NEURAL NETWORK APPROXIMATIONS OF CALABI–YAU METRICS

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String Pheno 2021

The Unreal World

- Our objective is to obtain the real world from a string compactification

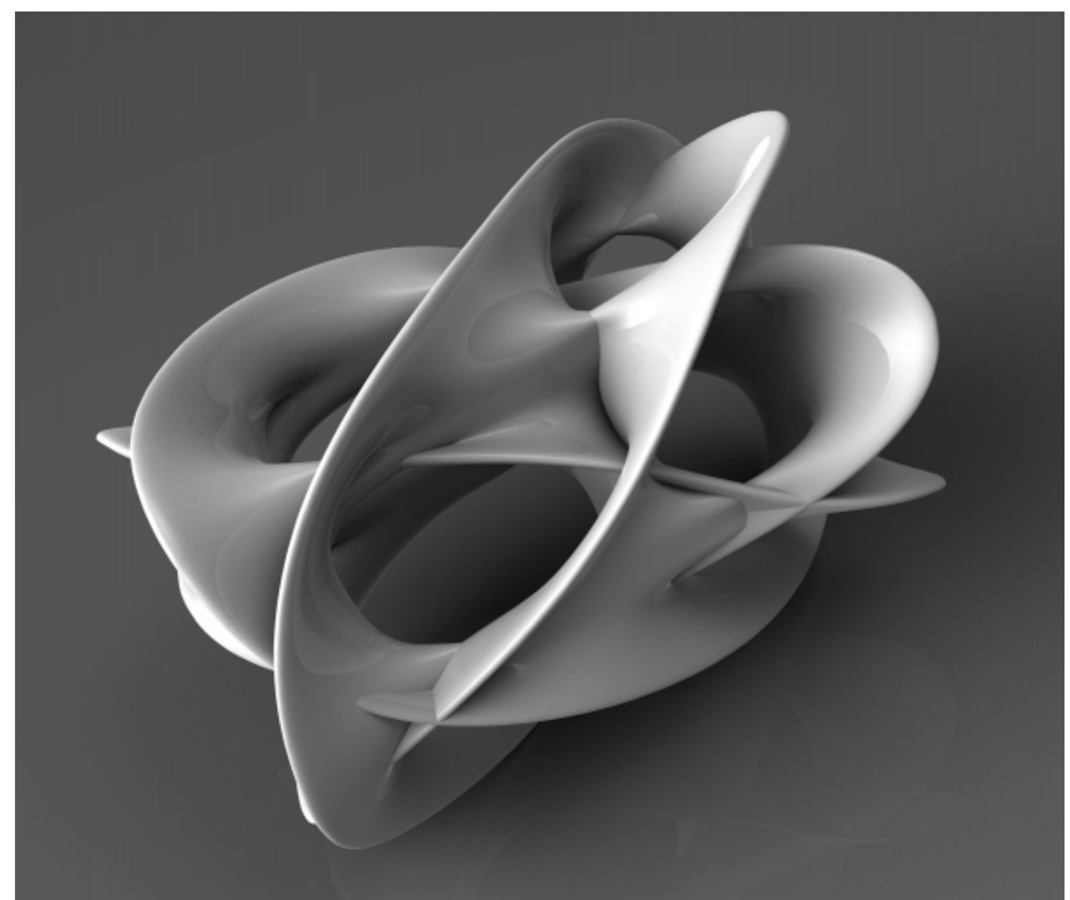
- We would happily start with a modestly unreal world

$$\boxed{\mathcal{N} = 1 \text{ supersymmetry in 4 dimensions}}$$

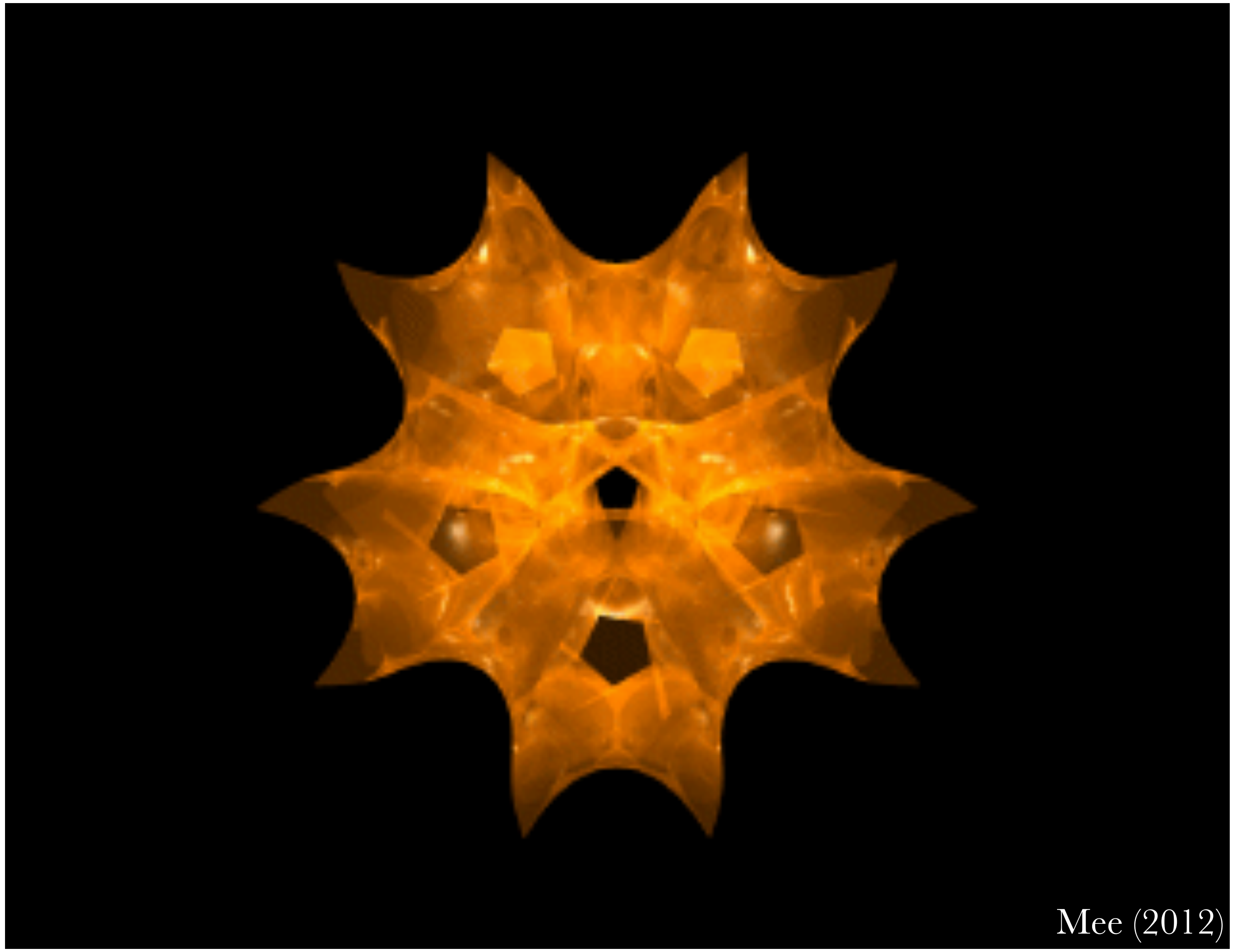
Because it is Ricci flat, the Calabi–Yau geometry ensures 4d supersymmetry

Use topological and geometric features of the Calabi–Yau to recover aspects of the real world

Candelas, Horowitz, Strominger, Witten (1985)



Calabi–Yau



$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0 \subset \mathbb{P}^4$

There is a nowhere vanishing holomorphic n-form

The canonical bundle is trivial

There is a Kähler metric with holonomy in $SU(n)$

Why a Metric?

- Numerous constructions of Standard Model — *i.e.*, correct matter spectrum and interactions — from knowing only topology

Braun, He, Ovrut, Pantev (2005)
Bouchard, Donagi (2005)

- Quantities relevant for phenomenology rely on knowing the metric

  – Yukawa couplings in heterotic compactifications

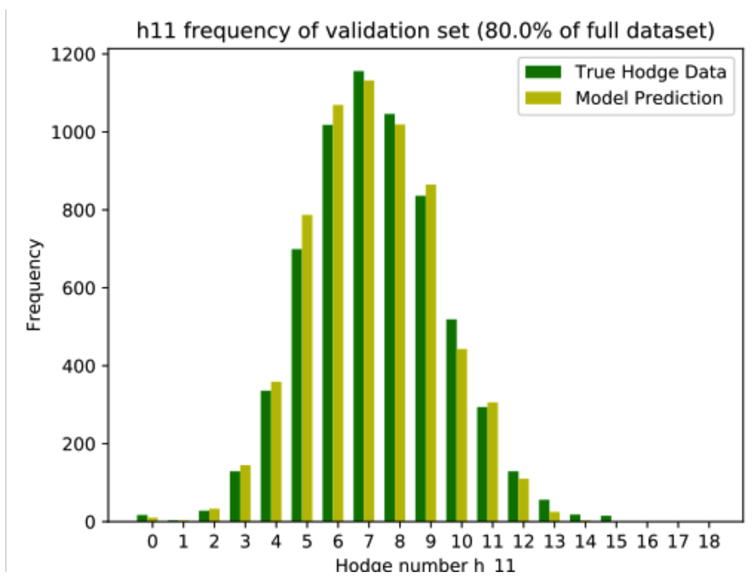
  – Kinetic terms from Kähler potential for matter

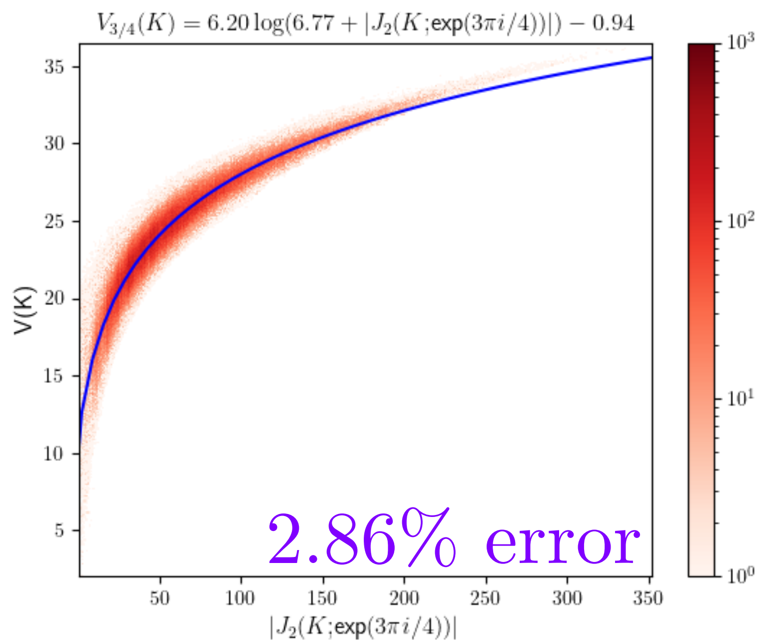
  – Moduli stabilization

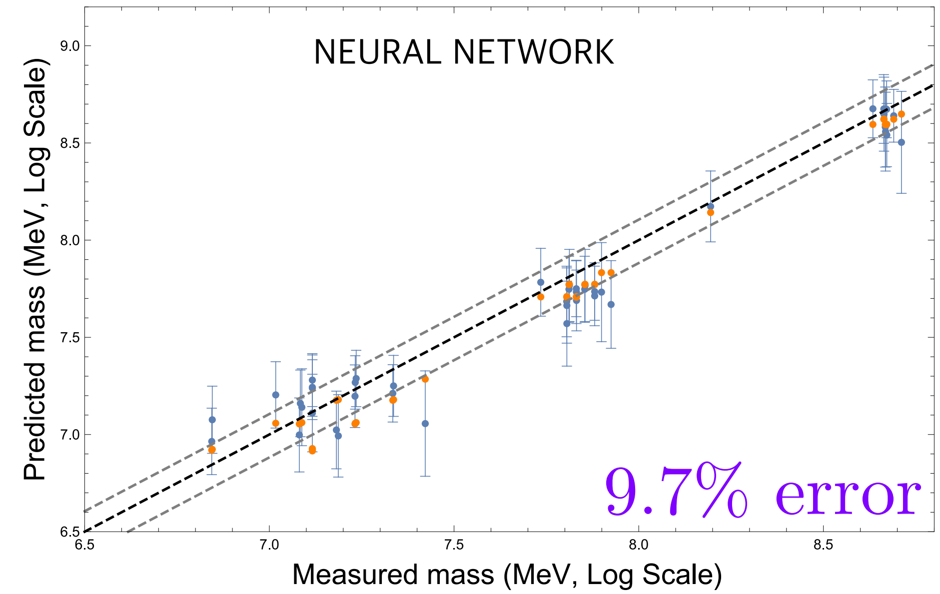
- To date, the best we can do is proceed with numerical approximations on Calabi–Yau threefolds

- Is it good enough to calculate, *e.g.*, the mass of the electron?

Machine Learning in hep-th







Topological Invariants of CICYs

Bull, He, VJ, Mishra (2018, 2019)
Erbin, Finotello (2020)

Knot Theory

Hughes (2016)
VJ, Kar, Parrikar (2019)
Levitt, Hajij, Sazdanovic (2019)
Gukov, Halverson, Ruehle, Sulkowski (2020)
Craven, VJ, Kar (2020)

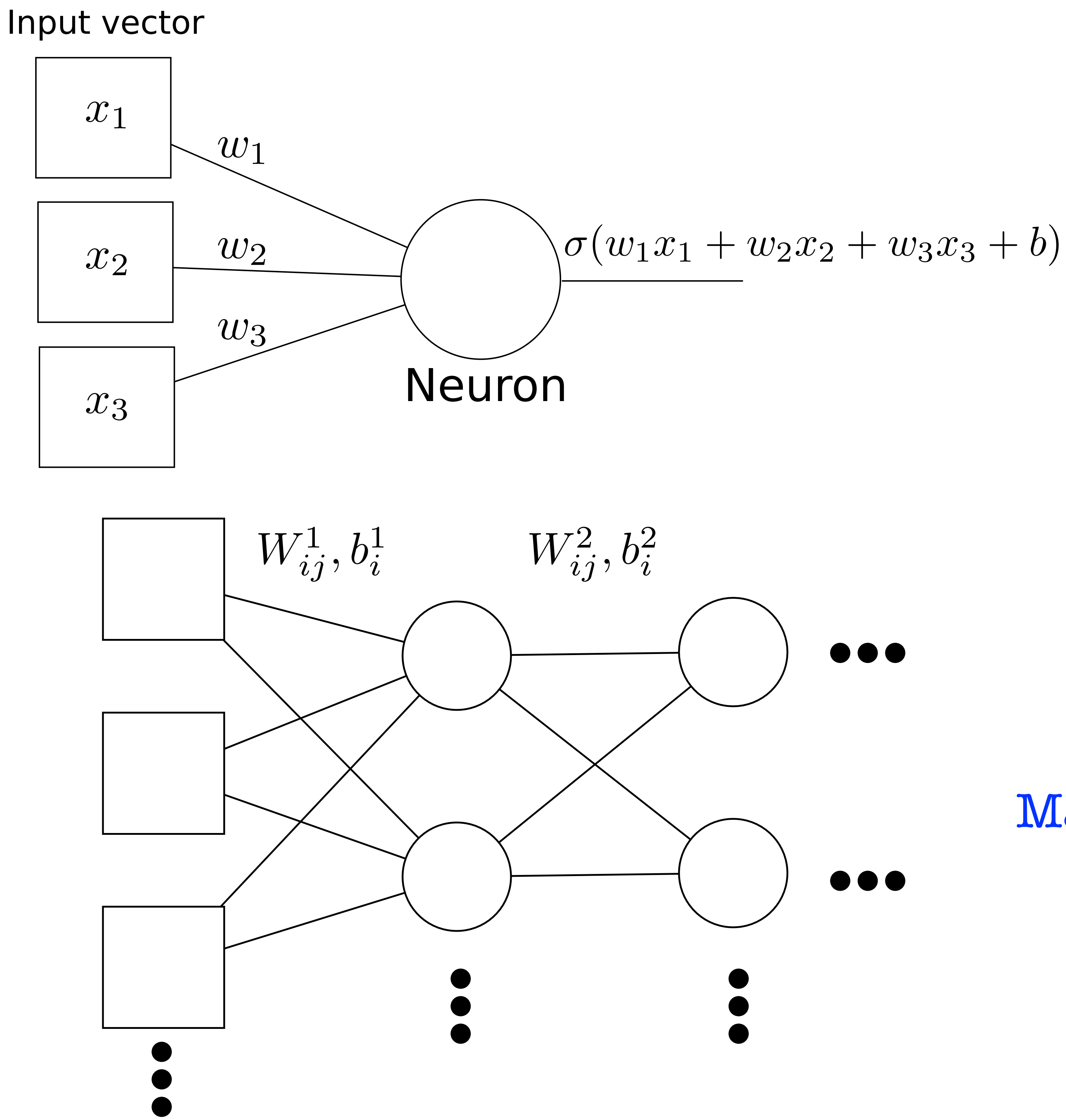
QCD

Hashimoto, Sugishita, Tanaka, Tomiya (2018)
Hashimoto (2019)
Akutagawa, Hashimoto, Sugimoto (2020)
Gal, VJ, Mayorga Peña, Mishra (2020)

- Many other **hep-th** and **math** results in this direction in recent years

He (2017)
Krefl, Seong (2017)
Ruehle (2017)
Carifio, Halverson, Krioukov, Nelson (2017)
⋮

Feedforward Neural Networks

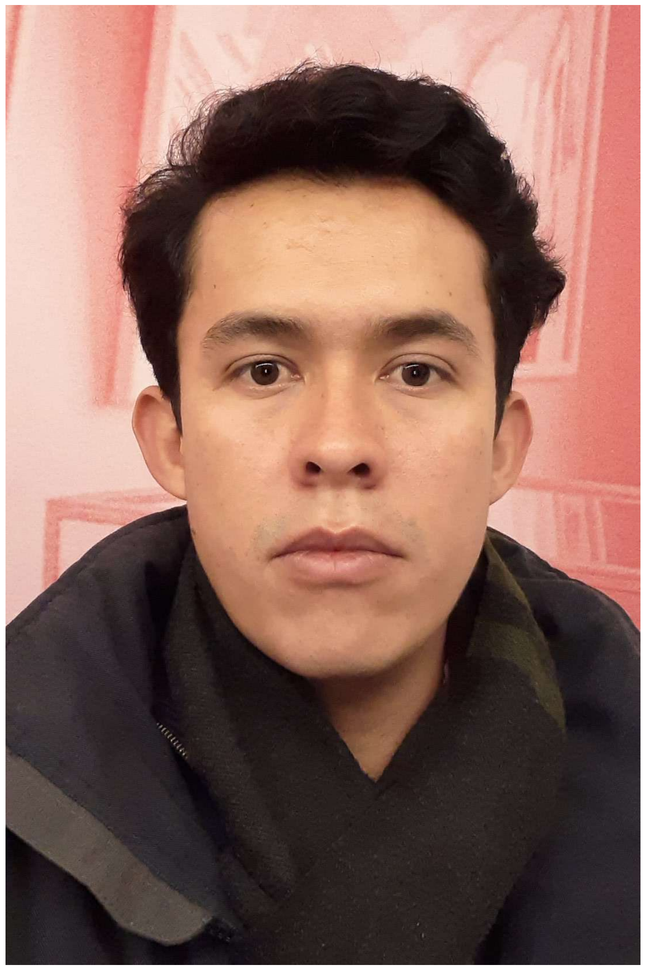


Rosenblatt (1957)

Mathematica 10+

Schematic representation of feedforward neural network. The top figure denotes the perceptron (a single neuron), the bottom, the multiple neurons and multiple layers of the neural network.

Collaborators



Damián Kaloni Mayorga Peña

UNIVERSITY OF THE WITWATERSRAND JOHANNESBURG

Challenger Mishra

UNIVERSITY OF CAMBRIDGE

arXiv:2012.15821, 21mm.nnnnn

Other Work

- Efforts in finding/using numerical metrics

  Headrick, Wiseman (2005)
  Donaldson (2005)
  Douglas, Karp, Lukic, Reinbacher (2006)
  Braun, Brelidze, Douglas, Ovrut (2007, 2008)
  Headrick, Nasser (2009)
  Anderson, Braun, Karp, Ovrut (2010)
  Anderson, Braun, Ovrut (2011)
  Cui, Gray (2019)
  Ashmore (2020)
  Ashmore, Ruehle (2021)

- Parallel Machine Learning successes in finding Calabi–Yau metrics

  Ashmore, He, Ovrut (2019)
  Anderson, Gerdes, Gray, Krippendorf, Raghuram, Ruehle (2020)
  Douglas, Lakshminarasimhan, Qi (2020)

- Analytic K3 metrics

  Kachru, Tripathy, Zimet (2018, 2020)

CICYs

- Torus: $z_1^3 + z_2^3 + z_3^3 = 0 \subset \mathbb{P}^2$

- K3: $z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0 \subset \mathbb{P}^3$

- Fermat quintic: $z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0 \subset \mathbb{P}^4$ $\qquad \left\{ \begin{array}{ccc} h^{1,1} & = & 1 \\ h^{1,2} & = & 101 \end{array} \right.$

- Dwork quintic: $z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 - 5\,\psi\, z_1 z_2 z_3 z_4 z_5 = 0 \subset \mathbb{P}^4\,, \qquad \psi \neq 1$

- Tian–Yau: $\left[\begin{array}{c|ccc} \mathbb{P}^3 & 3 & 0 & 1 \\ \mathbb{P}^3 & 0 & 3 & 1 \end{array} \right]^{14,23}_{\chi=-18} \iff \left\{ \begin{array}{ccc} \alpha^{ijk} z_i z_j z_k & = & 0 \\ \beta^{ijk} w_i w_j w_k & = & 0 \\ \gamma^{ij} z_i w_j & = & 0 \end{array} \right.$

a freely acting $\mathbb{Z}_3$ quotient gives a Calabi–Yau with $\chi = -6$

Greene, Kirklin, Miron, Ross (1986)

The Metric

- Calabi–Yau manifold $\mathcal{M}$ is complex and Kähler

- Metric obtained from Kähler potential

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \log K(z, \bar{z})$$

- Metric can be used to construct $(1,1)$ Kähler form

$$J = \frac{i}{2} g_{a\bar{b}}\, dz^a \wedge d\bar{z}^{\bar{b}} \longleftarrow$$

closed: $dJ = 0$

$J \wedge \Omega = 0$

- Ricci tensor is

$$R_{a\bar{b}} = \partial_a \partial_{\bar{b}} \log \det g$$

Yau's Theorem

- To every closed (1,1)-form $\frac{1}{2\pi}C_1(\mathcal{M})$ representing the first Chern class $c_1(\mathcal{M})$ of a Kähler manifold $\mathcal{M}$, there is a unique Kähler metric in the same Kähler class whose Ricci tensor (form) is the closed (1,1)-form $C_1(\mathcal{M})$

Calabi (1957)
Yau (1977)

- Yau's proof of Calabi's conjecture is not constructive

- Solve classical Monge–Ampère equation for real valued φ

$$\det\left(g_{a\bar{b}} + \partial_a \partial_{\bar{b}} \varphi\right) = e^f \det g_{a\bar{b}}$$

positive definite

any smooth function of average 1

- Then $J + i\partial_a\partial_{\bar{b}}\varphi$, Kähler metric, can attain any Ricci (curvature) form in the class referred to in Calabi conjecture

Donaldson's Algorithm

- Start from Fubini–Study metric on projective space, which is Kähler

$$K(z,\bar{z}) = \frac{1}{\pi}\log|z|^2\ , \quad |z|^2 = \sum_{a=1}^{n+1} z^a \bar{z}^{\bar{a}}$$

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K = \frac{|z|^2 \delta^{a\bar{b}} - z^a \bar{z}^{\bar{b}}}{\pi |z|^4}$$

- Pullback of the Fubini–Study metric onto the embedded CICY hypersurface is not Ricci flat

- However, it supplies a starting point for an iterative construction

- A valid Kähler potential is $K^{(k)}(z,\bar{z}) = \dfrac{1}{k\pi}\log(h^{a\bar{b}} s_a \bar{s}_{\bar{b}})$

element of a basis of degree k holomorphic polynomials over $\mathcal{M}$

- Kodaira embedding

Donaldson's Algorithm

- Let N_k be the dimension of such a basis and define

$$H_{a\bar{b}} = \frac{N_k}{\mathrm{Vol}_\Omega} \int_{\mathcal{M}} d\mathrm{Vol}_\Omega \, \frac{s_a s_{\bar{b}}}{h^{c\bar{d}} s_c s_{\bar{d}}}$$

- At level k use, $h^{a\bar{b}} = (H_{a\bar{b}})^{-1}$ and proceed iteratively until achieving a "balanced metric" $H_{a\bar{b}}$

- As k increases, the balanced metric approaches the Ricci flat metric

Tian (1990)
Donaldson (2005)

- Numerical implementations of this are computationally expensive

Douglas, Karp, Lukic, Reinbacher (2006)
Braun, Brelidze, Douglas, Ovrut (2007)
Ashmore, He, Ovrut (2019)

- Baseline for comparison

Flatness Measure

- We use σ-measure

$$\sigma = \frac{1}{\mathrm{Vol}_\Omega} \int_{\mathcal{M}} d\mathrm{Vol}_\Omega \left| 1 - \frac{\mathrm{Vol}_\Omega}{\mathrm{Vol}_J} \cdot \frac{J \wedge \cdots \wedge J}{\Omega \wedge \bar{\Omega}} \right|$$

- Two expressions for volume

$$\mathrm{Vol}_\Omega = \int_{\mathcal{M}} \Omega \wedge \bar{\Omega}$$

in terms of top holomorphic and antiholomorphic forms

$$\mathrm{Vol}_J = \int_{\mathcal{M}} J \wedge \cdots \wedge J$$

in terms of (1,1) Kähler form

- Upon rescaling, this measure becomes zero when

$$\Omega \wedge \bar{\Omega} \propto J \wedge \cdots \wedge J$$

Other Measures

- Kähler measure

$$\kappa = \frac{\mathrm{Vol}_J^{1/n}}{\mathrm{Vol}_\Omega} \int_{\mathcal{M}} d\mathrm{Vol}_J \; |k|^2$$

closed: $dJ = 0$

$$|k|^2 = \sum_{a,b,\bar{c}} |k_{ab\bar{c}}|^2 \;, \quad k_{ab\bar{c}} = \partial_a g_{b\bar{c}} - \partial_b g_{a\bar{c}}$$

- Compatibility measure

$$\mu = \frac{1}{N_p!} \sum_{m,n} \sum_{m',n'} \frac{1}{\mathrm{Vol}_\Omega} \int_{\mathcal{M}} d\mathrm{Vol}_J \; |M(m,n;m'n')|$$

number of patches

label patches

$z_m \neq 0$

z_n dependent coordinate

$M = g^{(m,n)} - \mathrm{Jac} \cdot g^{(m',n')} \cdot \overline{\mathrm{Jac}}$

measures disagreement of numerical metric on different coordinate patches of manifold

Strategy

- Decompose metric as $g = L \cdot D \cdot L^\dagger$

diagonal matrix

lower triangular
1s along diagonal

- Give the neural network the metric at points on manifold and query the trained neural network for the metric at new points

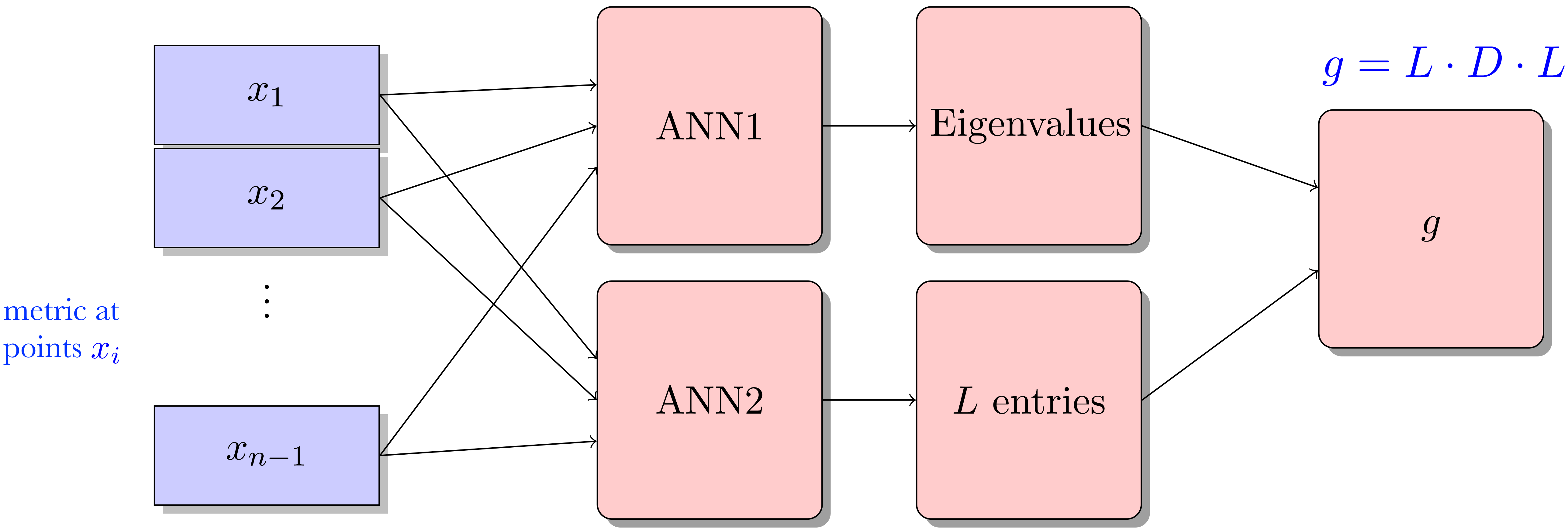
- Initialize metric at random

- Use gradient descent to minimize loss function

$$\mathrm{Loss} = \alpha_\sigma \sigma + \alpha_\kappa \kappa + \alpha_\mu \mu$$

Neural Network

- Neural network architecture



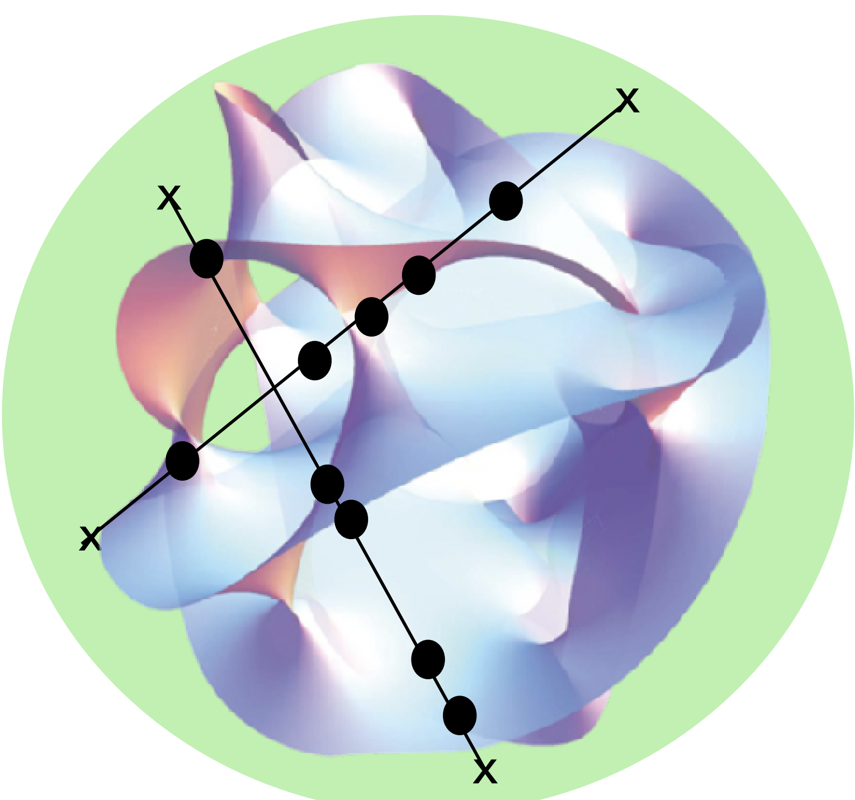
- Implemented using **pytorch**

Point Selection

- CICY hypersurfaces (torus, K3, quintics)

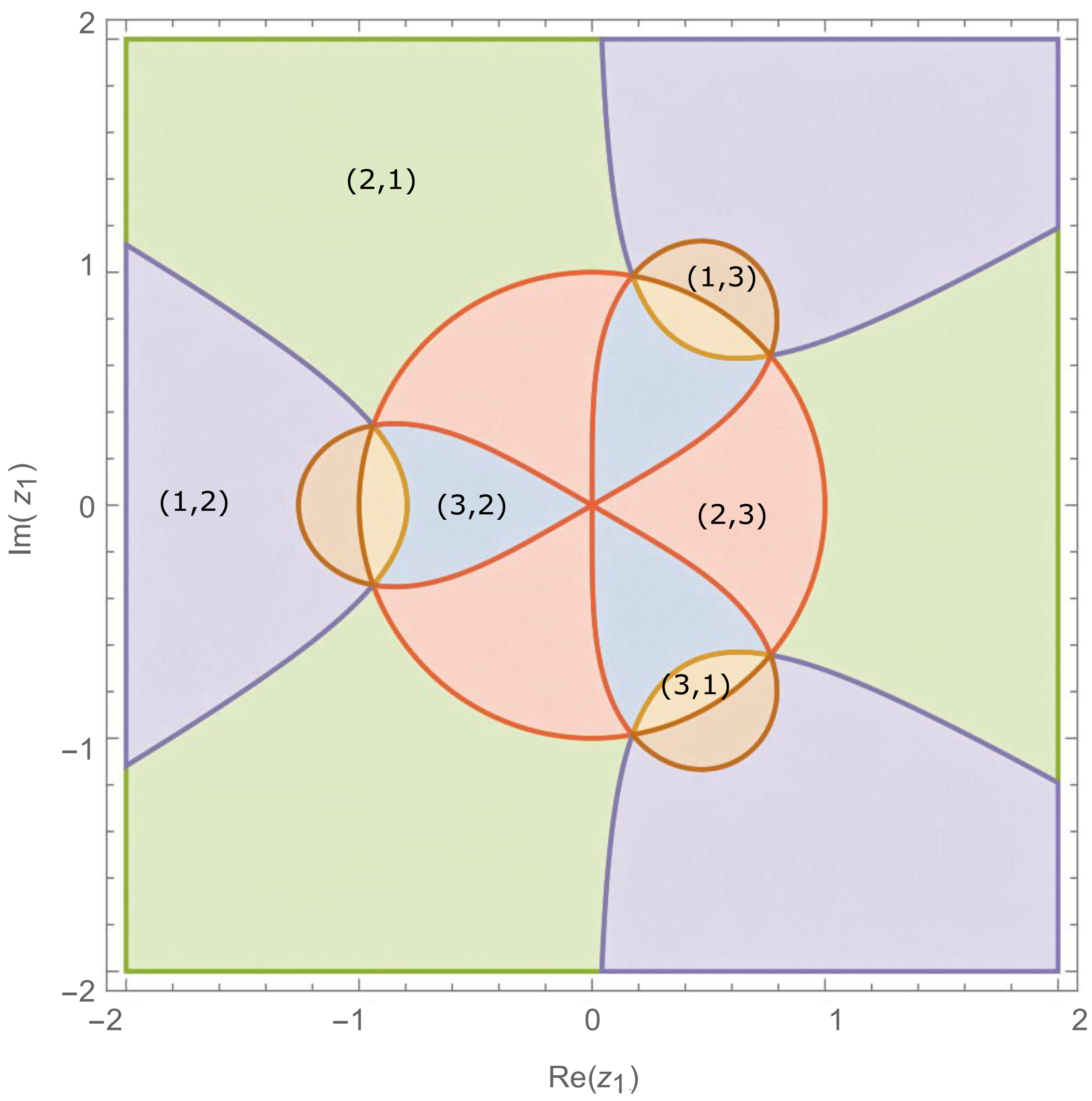
- Lines in $\mathbb{P}^n$ are uniformly distributed with respect to $SU(n+1)$ symmetry of Fubini–Study metric

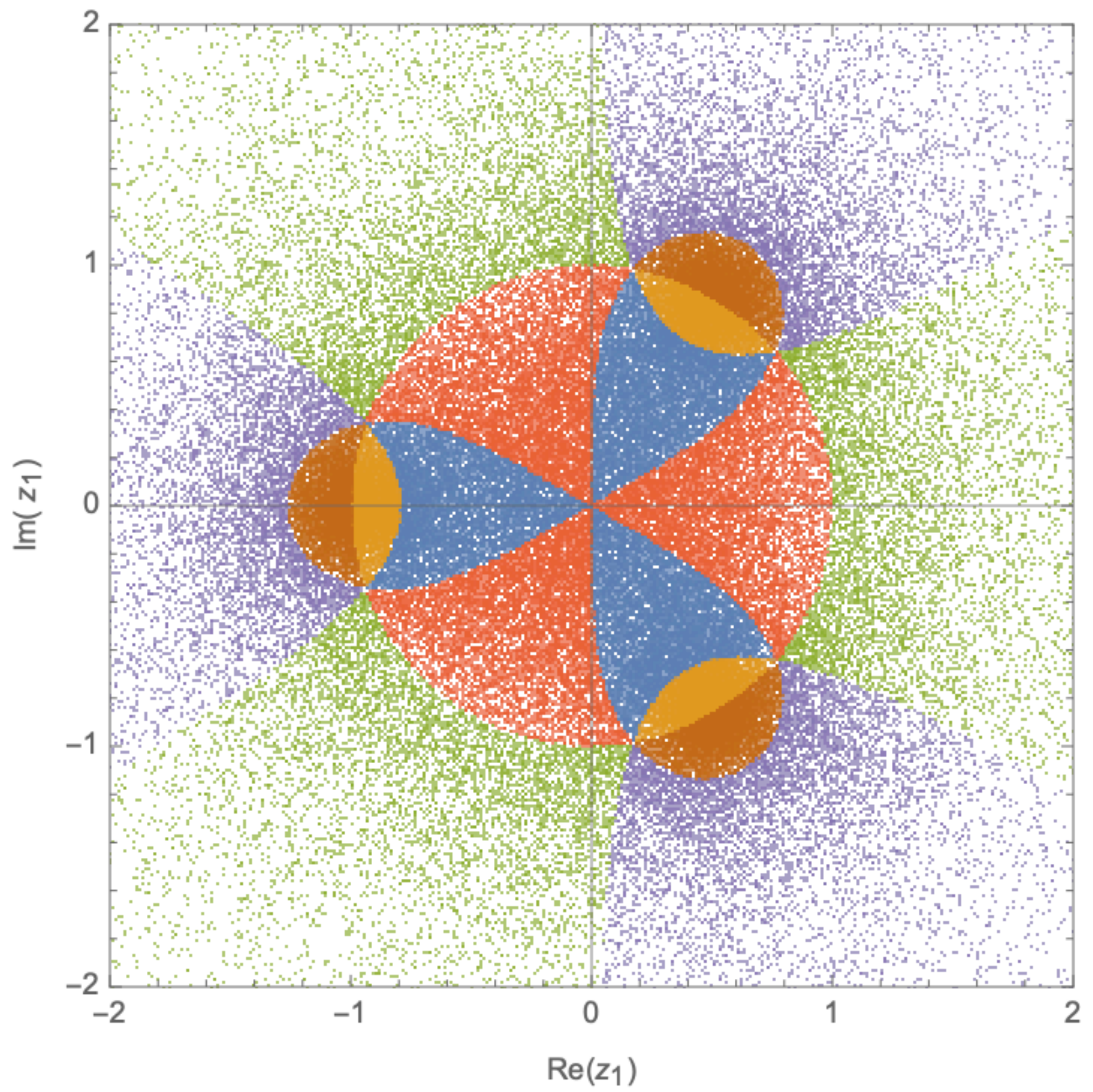
- Choose points on manifold from intersection of each line in the ambient space with the hypersurface



Braun, Brelidze, Douglas, Ovrut (2007)
Anderson, Braun, Karp, Ovrut (2010)
Ashmore, He, Ovrut (2019)

Points on Torus

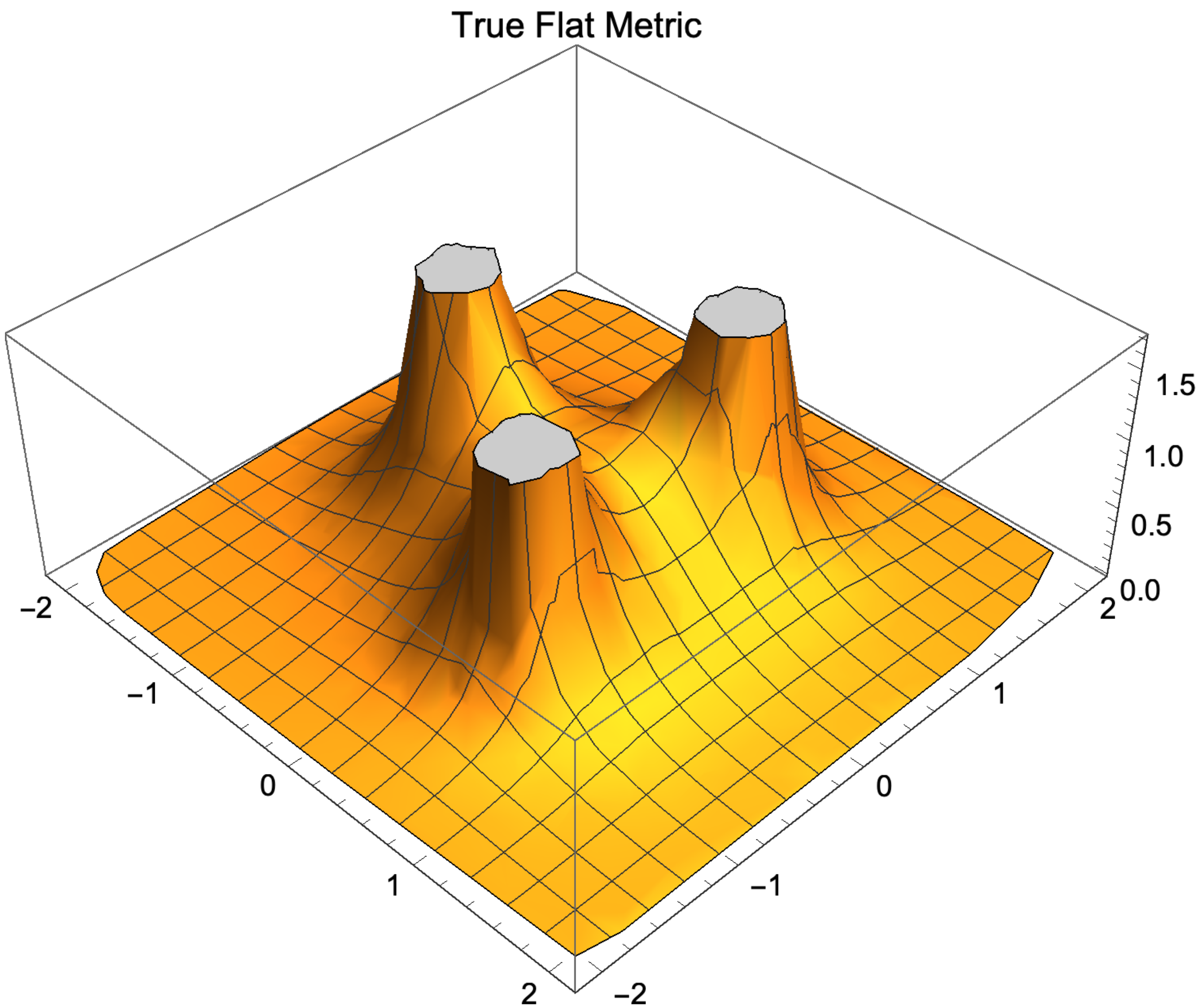


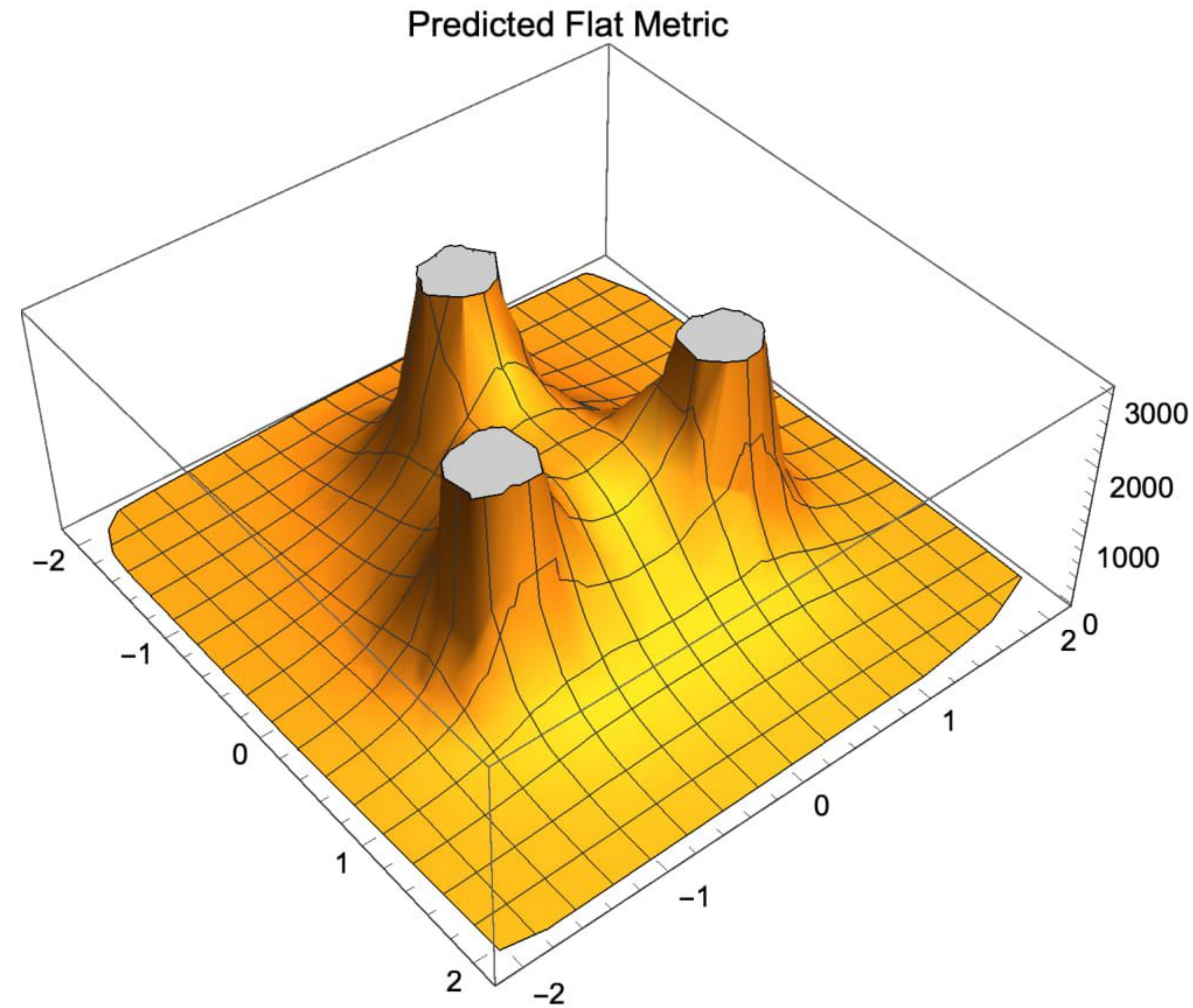


$z_m \neq 0$

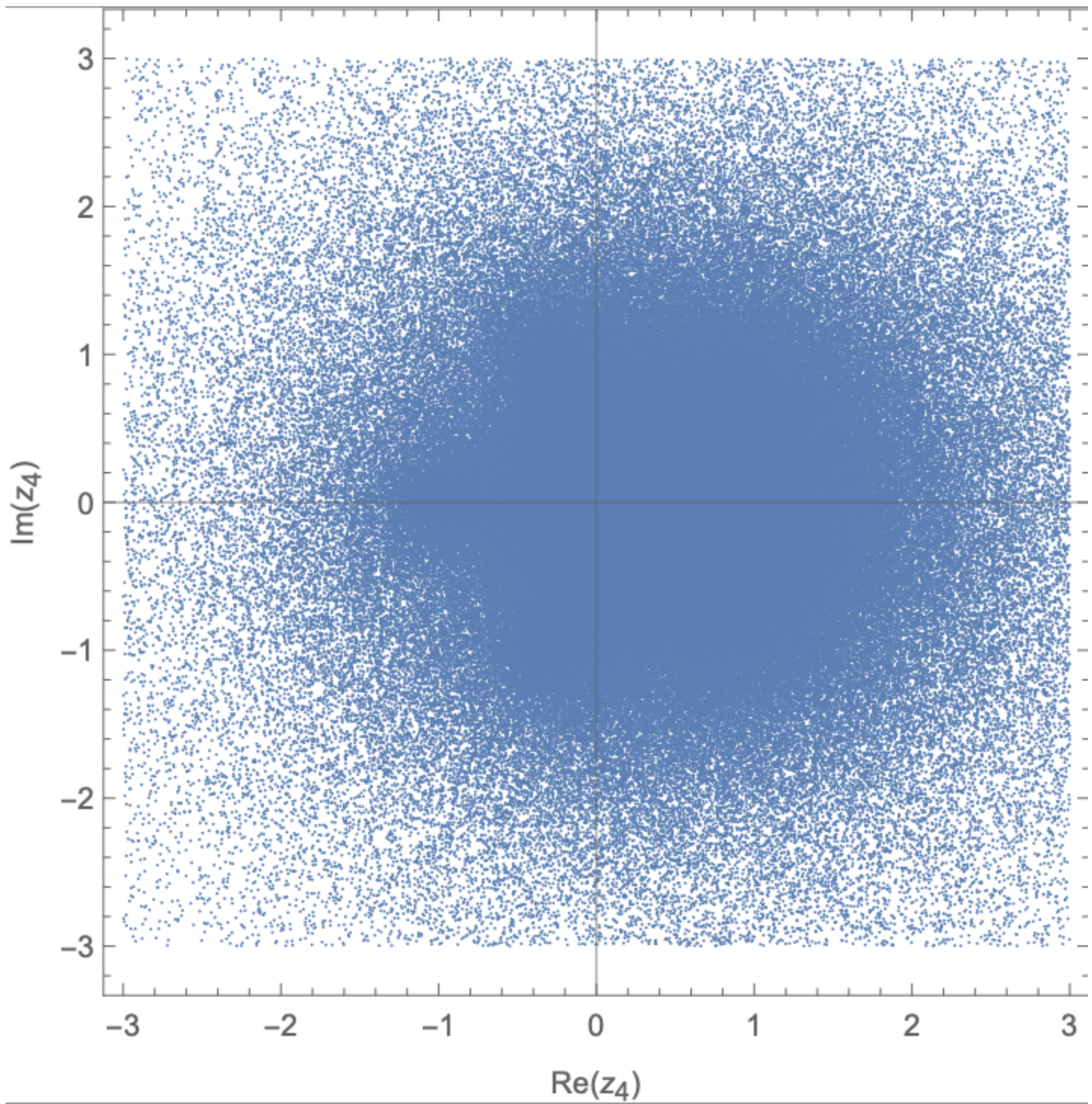
z_n dependent coordinate

Flat Metric

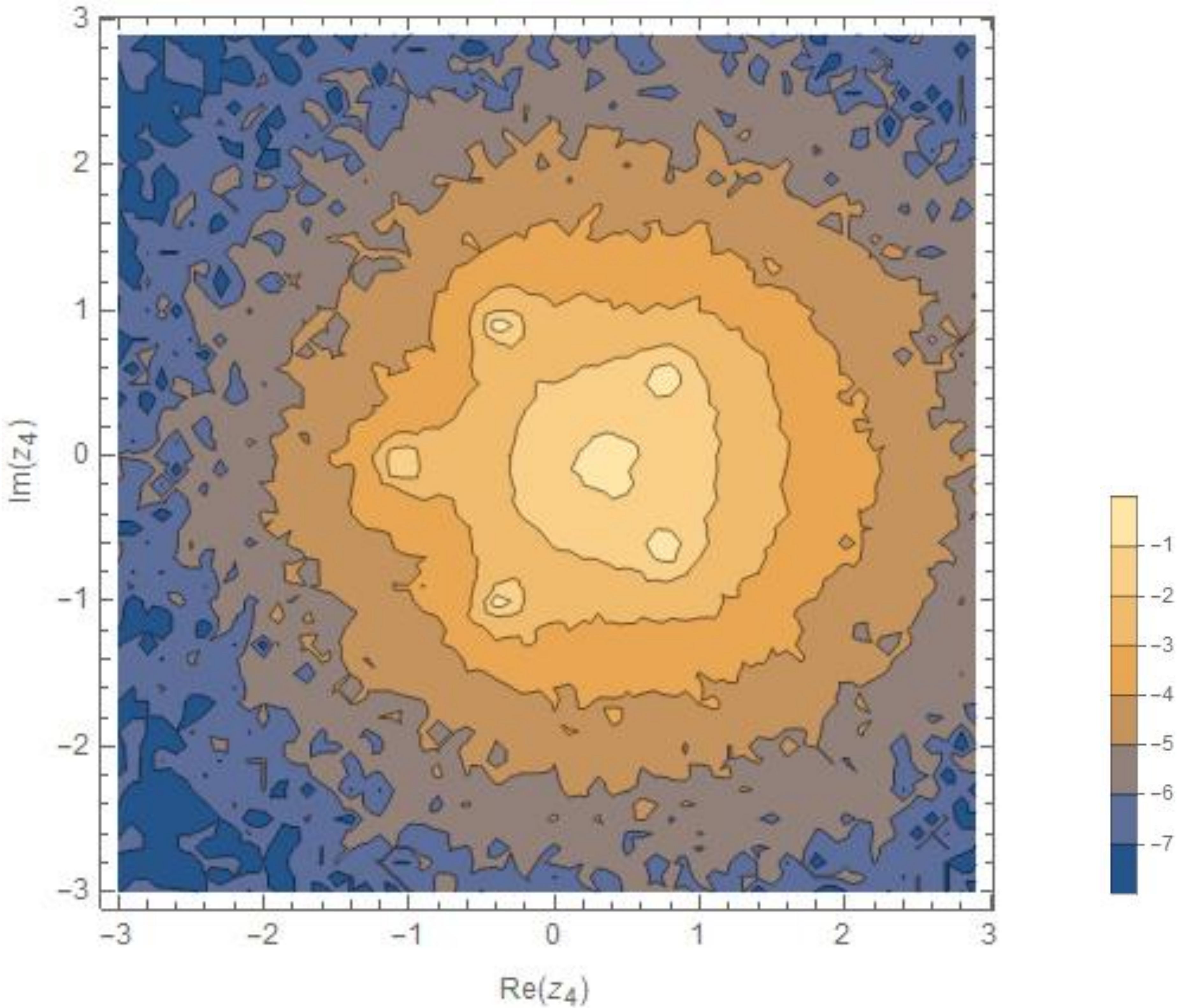

True Flat Metric
1.5
1.0
0.5
0.0
−2
−1
0
1
2
−2
−1
0
1
2


Predicted Flat Metric
3000
2000
1000
0
−2
−1
0
1
2
−2
−1
0
1
2

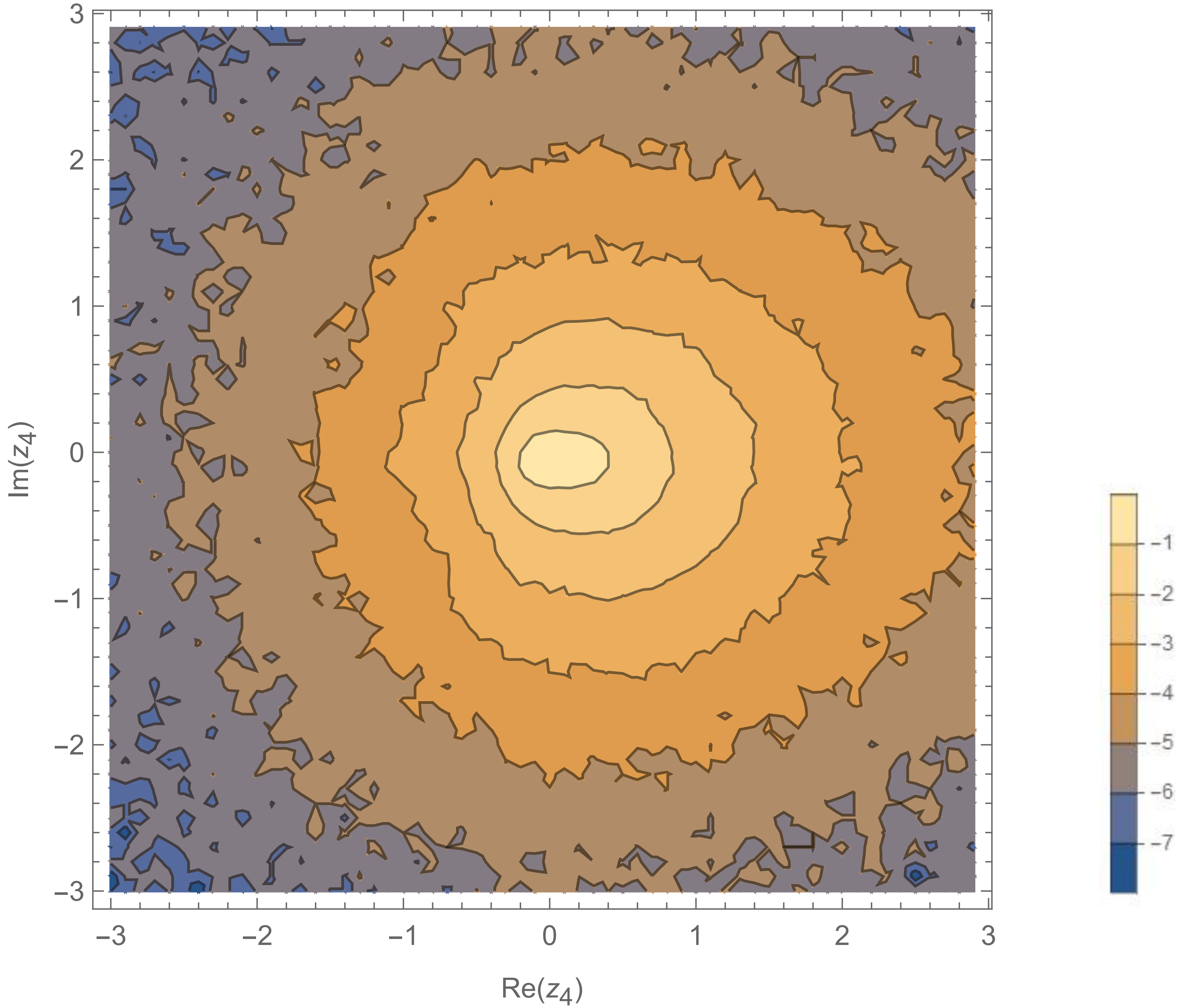
Points on Fermat Quintic



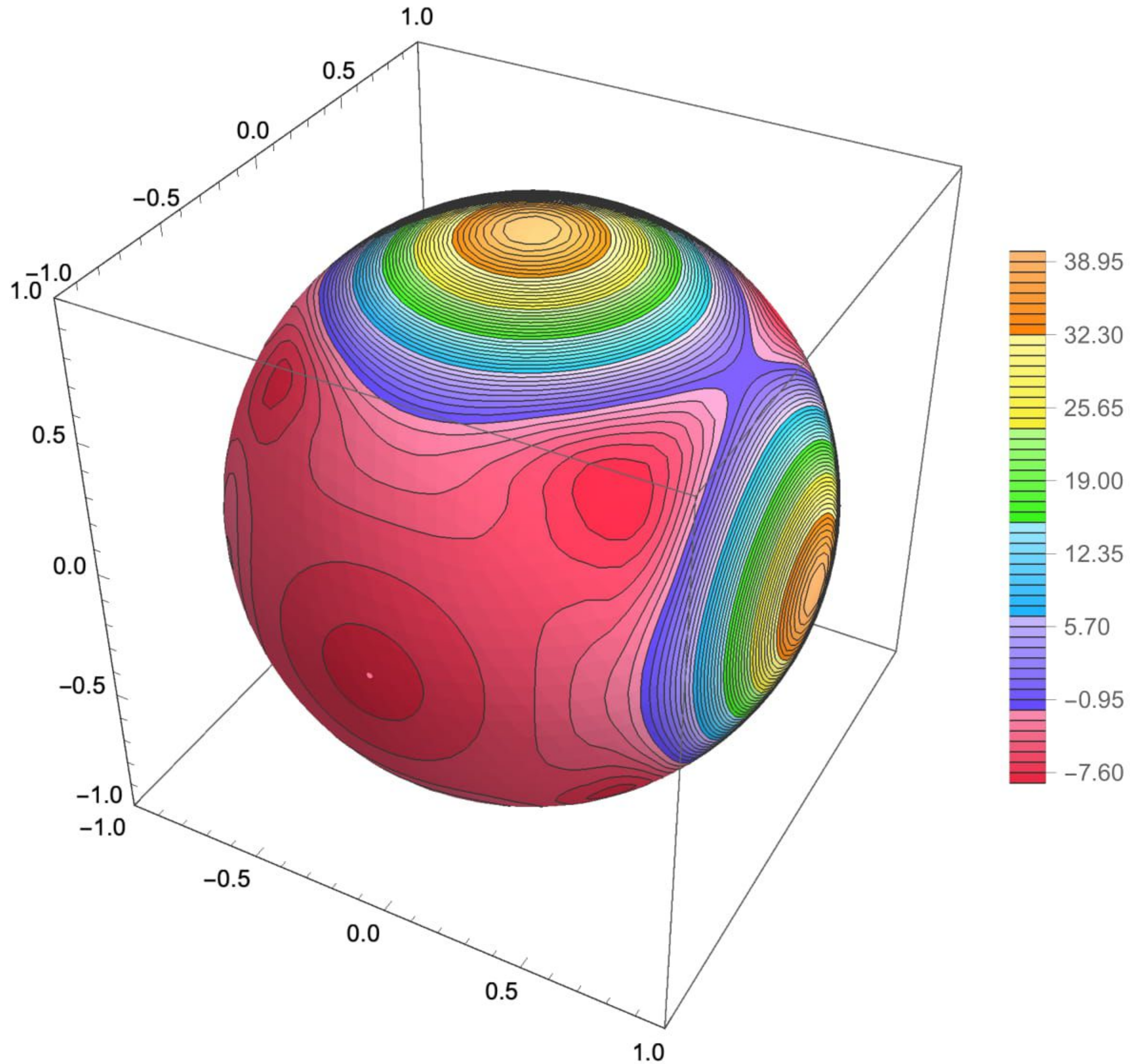
Points on Fermat Quintic



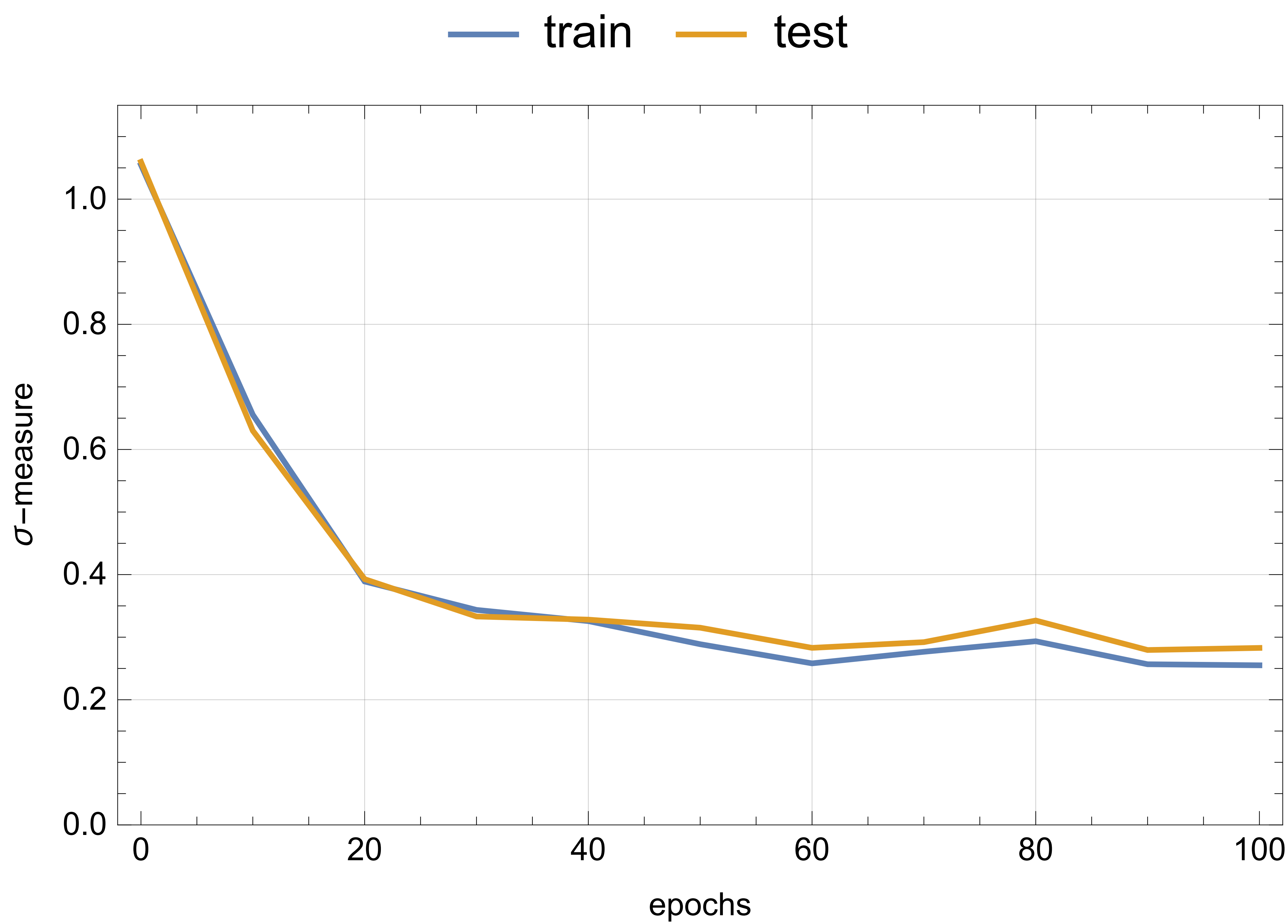
Points on Dwork Quintic



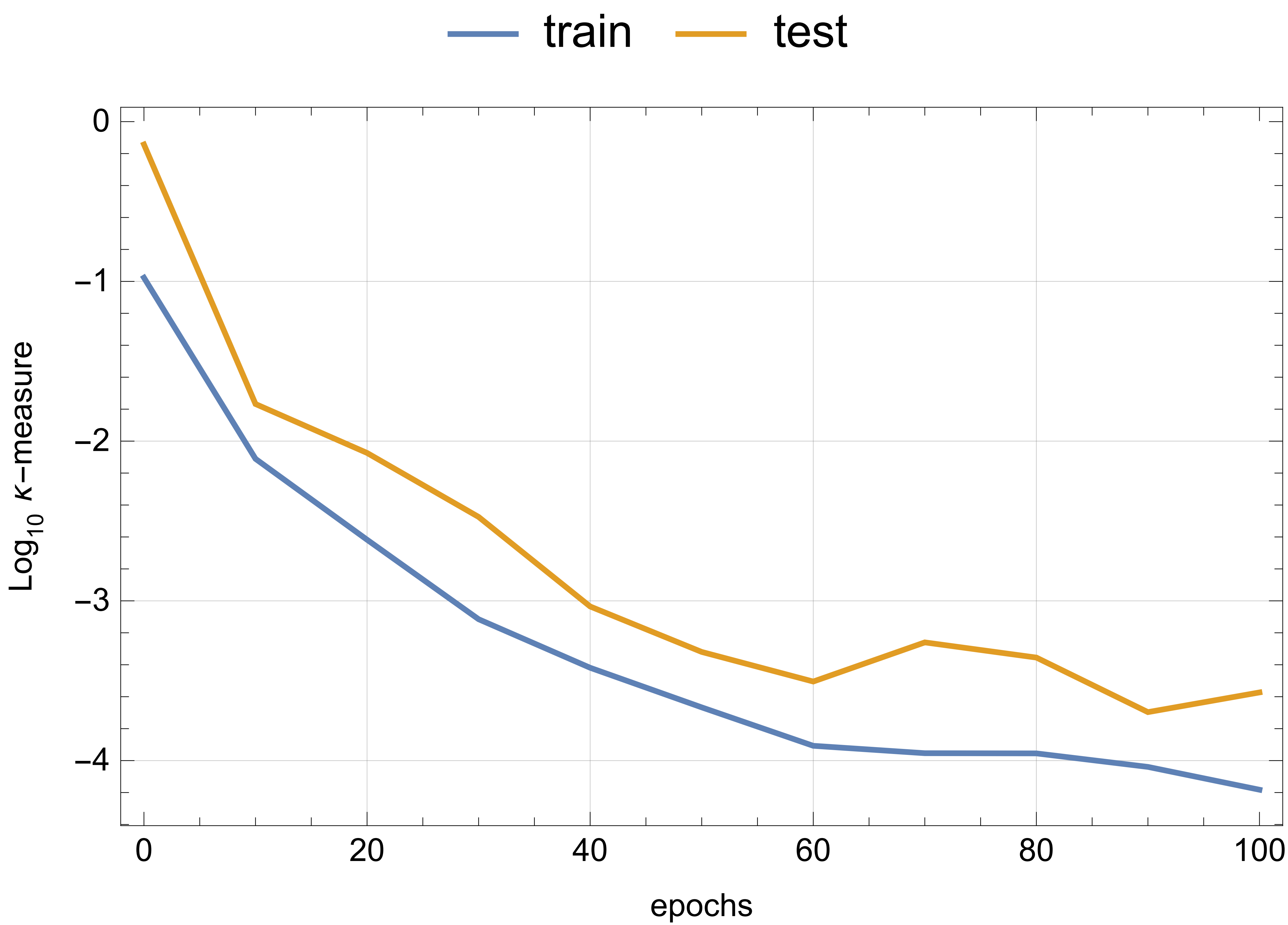
Curvature on Fermat Quintic



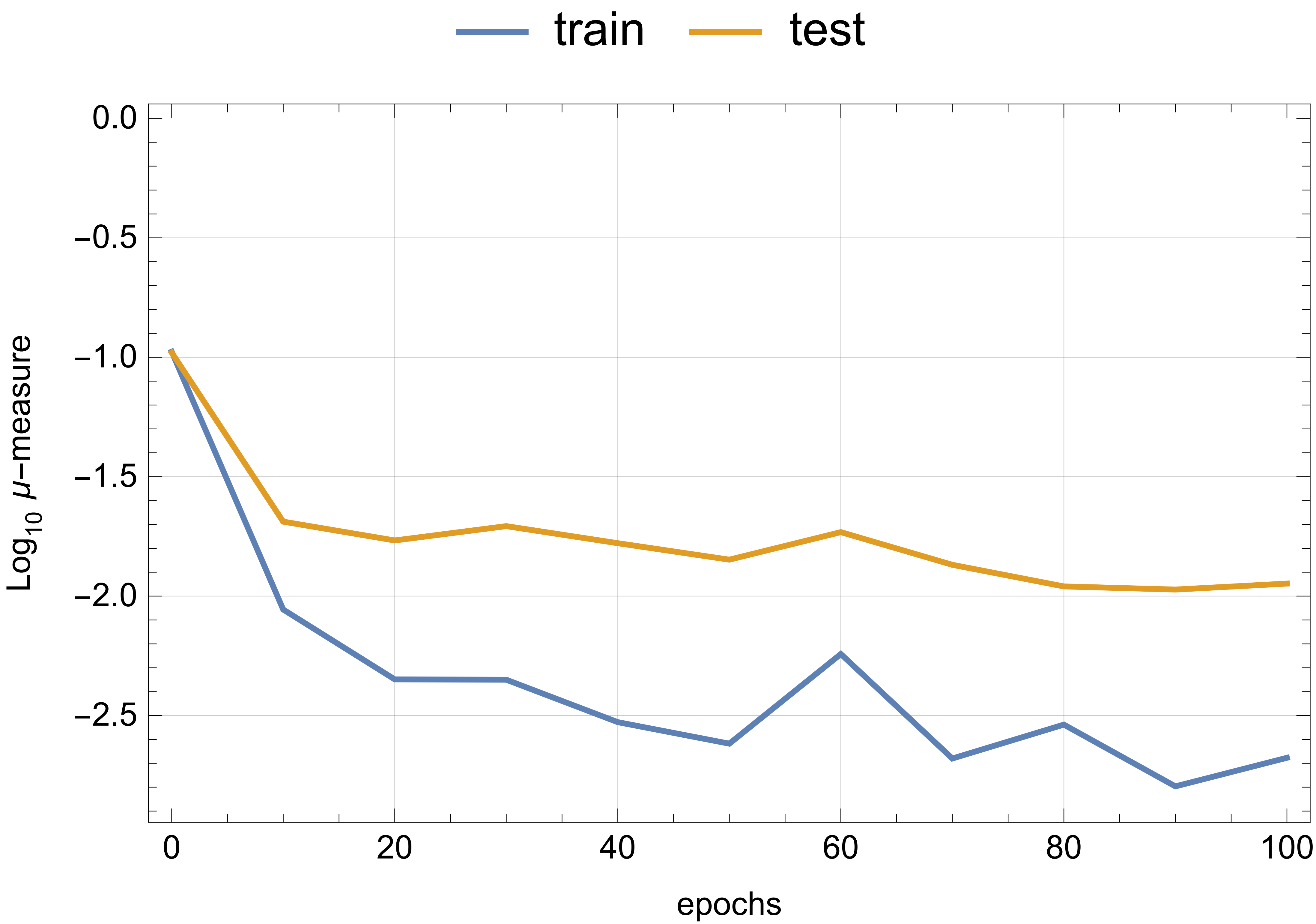
Fermat Quintic



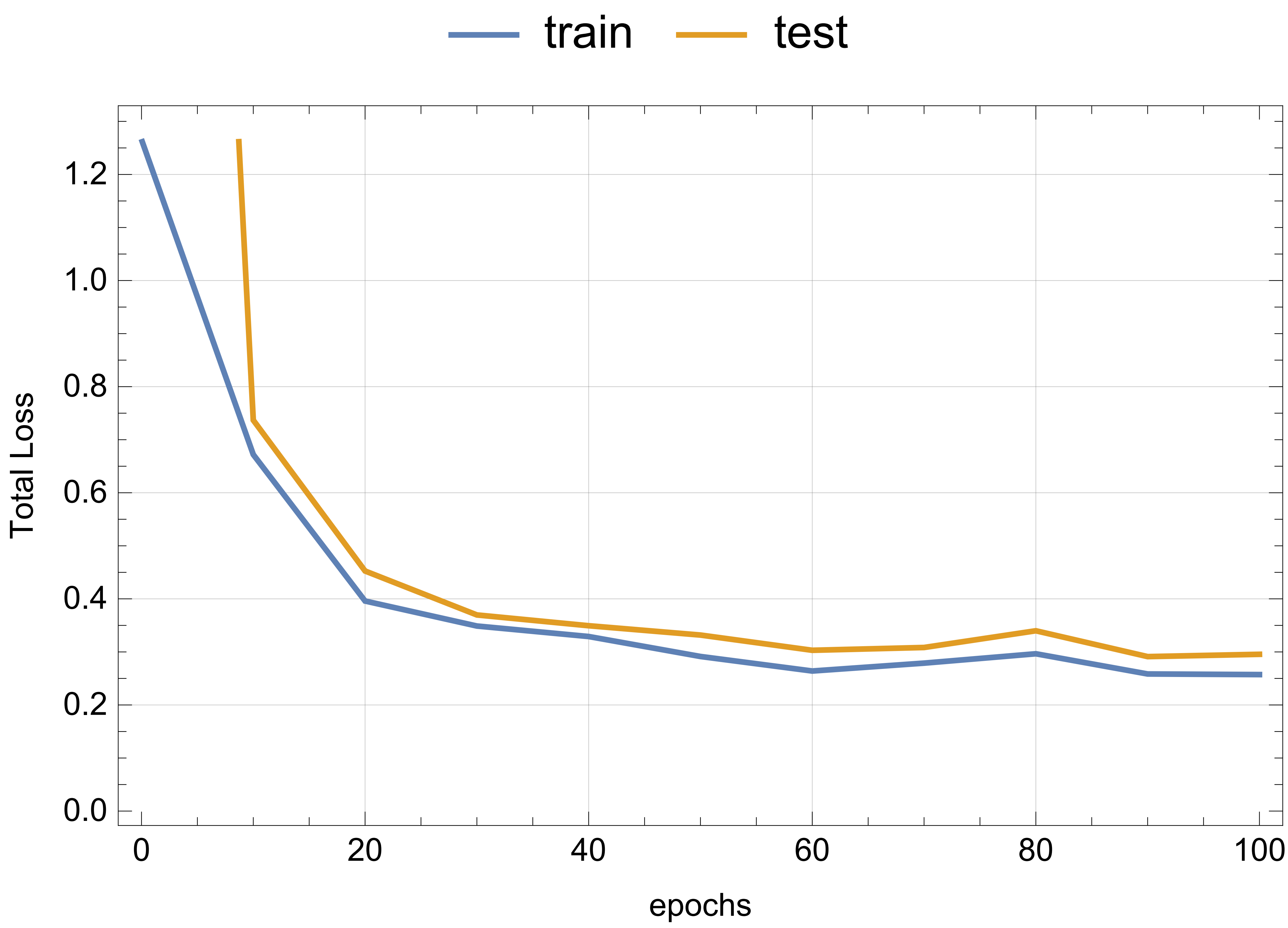
Fermat Quintic



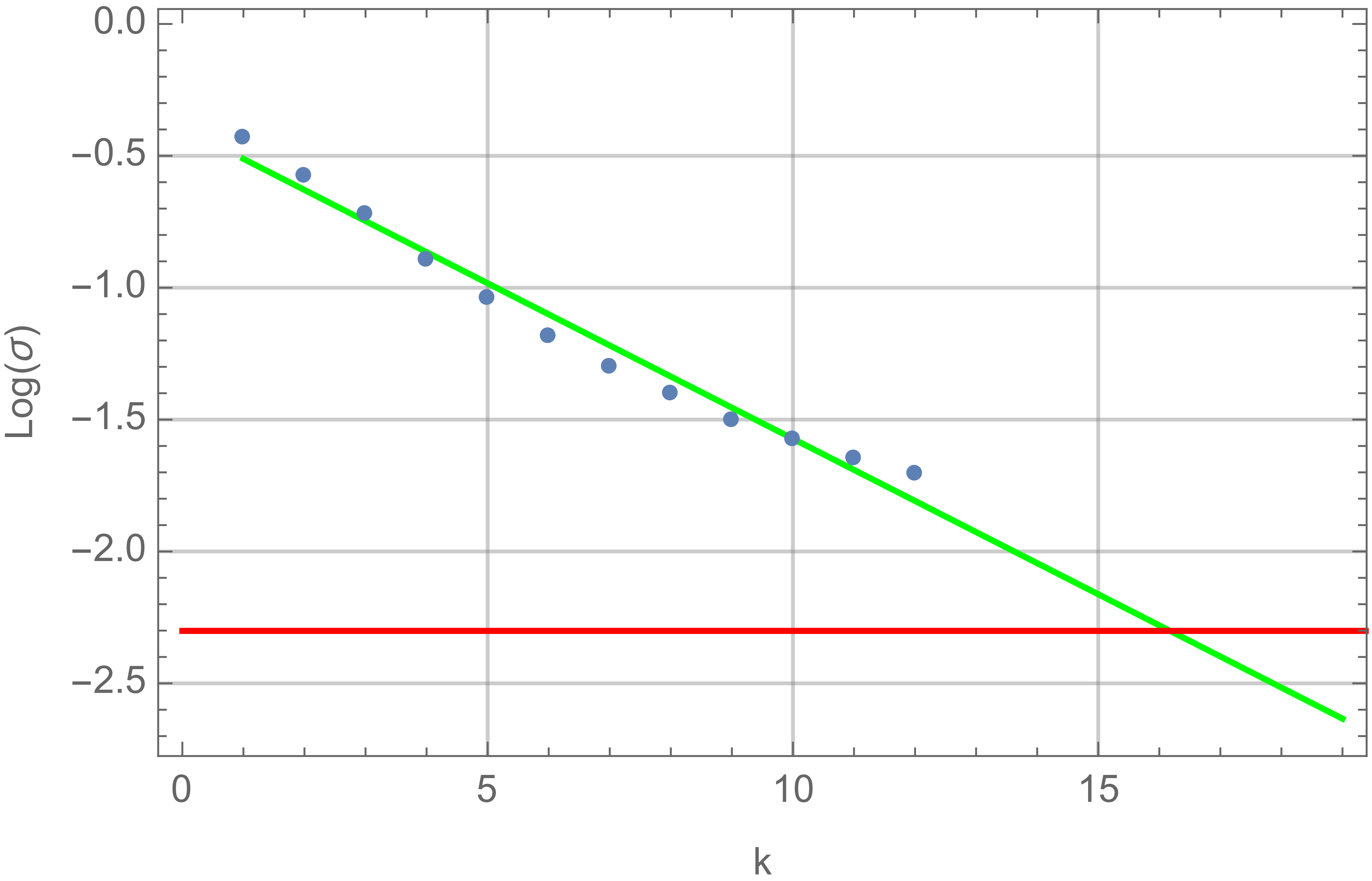
Fermat Quintic



Fermat Quintic

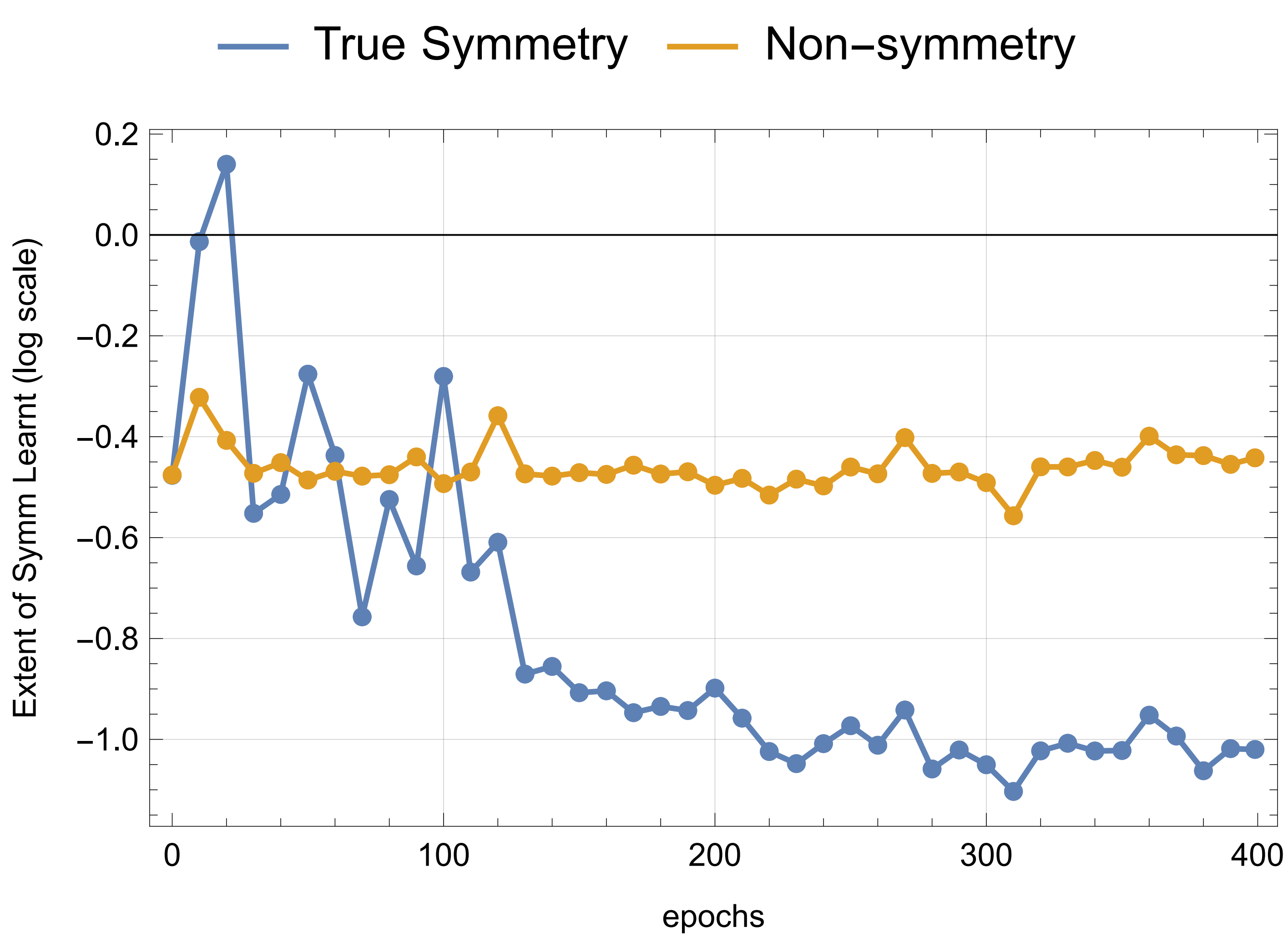


Comparison to Donaldson

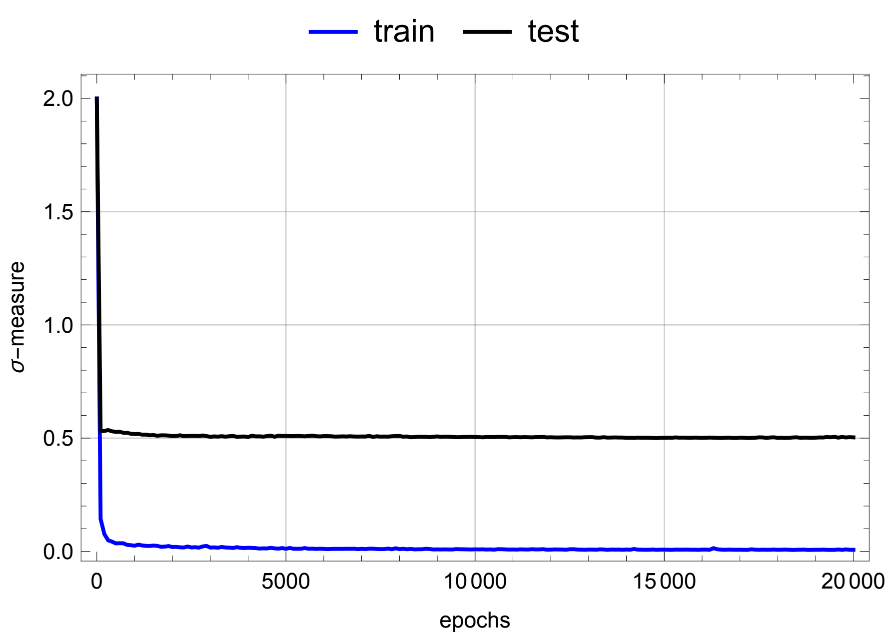


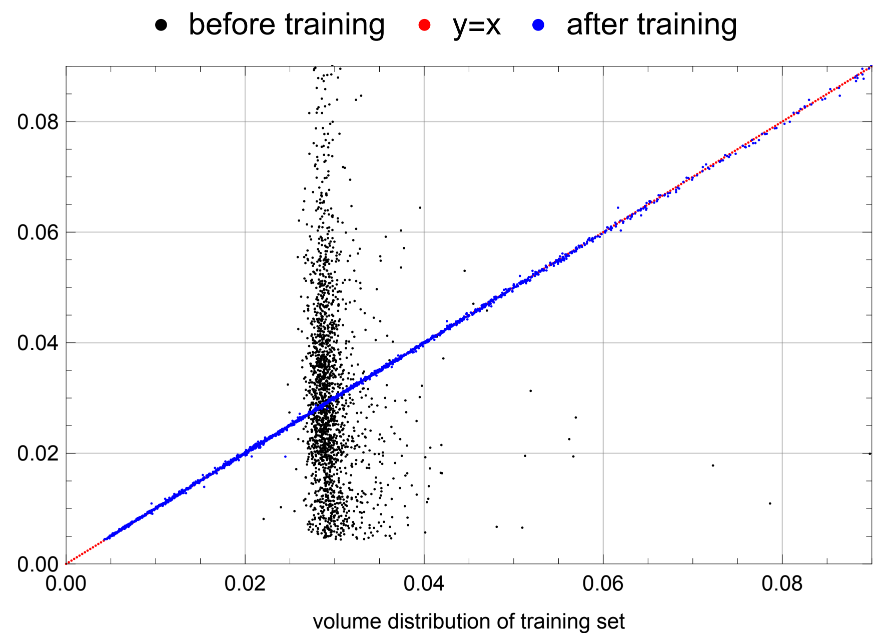
Discrete Symmetries

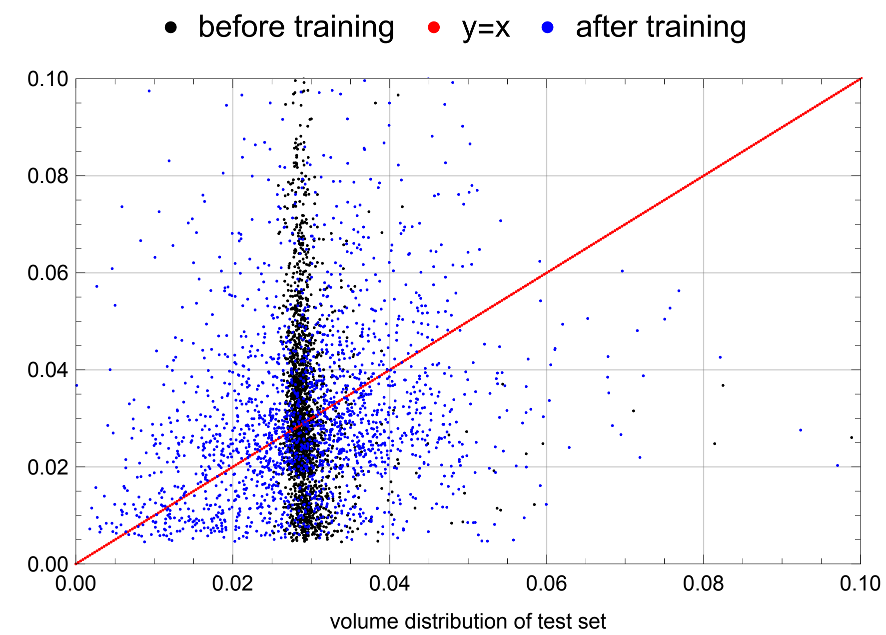
- Fermat quintic invariant under $\tau : z_i \mapsto \omega_5^i z_i \ , \quad \omega_5 = e^{2\pi i/5}$
- Measure using $\delta_n(\tau) := \frac{1}{N}\sum_z \left| \frac{g_{NN}(\theta_n; z) - g_{NN}(\theta_n; \tau \cdot z)}{g_{NN}(\theta_n; z)} \right|$
- Consistency check, but also learning while performing an auxiliary task



Tian–Yau







- σ-measure only

- $h^{1,1} = 14$; we don't have control of which Kähler class we land in

- Can augment by extra condition in loss function

Ricci Flow

- Every simply connected, closed three manifold is homeomorphic to S^3

Poincaré (1904)
Perelman (2002, 2003)

In fact, Perelman's result is more general: it proves geometrization conjecture — analogue of uniformization theorem in 2d

Thurston (1982)
Hamilton (1982-1999)

- Solved using Ricci flow with surgery

$$\frac{d}{d\lambda} g_{ab} = -2 \, \mathrm{Ric}_{ab}$$

fixed points are Ricci flat

- In fact, Perelman considered gradient flow equations associated to entropy functional

$$\mathcal{F} = \int_M dV \ e^{-f} \ (R + (\nabla f)^2)$$

- Ricci flow is like RG evolution of non-linear sigma model on string worldsheet with target space metric $g_{\mu\nu}$

Friedan (1980)

Kähler–Ricci Flow

- Equations of motion

$$\frac{\partial}{\partial\lambda} g_{a\bar{b}} = -(\mathrm{Ric}_{a\bar{b}} + \nabla_a \nabla_{\bar{b}} f) \longleftarrow \text{modified flow equation}$$

$$\frac{\partial}{\partial\lambda} f = -\Delta f - R \longleftarrow \text{backward heat equation}$$

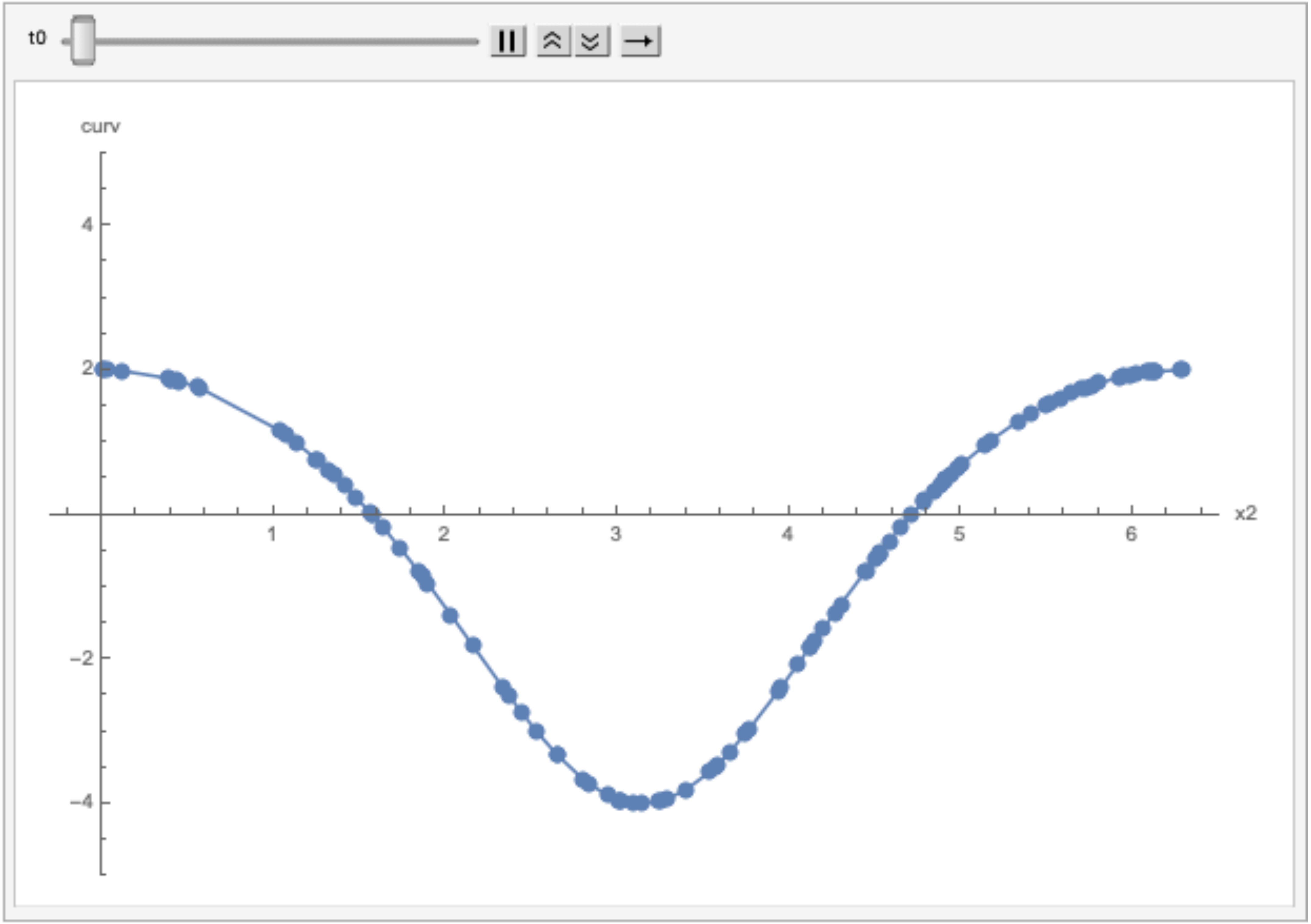
- Flow preserves Kähler class of metric

- Start with pullback of Fubini–Study metric and solve Ricci flow equations

- Loss function encodes equations of motion

- Preliminary results

Ricci Flow on Torus



Work in Progress

- How metric changes as a function of complex structure parameters in, for example, mirror quintic

- Tian–Yau, split bicubic, and other manifolds with $h^{1,1} > 1$

- Generalized CICYs

  Anderson, Apruzzi, Gao, Gray, Lee (2015)

  with Berglund, Hübsch

- Toric Calabi–Yau from Kreuzer–Skarke and G_2

  Altman, Gray, He, VJ, Nelson (2014)
  Demirtas, McAllister, Rios-Tascon (2020)

- Top down look at low energy effective action in 4d in realistic string compactification

STRING DATA 2021

13–17 December

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THANK YOU!