mathcal{H}_{\text{eff}}^{\Delta L_\mu=2} | M_\mu \rangle$$

- Suppressed by Λ^2

- Muonium is a non-relativistic Coulombic bound state

$$|M_\mu\rangle = \sqrt{\frac{2M_M}{2m_\mu 2m_e}} \int \frac{d^3p}{(2\pi)^3} \tilde{\varphi}(p) |p, p'\rangle$$

- Perturbative QED bound state: can calculate!



Calculation of Matrix Elements

- Direct computation is equivalent to vacuum saturation (factorization)
- Factorization approach,

$$\langle 0 | \bar{\mu} \gamma^\alpha \gamma^5 e | M_\mu^P \rangle = i f_P p^\alpha, \quad \langle 0 | \bar{\mu} \gamma^\alpha e | M_\mu^V \rangle = f_V M_M \epsilon^\alpha(p),$$

$$\langle 0 | \bar{\mu} \sigma^{\alpha\beta} e | M_\mu^V \rangle = i f_T (\epsilon^\alpha p^\beta - \epsilon^\beta p^\alpha)$$

- Where in the non-relativistic limit f_M

$$f_P = f_V = f_T = f_M$$

and

$$f_M^2 = 4 \frac{|\varphi(0)|^2}{M_M}$$

Coulombic bound state wave function

$$\varphi(r) = \frac{1}{\sqrt{\pi a_{M_\mu}^3}} e^{-\frac{r}{a_{M_\mu}}}$$

$$\text{Then, } |\varphi(0)|^2 = \frac{(m_{red}\alpha)^3}{\pi} = \frac{1}{\pi} (m_{red}\alpha)^3$$



Calculation of Matrix Elements Para-muonium and Ortho-muonium

- **Matrix elements** from operators - spin-singlet states

$$\begin{aligned}\langle \bar{M}_\mu^P | Q_1 | M_\mu^P \rangle &= f_M^2 M_M^2, & \langle \bar{M}_\mu^P | Q_2 | M_\mu^P \rangle &= f_M^2 M_M^2, \\ \langle \bar{M}_\mu^P | Q_3 | M_\mu^P \rangle &= -\frac{3}{2} f_M^2 M_M^2, & \langle \bar{M}_\mu^P | Q_4 | M_\mu^P \rangle &= -\frac{1}{4} f_M^2 M_M^2, \\ \langle \bar{M}_\mu^P | Q_5 | M_\mu^P \rangle &= -\frac{1}{4} f_M^2 M_M^2\end{aligned}$$

- x_P for para-muonium state in terms of the Wilson coefficients

$$x_P = \frac{4 (m_{red}\alpha)^3}{\pi\Lambda^2\Gamma} \left[C_1^{\Delta L=2} + C_2^{\Delta L=2} - \frac{3}{2} C_3^{\Delta L=2} - \frac{1}{4} (C_4^{\Delta L=2} + C_5^{\Delta L=2}) \right]$$

- **Matrix elements** from operators - spin-triplet state

$$\begin{aligned}\langle \bar{M}_\mu^V | Q_1 | M_\mu^V \rangle &= -3f_M^2 M_M^2, & \langle \bar{M}_\mu^V | Q_2 | M_\mu^V \rangle &= -3f_M^2 M_M^2 \\ \langle \bar{M}_\mu^V | Q_3 | M_\mu^V \rangle &= -\frac{3}{2} f_M^2 M_M^2, & \langle \bar{M}_\mu^V | Q_4 | M_\mu^V \rangle &= -\frac{3}{4} f_M^2 M_M^2 \\ \langle \bar{M}_\mu^V | Q_5 | M_\mu^V \rangle &= -\frac{3}{4} f_M^2 M_M^2\end{aligned}$$

- x_V for ortho-muonium state in terms of the Wilson coefficients

$$x_V = -\frac{12 (m_{red}\alpha)^3}{\pi\Lambda^2\Gamma} \left[C_1^{\Delta L=2} + C_2^{\Delta L=2} + \frac{1}{2} C_3^{\Delta L=2} + \frac{1}{4} (C_4^{\Delta L=2} + C_5^{\Delta L=2}) \right]$$

Results hold true for any NP model that can be matched onto a set of local $\Delta L_\mu = 2$ interactions

Δm calculated for Q_2
[Feinberg and Weinberg, (1961)]



Width Difference $\Delta\Gamma$

- Calculated Δm from operators that change the lepton quantum flavor number by two units, $\Delta L_\mu = 2$
- Mass eigenstates have width difference ($\Delta\Gamma$)
- Particles oscillate by two insertions of operators that change lepton flavor number by one unit, $\Delta L_\mu = 1$, highly suppressed by Λ^4
- What about oscillations when a SM $\Delta L_\mu = 0$ decay interferes with a LFV $\Delta L_\mu = 2$ process?
- Leading contribution to $\Delta\Gamma$?

$\Delta\Gamma$ calculated for the first time!



Width Difference $\Delta\Gamma$

- Contribution from $\Delta L_\mu = 1$ and $\Delta L_\mu = 2$ operators

- ▶ Two insertions of $\Delta L_\mu = 1$ operators

Remember

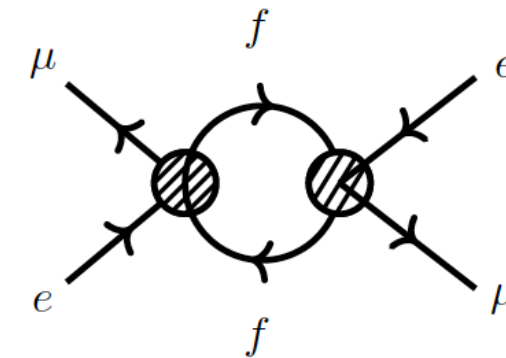
$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{\Delta L_\mu=0} + \mathcal{L}_{\text{eff}}^{\Delta L_\mu=1} + \mathcal{L}_{\text{eff}}^{\Delta L_\mu=2}$$

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{\Delta L_\mu=1} = & -\frac{1}{\Lambda^2} \sum_f \left[\left(C_{VR}^f \bar{\mu} \gamma^\alpha P_{Re} + C_{VL}^f \bar{\mu} \gamma^\alpha P_{Le} \right) \bar{f} \gamma_\alpha f \right. \\ & + \left(C_{AR}^f \bar{\mu} \gamma^\alpha P_{Re} + C_{AL}^f \bar{\mu} \gamma^\alpha P_{Le} \right) \bar{f} \gamma_\alpha \gamma_5 f \\ & + m_e m_f G_F \left(C_{SR}^f \bar{\mu} P_{Le} + C_{SL}^f \bar{\mu} P_{Re} \right) \bar{f} f \\ & + m_e m_f G_F \left(C_{PR}^f \bar{\mu} P_{Le} + C_{PL}^f \bar{\mu} P_{Re} \right) \bar{f} \gamma_5 f \\ & \left. + m_e m_f G_F \left(C_{TR}^f \bar{\mu} \sigma^{\alpha\beta} P_{Le} + C_{TL}^f \bar{\mu} \sigma^{\alpha\beta} P_{Re} \right) \bar{f} \sigma_{\alpha\beta} f + h.c. \right] \end{aligned}$$

$$\left(m - \frac{i}{2} \Gamma \right)_{12} = \frac{1}{2M_M} \langle \bar{M}_\mu | \mathcal{H}^{\Delta L_\mu=2} | M_\mu \rangle + \frac{1}{2M_M} \sum_n \frac{\langle \bar{M}_\mu | \mathcal{H}^{\Delta L_\mu=1} | n \rangle \langle n | \mathcal{H}^{\Delta L_\mu=1} | M_\mu \rangle}{M_M - E_n + i\epsilon}$$

$$|\Delta\Gamma| = 2 |\Gamma_{12}| \longrightarrow y = \frac{\Delta\Gamma}{2\Gamma}$$

$$\implies y = \frac{1}{\Gamma} \sum_n \rho_n \langle \bar{M}_\mu | \mathcal{H}_{\text{eff}} | n \rangle \langle n | \mathcal{H}_{\text{eff}} | M_\mu \rangle$$



$\Delta\Gamma^{ff}$ is generated by on-shell degrees of freedom,
 $f = e, \nu$

Where, $\mathcal{H}_{\text{eff}} = \mathcal{H}_{\text{eff}}^{\Delta L_\mu=1}$

Suppressed by a factor of $\frac{1}{\Lambda^4}$, i.e. $y \ll x$

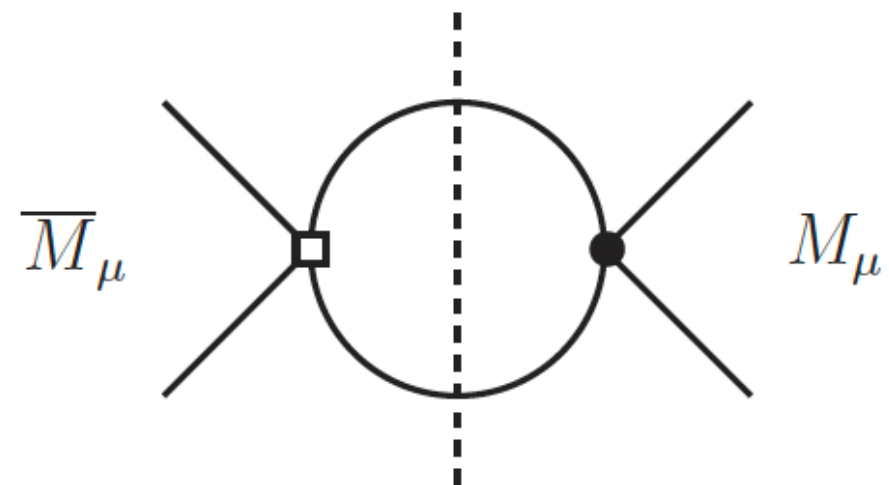


Width Difference $\Delta\Gamma$

- Contribution from $\Delta L_\mu = 2$ operators?
- Standard model $\Delta L_\mu = 0$ decay interferes with $\Delta L_\mu = 2$ process

$$M_\mu \rightarrow \bar{\nu}_\mu \nu_e \rightarrow \bar{M}_\mu$$

- Only suppressed by $\Lambda^2 M_W^2$



Leading contribution to $y!$



Width Difference $\Delta\Gamma$

- Writing y in terms of the correlation function

$$y = \frac{1}{2M_M\Gamma} \text{Im} \left[\left\langle \bar{M}_\mu \left| i \int d^4x \mathcal{T} [\mathcal{H}_{\text{eff}}(x)\mathcal{H}_{\text{eff}}(0)] \right| M_\mu \right\rangle \right]$$

$$= \frac{1}{M_M\Gamma} \text{Im} \left[\left\langle \bar{M}_\mu \left| i \int d^4x \mathcal{T} [\mathcal{H}_{\text{eff}}^{\Delta L_\mu=2}(x)\mathcal{H}_{\text{eff}}^{\Delta L_\mu=0}(0)] \right| M_\mu \right\rangle \right]$$

- Here, $\mathcal{H}_{\text{eff}}^{\Delta L_\mu=0} = -\mathcal{L}_{\text{eff}}^{\Delta L_\mu=0}$ is the ordinary SM Lagrangian

$$\mathcal{L}_{\text{eff}}^{\Delta L_\mu=0} = -\frac{4G_F}{\sqrt{2}} (\bar{\mu}_L \gamma_\alpha e_L) (\bar{\nu}_{eL} \gamma^\alpha \nu_{\mu L})$$

- And $\mathcal{H}_{\text{eff}}^{\Delta L_\mu=2}$ only contributes through the operators Q_6 and Q_7

$$Q_6 = (\bar{\mu}_L \gamma_\alpha e_L) (\bar{\nu}_{\mu L} \gamma^\alpha \nu_{eL}), \quad Q_7 = (\bar{\mu}_R \gamma_\alpha e_R) (\bar{\nu}_{\mu L} \gamma^\alpha \nu_{eL})$$

◆ Para-muonium,

$$y_P = \frac{G_F}{\sqrt{2}\Lambda^2} \frac{M_M^2}{\pi^2\Gamma} (m_{red}\alpha)^3 (C_6^{\Delta L=2} - C_7^{\Delta L=2})$$

◆ Ortho-muonium,

$$y_V = -\frac{G_F}{\sqrt{2}\Lambda^2} \frac{M_M^2}{\pi^2\Gamma} (m_{red}\alpha)^3 (5C_6^{\Delta L=2} + C_7^{\Delta L=2})$$

Leading contributions to muonium lifetime difference, only suppressed by a power of Λ^2



Experimental Constraints

- Have expressions for x and y for both para and ortho-muonium
- Want to place constraints on the BSM scale Λ
- Can do this using experimental constraints on the muonium anti-muonium parameters
- Oscillation probability from an experiment performed at PSI [L. Willmann et. al. (1999)]

$$P \left(M_\mu \rightarrow \bar{M}_\mu \right) \leq 8.3 \times 10^{-11} / S_B (B_0)$$

- $S_B(B_0)$ describes the suppression of the oscillation due to the external magnetic field and depends on the interaction type, or the Lorentz structure of the operators

Interaction type	2.8 μ T	0.1 T	100 T
SS	0.75	0.50	0.50
PP	1.0	0.9	0.50
$(V \pm A) \times (V \pm A)$ or $(S \pm P) \times (S \pm P)$	0.75	0.35	0.0
$(V \pm A) \times (V \mp A)$ or $(S \pm P) \times (S \mp P)$	0.95	0.78	0.67

We will use
 $S_B(B_0) = 2.8\mu\text{T}$



Experimental Constraints - Results

- Oscillation probability is averaged over number of polarization degrees of freedom

$$P \left(M_\mu \rightarrow \bar{M}_\mu \right)_{\text{exp}} = \sum_{i=P,V} \frac{1}{2S_i + 1} P \left(M_\mu^i \rightarrow \bar{M}_\mu^i \right)$$

- Recall, $P \left(M_\mu \rightarrow \bar{M}_\mu \right) = R_M(x, y)$

- Constraints on Λ probed by different $\Delta L_\mu = 2$ operators, we set the corresponding Wilson coefficient $C_i = 1$

Operator	Interaction type	$S_B(B_0)$ (from [9])	Constraints on the scale Λ , TeV
Q_1	$(V - A) \times (V - A)$	0.75	5.4
Q_2	$(V + A) \times (V + A)$	0.75	5.4
Q_3	$(V - A) \times (V + A)$	0.95	5.4
Q_4	$(S + P) \times (S + P)$	0.75	2.7
Q_5	$(S - P) \times (S - P)$	0.75	2.7
Q_6	$(V - A) \times (V - A)$	0.75	0.58×10^{-3}
Q_7	$(V + A) \times (V - A)$	0.95	0.38×10^{-3}



Summary

- * LFV muonium oscillations are a highly effective way to search for NP
- * Set up muonium oscillation formalism
- * Calculated the parameters x and y for spin-0 and spin-1, here y was calculated for the first time.

Only suppressed by $\Lambda^2 M_W^2$!

- * Used expressions for x and y and the current experimental bounds on muonium oscillations to put a constraint on the NP scale Λ



Questions?

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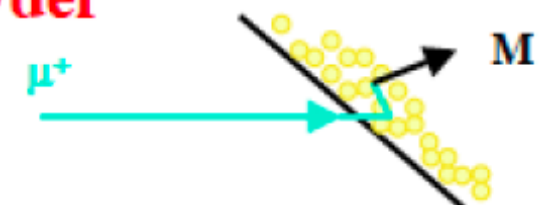
Searches for Muonium Oscillations

- Most recent search for muonium oscillations was at the Paul Scherrer Institute (PSI)

- Probability of $P_{M\bar{M}} \leq 8.2 \times 10^{-11}$ @ 90 % C.L.

- $$P(M \rightarrow \bar{M}) = \int_0^\infty \frac{dt}{\tau} e^{-t/\tau} |\langle \bar{M} | M(t) \rangle|^2$$

• SiO₂ Powder



- Beam of μ^+ at SiO₂ powder target, electron capture to form muonium
- If $M(\mu^+e^-) \rightarrow \bar{M}(\mu^-e^+)$ then an energetic e^- and an e^+ would be detected, with the e^- from the $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$ decay
- Background
 - Bhabha scattering of the e^- and a e^+ from the μ^+ decay
 - $\mu^+ \rightarrow e^- e^+ e^+ \nu_e \bar{\nu}_\mu$



Muonium Oscillation Formalism

- In the Schrödinger picture start with a flavor eigenstate

$$|\psi\rangle = |\psi, t = 0\rangle \quad |\psi, t\rangle = U(t,0) |\psi\rangle = e^{-iHt} |\psi\rangle$$

- We get the Schrödinger equation

$$i \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

- Generalizing to a two state system describing muonium oscillations

$$i \frac{d}{dt} \begin{pmatrix} |M(t)\rangle \\ |\bar{M}(t)\rangle \end{pmatrix} = \begin{pmatrix} m - i\frac{\Gamma}{2} \end{pmatrix} \begin{pmatrix} |M(t)\rangle \\ |\bar{M}(t)\rangle \end{pmatrix}$$

- With matrix Hamiltonian

$$H = \begin{pmatrix} m - i\frac{\Gamma}{2} \end{pmatrix}$$



Calculation of Matrix Elements

Example: $Q_1 = (\bar{\mu}\gamma_\alpha P_L e) (\bar{\mu}\gamma^\alpha P_L e)$

$$\langle \bar{M}_\mu | Q_1 | M_\mu \rangle = 4(\bar{u}\gamma^\alpha P_L v)(\bar{v}\gamma_\alpha P_L u) \left| \int \frac{d^3 p}{(2\pi)^3} \tilde{\psi}(p) \right|^2$$

Where, $\left| \int \frac{d^3 q}{(2\pi)^3} \tilde{\psi}(q) \right|^2 = |\psi(0)|^2$

spatial wavefunction at the origin

non-relativistic spinors

$$u = \sqrt{m_e} \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \quad v = \sqrt{m_e} \begin{pmatrix} \eta \\ -\eta \end{pmatrix},$$

$$\bar{u} = \sqrt{m_e} (\xi^\dagger, \xi^\dagger) \gamma^0, \quad \bar{v} = \sqrt{m_e} (\eta^\dagger, -\eta^\dagger) \gamma^0$$

spinor products: spin-0, spin-1
Spin-1 with 3 possible polarizations

$$\xi \eta^\dagger = \frac{1}{\sqrt{2}} \mathbf{1}_{2 \times 2} \quad \xi \eta^\dagger = \frac{1}{\sqrt{2}} \vec{e}^* \cdot \vec{\sigma}$$

$$\langle \bar{M}_\mu | Q_1 | M_\mu \rangle_{spin=0} = 2 |\psi(0)|^2$$

$$\langle \bar{M}_\mu | Q_1 | M_\mu \rangle_{spin=1} = -6 |\psi(0)|^2$$



Matrix Element - Vacuum Insertions

- Operator Q_1 as example

$$Q_1 = (\bar{\mu}\gamma_\alpha P_L e) (\bar{\mu}\gamma^\alpha P_L e)$$

- Matrix element

$$\langle \bar{M}_\mu | -\mathcal{L}_{\text{eff}}^{\Delta L_\mu=2} | M_\mu \rangle = \langle \bar{M}_\mu | \frac{1}{\Lambda_1^2} \sum_i C_i(\mu) Q_i(\mu) | M_\mu \rangle$$

$$\begin{aligned} \langle \bar{M}_\mu | Q_1 | M_\mu \rangle &= \langle \bar{M}_\mu | (\bar{\mu}\gamma_\alpha P_L e) (\bar{\mu}\gamma^\alpha P_L e) | M_\mu \rangle \\ &\simeq \langle \bar{M}_\mu | (\bar{\mu}\gamma_\alpha P_L e) | 0 \rangle \underbrace{\langle 0 | (\bar{\mu}\gamma^\alpha P_L e) | M_\mu \rangle} \end{aligned}$$

Projection operators

$$P_{R,L} = \frac{1}{2} (1 \pm \gamma^5)$$

$$\langle 0 | (\bar{\mu}\gamma^\alpha P_L e) | M_\mu \rangle = \frac{1}{2} \langle 0 | (\bar{\mu}\gamma^\alpha (1 - \gamma^5) e) | M_\mu \rangle$$

$$= \frac{1}{2} \langle 0 | (\bar{\mu}\gamma^\alpha e) | M_\mu \rangle - \frac{1}{2} \langle 0 | (\bar{\mu}\gamma^\alpha \gamma^5 e) | M_\mu \rangle$$



Width Difference $\Delta\Gamma$

- operator product expansion

$$\begin{aligned}
 T &= i \int d^4x \mathcal{T} \left[\mathcal{H}_{\text{eff}}^{\Delta L_\mu=2}(x) \mathcal{H}_{\text{eff}}^{\Delta L_\mu=0}(0) \right] \\
 &= i \int d^4x \mathcal{T} \left[(\bar{\mu} \Gamma_\alpha e) \left(\bar{\nu}_{\mu_L} \gamma^\alpha \nu_{eL} \right)(x) (\bar{\mu} \gamma_\beta P_L e) \left(\bar{\nu}_{eL} \gamma^\beta \nu_{\mu_L} \right)(0) \right]
 \end{aligned}$$

- Contract neutrino fields into propagators
- Cutkoski rules to calculate the discontinuity (imaginary part) of T

$$\text{Disc } T = \frac{G_F}{\sqrt{2}\Lambda^2} \frac{M_M^2}{3\pi} \left[C_6^{\Delta L=2} (Q_1 + Q_5) + \frac{1}{2} C_7^{\Delta L=2} Q_3 \right]$$

- T and the matrix elements we found earlier, compute y

