One-loop determinant for massive vector field in large dimension limit

Outline

1. Introduction
    - Big picture
    - Denef-Hartnoll-Sachdev method
    - Large dimension limit
2. Massive scalar field
3. Massive vector field
    - Action and equation of motion
    - Decoupled modes
    - Result
4. Comments

Introduction

- Big picture

The AdS/CFT correspondence relates a weakly couple system with dynamical gravity in a curved spacetime and a strongly coupled non-gravitational system in flat spacetime

Gravity partition function in asymptotically AdS $\Leftrightarrow$ partition function of strongly coupled CFT in large N limit

One-loop contribution to gravity partition function $\Leftrightarrow$ $1/N$ corrections to the partition function of boundary field theory

Introduction

- Denef-Hartnoll-Sachdev(DHS) method

(arXiv:0908.2657v2 2010)

$$Z(\kappa) = \frac{1}{\det(\mathcal{D}(\kappa))}$$ is a meromorphic function

Based on Weierstrass factorization theorem

$$Z(\kappa) = e^{Pol(\kappa)} \prod_i \frac{1}{(\kappa - \kappa_i)^{d_i}}$$

where κ_i makes sure $\mathcal{D}(\kappa_i)\psi = 0$ for ψ that is smooth and satisfies boundary conditions

Introduction

- DHS method

Example: Harmonic oscillator

$$Z(\kappa) = \int \mathfrak{D}\phi\, \mathfrak{D}\bar{\phi}\, e^{\int_0^{\frac{1}{T}} d\tau\, (|\partial_\tau \phi|^2 + \kappa|\phi|^2)} = \frac{1}{\det(-\partial_\tau^2 + \kappa)}$$

$\mathcal{D}(\kappa) = -\partial_\tau^2 + \kappa, \mathcal{D}(\kappa)\phi = 0$ with $\phi(\tau) = \phi(\tau + \frac{1}{T})$

$\Longrightarrow \phi(\tau) = e^{\pm\sqrt{\kappa}\,\tau}, \pm\sqrt{\kappa} = i\, 2\pi n T, n \in \mathbb{Z}$

$\Longrightarrow Z(\kappa) = e^{Pol(\kappa)} \dfrac{1}{\sinh^2(\frac{\sqrt{\kappa}}{2T})}$

When $\kappa \to \infty, Z(\kappa) \to e^{-E_0/T} = e^{-\sqrt{\kappa}/T} \Longrightarrow Z(\kappa) = \frac{1}{4\sinh^2(\frac{\sqrt{\kappa}}{2T})}$

Introduction

- DHS method

Asymptotically AdS black brane, real massive field

$$\kappa \to \Delta(m^2)$$

$$\log\big(Z(\Delta)\big) = Pol(\Delta) - \frac{1}{2}\sum_i d_i \log(\Delta - \Delta_i)$$

$Pol(\Delta)$ usually fixed by matching with the heat kernel calculation

heat kernel expansion: expansion in $1/m^2 \Longrightarrow 1/\Delta^2$ when $\Delta \to \infty$

$$\Longrightarrow Pol(\Delta) \text{ when } \Delta \to \infty$$

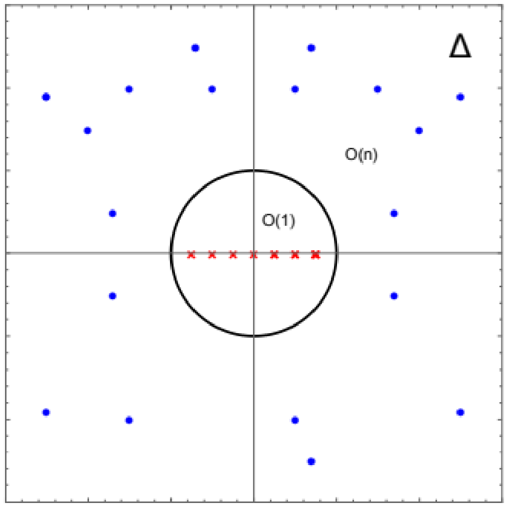
Difficulty: full expression of Δ_i is usually not known!

Introduction

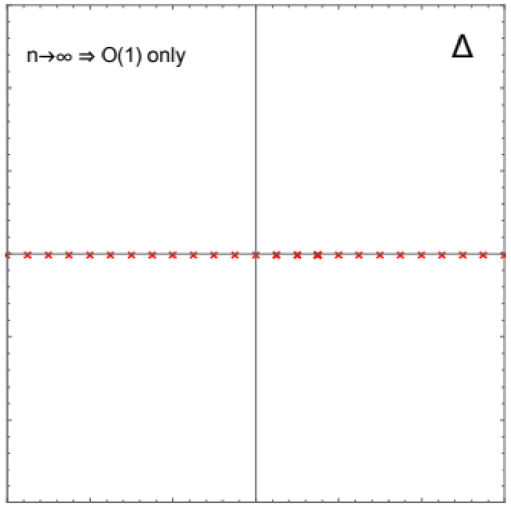
- Large dimension limit

$$\mathcal{D}(\Delta)\psi = 0$$

Picture taken from 2020 paper by Keeler and Priya(arXiv:1904.09299v2)



(a) A cartoon of poles in the Δ plane.



(b) A cartoon of the poles in the large $D = n - 3$ limit of the Δ plane.

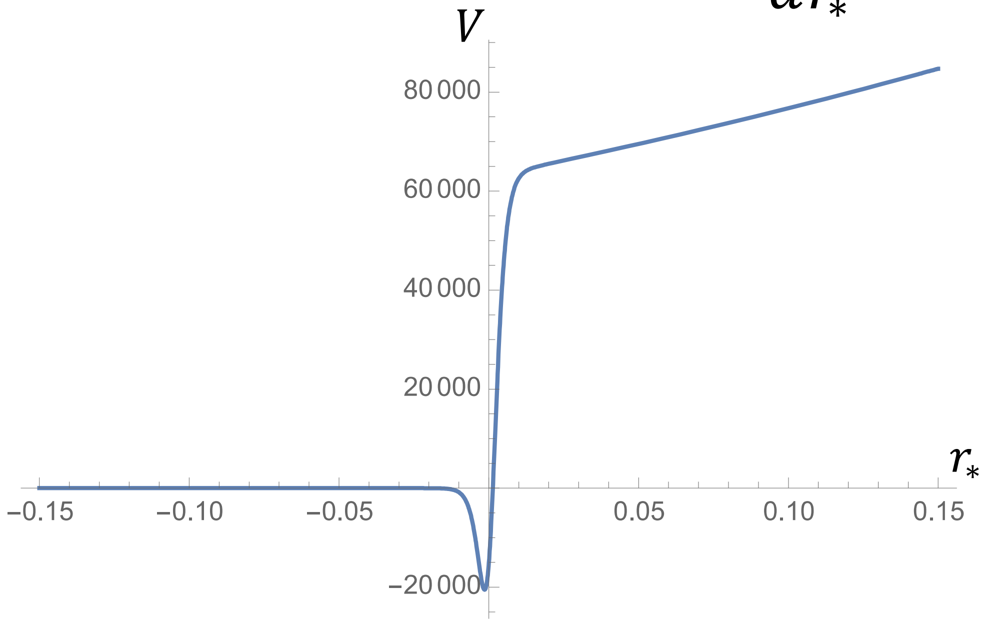
$$\log\big(Z(\Delta)\big) = Pol(\Delta) - \frac{1}{2}\sum_i d_i \log(\Delta - \Delta_i)$$

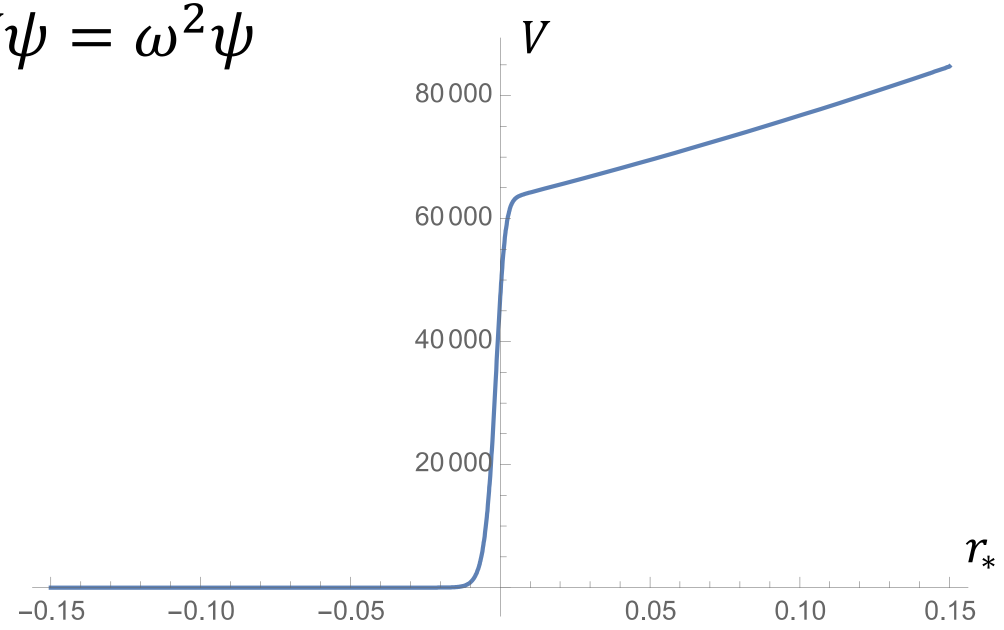
Introduction

- Large dimension limit

Schrödinger-like form

$$-\frac{d^2}{dr_*^2}\psi + V\psi = \omega^2\psi$$





Massive scalar field

- Action and equation of motion

$$S = \int d^D x \sqrt{-g}(\frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi + \frac{1}{2} m^2 \phi^2), D = d + 1$$

Metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2} d\vec{x}^2, f(r) = \frac{r^2}{L^2}(1 - \frac{r_h^d}{r^d})$$

Equation of motion

$$\Box \phi = m^2 \phi$$

Massive scalar field

- Action and equation of motion

$$f(r)\phi''(r) + \left(f'(r) + (d-1)\frac{f(r)}{r}\right)\phi'(r) + \left(-m^2 + \frac{\omega^2}{f(r)} - \frac{k^2}{r^2}\right)\phi(r) = 0$$

Field redefinition and coordinate transformation

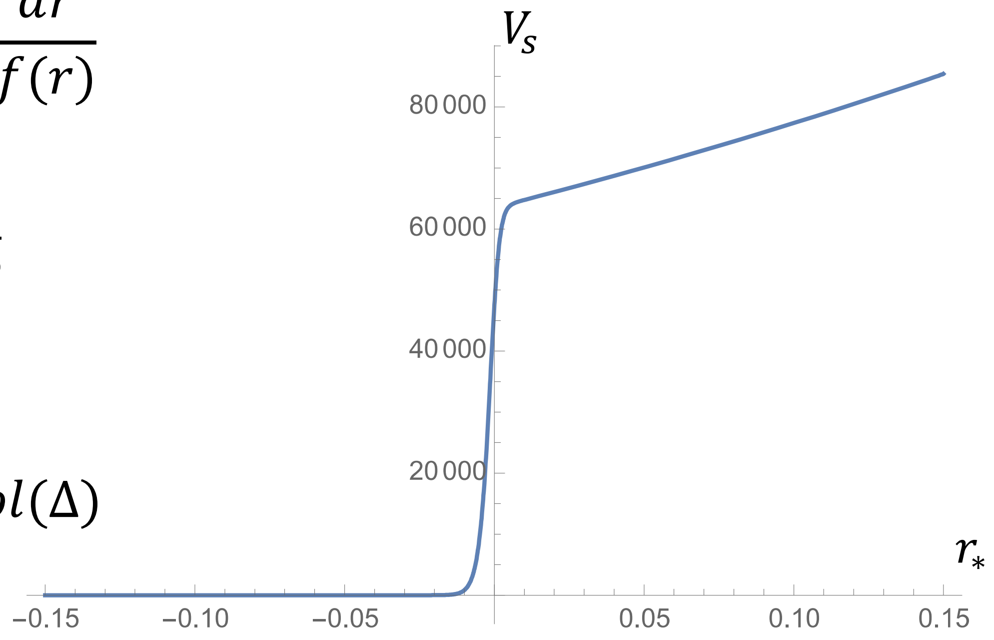
$$\phi(r) = r^{-\frac{d-1}{2}}\bar{\phi}(r), dr_* = \frac{dr}{f(r)}$$

Schrödinger-like form

$$-\frac{d^2}{dr_*^2}\bar{\phi} + V_s\bar{\phi} = \omega^2\bar{\phi}$$

Where $V_s = f(r)\left(\frac{(d-1)(d-3)}{4r^2}f(r) + \frac{d-1}{2r}f'(r) + \frac{k^2}{r^2} + m^2\right)$

$\Longrightarrow$ No decoupled modes! Contribution will be captured by $Pol(\Delta)$



Massive vector field

- Action and equation of motion

$$S = \int d^D x \sqrt{-g}(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu)\,, D = d + 1$$

Metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}d\vec{x}^2, f(r) = \frac{r^2}{L^2}(1 - \frac{r_h^d}{r^d})$$

Equation of motion

$$\Box A_\mu - R_{\mu\nu}A^\nu = m^2 A_\mu$$

Massive vector field

- Action and equation of motion

Consider a basis function $e^{i\omega t - i\vec{k}\cdot\vec{x}}A_\mu(r)$

Define the propagation direction to be x direction, $\vec{x} = (x, \vec{x}_\perp)$

$$A_\mu = (A_t, A_r, A_x, \vec{A}_{x_\perp})$$

$\Longrightarrow (d-2)$ decoupled equations for $\vec{A}_{x_\perp}$

3 coupled equations for A_t, A_r and A_x

Massive vector field

- Action and equation of motion

Decoupled equations for $\vec{A}_{x_\perp}$

$$f(r)\vec{A}''_{x_\perp}(r) + \left(f'(r) + (d-3)\frac{f(r)}{r}\right)\vec{A}'_{x_\perp}(r) + \left(-m^2 + \frac{\omega^2}{f(r)} - \frac{k^2}{r^2}\right)\vec{A}_{x_\perp}(r) = 0$$

Compare with the equation for scalar field

$$f(r)\phi''(r) + \left(f'(r) + (d-1)\frac{f(r)}{r}\right)\phi'(r) + \left(-m^2 + \frac{\omega^2}{f(r)} - \frac{k^2}{r^2}\right)\phi(r) = 0$$

In large dimension limit, they are essentially the same!

$\Longrightarrow$ contribution will be captured by $Pol(\Delta)$

Massive vector field

- Action and equation of motion

Coupled equations for A_t, A_r and A_x

$$\begin{cases} f(r)A_t''(r) + (d-1)\frac{f(r)}{r}A_t'(r) + \left(\frac{\omega^2}{f(r)} - \frac{k^2}{r^2} - m^2\right)A_t(r) - i\omega f'(r)A_r(r) = 0 \\ f(r)A_r''(r) + \left(2f'(r) + (d-1)\frac{f(r)}{r}\right)A_r'(r) - i\omega\frac{f'(r)}{f(r)^2}A_t(r) + \left(-m^2 + f''(r) + \frac{\omega^2}{f(r)} + (d-1)\frac{f'(r)}{r} - \frac{k^2}{r^2} - (d-1)\frac{f(r)}{r^2}\right)A_r(r) - 2i\frac{k}{r^3}A_x(r) = 0 \\ f(r)A_x''(r) + \left(f'(r) + (d-3)\frac{r(r)}{r}\right)A_x'(r) + 2ik\frac{f(r)}{r}A_r(r) + \left(\frac{\omega^2}{f(r)} - \frac{k^2}{r^2} - m^2\right)A_x(r) = 0 \end{cases}$$

With divergence free condition

$$\frac{ikA_x(r)}{r^2} + \frac{i\omega A_t(r)}{f(r)} + f(r)A_r'(r) + \left((d-1)\frac{f(r)}{r} + f'(r)\right)A_r(r) = 0$$

field redefinition and coordinate transformation

$$A_r(r) = \frac{r^{-\frac{d-1}{2}}}{f(r)}\bar{A}_r(r), A_x(r) = r^{-\frac{d-3}{2}}\bar{A}_x(r), dr_* = \frac{dr}{f(r)}$$

Massive vector field

- Action and equation of motion

Schrödinger-like equations for $\bar{A}_r$ and $\bar{A}_x$

$$-\frac{d^2}{dr_*^2}\begin{pmatrix}\bar{A}_r\\ \bar{A}_x\end{pmatrix}+\begin{pmatrix}V_{rr} & V_{rx}\\ V_{xr} & V_{xx}\end{pmatrix}\begin{pmatrix}\bar{A}_r\\ \bar{A}_x\end{pmatrix}=\omega^2\begin{pmatrix}\bar{A}_r\\ \bar{A}_x\end{pmatrix}$$

With

$$\begin{pmatrix}V_{rr} & V_{rx}\\ V_{xr} & V_{xx}\end{pmatrix}=\begin{pmatrix}\frac{f(r)(4(k^2+m^2r^2)+(d^2-1)f(r)-2(d-1)rf'(r))}{4r^2} & -\frac{ikf(r)(rf'(r)-2f(r))}{r^2}\\ -\frac{2ikf(r)}{r^2} & \frac{f(r)(4(k^2+m^2r^2)+(d-3)(d-5)f(r)+2(d-3)rf'(r))}{4r^2}\end{pmatrix}$$

$V_{rr}, V_{xx} \sim d^2$

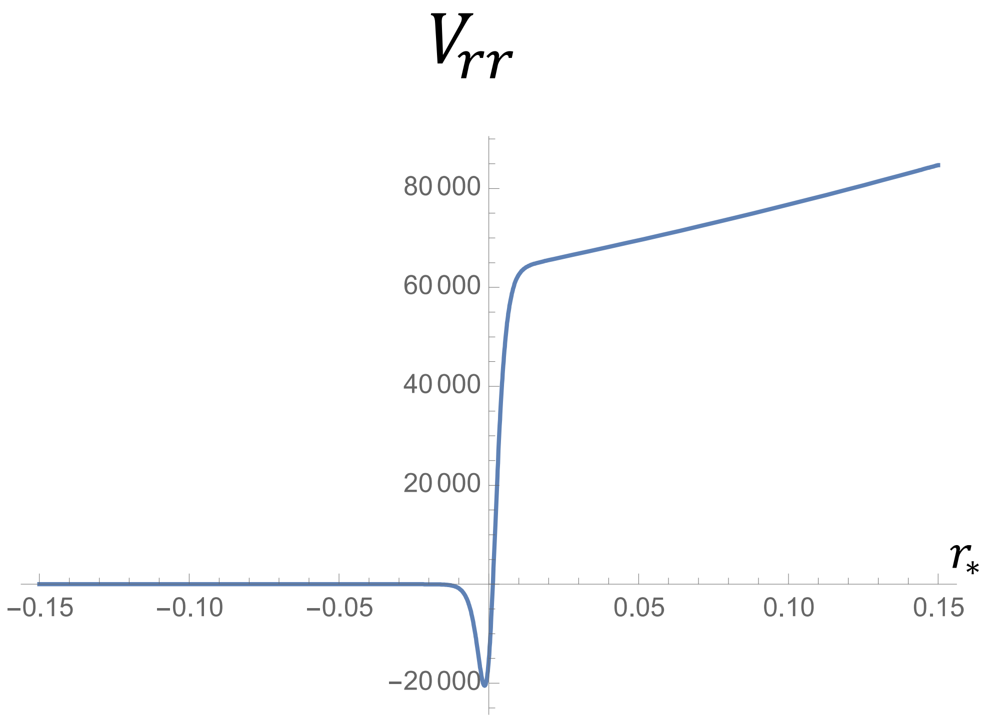
$V_{rx} \sim d$

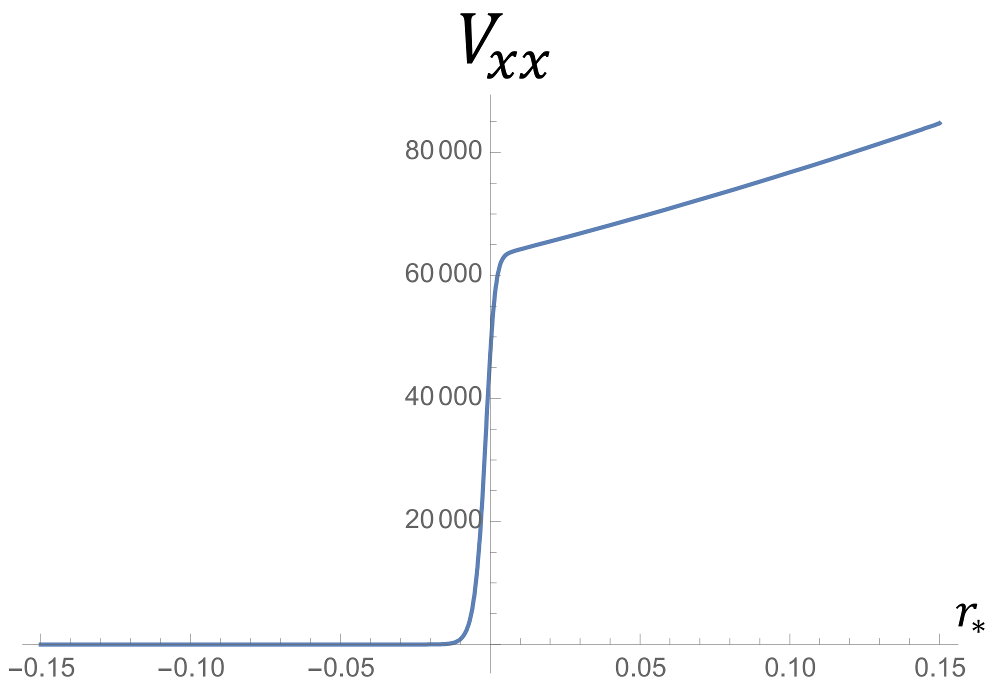
$V_{xr} \sim 1$

$\Longrightarrow$ equations for $\bar{A}_r$ and $\bar{A}_x$ are decoupled in large dimension limit!

Massive vector field

- Action and equation of motion





Massive vector field

- Decoupled modes

$$\frac{d^2}{dr_*^2}\bar{A}_r - V_{rr}\bar{A}_r = -\omega^2\bar{A}_r$$

Use near-horizon coordinate $\rho = r^d$

Define

$$F(\rho) = f(r) = \rho^{\frac{2}{d}-1}(\rho - 1)$$

$$\mathcal{L} = -\frac{1}{d^2}F(\rho)\frac{d}{d\rho}\left(F(\rho)\frac{d}{d\rho}\bar{A}_r\right)$$

$$U = (\frac{1}{4}\rho^{-\frac{2}{d}}F(\rho)\left(4\frac{k^2}{d^2} + \left(1 - \frac{1}{d^2}\right)F(\rho) - 2\left(1 - \frac{1}{d}\right)\rho F'(\rho)\right) - \frac{\omega^2}{d^2})$$

$$\bar{\Delta} = \sqrt{\left(\frac{d-2}{2}\right)^2 + m^2} - \frac{d-2}{2}$$

Massive vector field

- Decoupled modes

$$\mathcal{L}\bar{A}_r + U\bar{A}_r + \rho^{\frac{2}{d}-1}(\rho-1)\frac{\bar{\Delta}}{d}\left(\frac{\bar{\Delta}-2}{d}+1\right)\bar{A}_r = 0$$

Solve the equation perturbatively in $1/d$ expansion

$$\mathcal{L} = \sum_{i=0}\frac{\mathcal{L}^{(i)}}{d^i}, U = \sum_{i=0}\frac{U^{(i)}}{d^i}, \bar{A}_r = \sum_{i=0}\frac{\bar{A}_r^{(i)}}{d^i}, \bar{\Delta} = d\left(\mu - \frac{1}{2}\right), \mu = \sum_{i=0}\frac{\mu^{(i)}}{d^i}, \omega = \sum_{i=0}\frac{\omega^{(i)}}{d^i}, k^2 \sim O(d)$$

With boundary condition $\bar{A}_r(\rho) = \mathcal{N}(\rho-1)^{-\frac{i\omega}{d}}$ when $\rho\sim 1$ for quasinormal modes

$$\Rightarrow \mu^{(0)} = \frac{1}{2}, \mu^{(1)} = -p - \frac{k^2}{d}$$

$$\Rightarrow \bar{\Delta}_* = -p - \frac{k^2}{d}, p \in \mathbb{Z}_{\geq 0}$$

In the case of anti-quasinormal modes, the boundary condition $\bar{A}_r(\rho) = \mathcal{N}(\rho-1)^{+\frac{i\omega}{d}}$ when $\rho\sim 1$

$$\Rightarrow \bar{\bar{\Delta}}_* = p - \frac{k^2}{d}, p \in \mathbb{Z}_{<0}$$

Massive vector field

- Result

$$\log(Z) = Pol(\bar{\Delta}) + \sum_{\vec{k},p\in\mathbb{Z}_{\geq 0}} \log(\bar{\Delta} + p + \frac{k^2}{d}) + \sum_{\vec{k},p\in\mathbb{Z}_{<0}} \log(\bar{\Delta} - p + \frac{k^2}{d})$$

Compactify $\vec{x}$ coordinates, then take decompactification limit

$$\sum_{\vec{k},p} \log\left(\bar{\Delta} + p + \frac{k^2}{d}\right) \to \sum_{p,\vec{n}\in\mathbb{Z}_{d-1}} \log\left(\bar{\Delta} + p + \frac{1}{d}\left(\frac{2\pi\vec{n}}{l}\right)^2\right)$$

$$\to \sum_{p,\vec{n}\in\mathbb{Z}_{d-1}} \int_0^\infty \frac{dt}{t} e^{-t\left(\bar{\Delta}+p+\frac{1}{d}\left(\frac{2\pi\vec{n}}{l}\right)^2\right)} \xrightarrow{\text{Poisson resummation}} \sum_{p,\vec{q}\in\mathbb{Z}_{d-1}} vol(\mathbb{R}^{d-1}) \int_0^\infty \frac{dt}{t} \left(\frac{d}{4\pi t}\right)^{\frac{d-1}{2}} e^{-t(\bar{\Delta}+p)} e^{-\frac{q^2}{t}l^2 d}$$

$$\xrightarrow{l\to\infty} \sum_p vol(\mathbb{R}^{d-1}) \Gamma\left(-\frac{d-1}{2}\right)\left(\frac{(\bar{\Delta}+p)d}{4\pi}\right)^{\frac{d-1}{2}}$$

Massive vector field

- Result

$$\log(Z) = Pol(\bar{\Delta}) + 2\sum_{p\in\mathbb{N}} vol(\mathbb{R}^{d-1})\Gamma\left(-\frac{d-1}{2}\right)\left(\frac{(\bar{\Delta}+p)d}{4\pi}\right)^{\frac{d-1}{2}} - vol(\mathbb{R}^{d-1})\Gamma\left(-\frac{d-1}{2}\right)\left(\frac{\bar{\Delta}d}{4\pi}\right)^{\frac{d-1}{2}}$$

With Hurwitz zeta function regularization $\sum_{p\in\mathbb{N}}(\bar{\Delta}+p)^{\frac{d-1}{2}} = \zeta(-\frac{d-1}{2},\bar{\Delta})$

$$\log(Z) = Pol(\bar{\Delta}) + vol(\mathbb{R}^{d-1})\Gamma\left(-\frac{d-1}{2}\right)\left(\frac{d}{4\pi}\right)^{\frac{d-1}{2}}\left(2\zeta\left(-\frac{d-1}{2},\bar{\Delta}\right) - \bar{\Delta}^{\frac{d-1}{2}}\right)$$

Comments

$$\log(Z) = Pol(\bar{\Delta}) + vol(\mathbb{R}^{d-1})\Gamma\left(-\frac{d-1}{2}\right)\left(\frac{d}{4\pi}\right)^{\frac{d-1}{2}}\left(2\zeta\left(-\frac{d-1}{2},\bar{\Delta}\right) - \bar{\Delta}^{\frac{d-1}{2}}\right)$$

- We might be able to fix $Pol(\bar{\Delta})$ by matching with heat kernel expansion, but the order of limit issue and convergence issue need careful investigation.
- Strictly speaking, the condition of this expression is $\bar{\Delta} \ll d$, in which case $\bar{\Delta} \to \frac{m^2}{d}$
- Once the replacement $\bar{\Delta} \to \frac{m^2}{d}$ is done, the expression has very similar structure as the heat kernel expansion, which may suggest the expression still being valid even outside its original valid region.
- Distinction between $(d+1)$ being even or odd.
- When $(d+1)$ is even, $\zeta\left(-\frac{d-1}{2}, \frac{m^2}{d}\right)$ can be expanded as a polynomial of m. Combined with the gamma function, it will produce non-polynomial term such $m^D \log(m)$.