Numerical Metrics and the Swampland Distance Conjecture

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Motivation

How many string vacua are physically acceptable?

- Usually focus on getting correct gauge group, matter spectrum, superpotential, etc.

Normalised couplings? Other non-BPS information

- KK spectrum for swampland
- Spectrum of CFTs

Heterotic compactifications

Minimal SUSY on $\mathbb{R}^{1,3} \times X$ with gauge bundle V [Candelas et al. ‘85]

- No H flux $\Rightarrow$ X is Calabi–Yau, V admits HYM connection

Physics in 4d determined by geometry of X – Kaluza–Klein reduction fixes 4d modes

- e.g. for KK scalars, masses in 4d c.f. eigenvalues of Laplacian in 6d

$$\Delta\phi_6 = \lambda\phi_6 \quad \Rightarrow \quad \Box_4\zeta_4 = \lambda\zeta_4 \equiv m^2\zeta_4$$

- Zero modes determine low-energy physics, e.g. matter fields c.f. $H^1(X, V_R)$ with harmonic basis $\{\psi_a\}$

Outline

Numerical Calabi–Yau metrics

The spectrum of the Laplacian

Application: the finite SDC

Numerical Calabi–Yau metrics

Calabi–Yau basics

Calabi–Yau manifolds are Kähler manifolds with a Ricci-flat metric

- Existence but no explicit constructions

Kähler $\Rightarrow$ Kähler potential K gives (real) closed two-form $J = \partial\bar{\partial}K$

$c_1(X) = 0 \Rightarrow$ (complex) nowhere-vanishing closed (3,0)-form Ω

$$J^3 = \text{vol}_J, \qquad |\Omega|^2 = \text{vol}_\Omega.$$

Example: Dwork quintics

Quintic hypersurface Q_ψ in $\mathbb{P}^4$

$$Q_\psi(z) \equiv z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi\, z_0 z_1 z_2 z_3 z_4 = 0$$

(3,0)-form Ω determined by Q_ψ, e.g. in $z_0 = 1$ patch

$$\Omega = \frac{\mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4}{\partial Q_\psi / \partial z_1}$$

Metric g (and J) completely determined by Kähler potential

$$g_{i\bar{j}}(z,\bar{z}) = \partial_i \bar{\partial}_{\bar{j}} K(z,\bar{z}), \qquad \mathrm{vol}_J \sim \det g_{i\bar{j}} \mathrm{d}^6 z$$

How do we measure accuracy?

The Ricci-flat metric is given by a K that satisfies (c.f. Monge–Ampère)

$$\left.\frac{\text{vol}_J}{\text{vol}_\Omega}\right|_p = \text{constant} \quad \Rightarrow \quad R_{i\bar{j}} = 0$$

Define a functional of K

$$\sigma(K) = \int_X \left|1 - \frac{\text{vol}_J}{\text{vol}_\Omega}\right| \text{vol}_\Omega$$

The exact CY metric has $\sigma = 0$

Finding the Ricci-flat metric reduces to finding a single function $K(z, \bar{z})$ that minimises σ

Algebraic metrics

Generalise by replacing coordinates z_i with homogeneous polynomials s_α of degree k

$$\text{e.g.} \quad k = 2: \quad s_\alpha = (z_0^2, z_0 z_1, z_0 z_2, \ldots)$$

Kähler potential is then

$$K(h) = \log \sum_{\alpha,\bar\beta=0}^{14} s_\alpha h^{\alpha\bar\beta} \bar s_{\bar\beta}, \qquad h^{\alpha\bar\beta} \sim 225 \text{ parameters}$$

At degree k have $\mathcal{O}(k^4)$ parameters, so can approximate the Ricci-flat metric to arbitrary precision

- Algebraic metrics [Tian '90] – higher k allows better precision (smaller σ)

How to fix $h^{\alpha\bar{\beta}}$?

Finding the “best” approximation to the Ricci-flat metric amounts to finding $h^{\alpha\bar{\beta}}$ so that σ is minimised

Three approaches:

- Iterative procedure [Donaldson ‘05; Douglas ‘06; Braun ‘07]
- Minimise σ directly [Headrick, Nassar ‘09]
- Treat σ as a loss function for a neural network [Douglas et al. 20; Anderson et al. ‘20]

One can also try to find $g_{i\bar{j}}$ directly (need to impose Kählerity, overlap conditions, etc.) [Anderson et al. ‘20; Jejjala ‘20]

In all cases, numerical integrals carried out by Monte Carlo

The spectrum of the Laplacian

The (p, q) Laplacian [Braun et al. ‘08, AA ‘20]

Important phenomenological details of models determined by eigenmodes on CY

Without gauge sector, eigenmodes are (p, q)-eigenforms of the Laplacian

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d}, \qquad \Delta|\phi_n\rangle = \lambda_n|\phi_n\rangle$$

where λ_n are real and non-negative and appear with multiplicity μ_n (c.f. continuous or finite symmetries)

- Need some way of computing both the spectrum and the harmonic modes themselves

The (p,q) Laplacian

For each (p,q), given a (non-orthonormal) basis of (p,q)-forms $\{\alpha_{p,q}\}$, we can expand the eigenmodes as

$$|\phi\rangle = \sum_A \langle \alpha_A|\tilde{\phi}\rangle\,|\alpha_A\rangle, \qquad A = 1,\ldots,\dim\{\alpha_{p,q}\}$$

so that $\Delta|\phi\rangle = \lambda|\phi\rangle$ becomes generalised eigenvalue problem for λ and $\tilde{\phi}_A$

$$\begin{aligned} \langle \alpha_A|\Delta|\alpha_B\rangle\langle \alpha_B|\tilde{\phi}\rangle &= \lambda\langle \alpha_A|\alpha_B\rangle\langle \alpha_B|\tilde{\phi}\rangle \\ \Rightarrow \quad \Delta_{AB}\tilde{\phi}_B &= \lambda O_{AB}\tilde{\phi}_B \end{aligned}$$

where

$$O_{AB} = \int \alpha_A \wedge \star\bar{\alpha}_B, \quad \text{etc.}$$

The (p, q) Laplacian

Basis $\{\alpha_{p,q}\}$ is infinite dimensional – truncate to a finite approximate basis at degree k_ϕ in z_i

$$\{\alpha_{p,q}\} = \frac{(\text{degree } k_\phi \ (p,0)\text{-form})(\text{degree } k_\phi \ (0,q)\text{-form})}{(|z_0|^2 + \ldots |z_4|^2)^{k_\phi}}$$

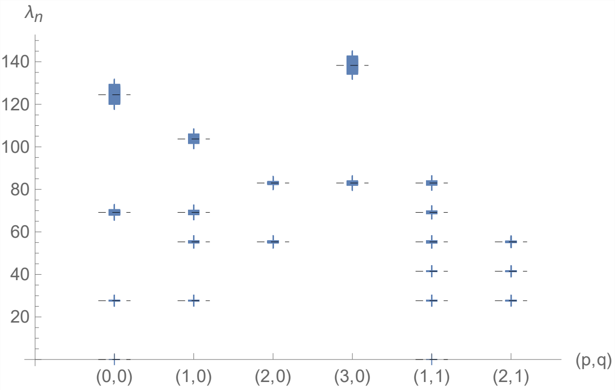
where we have (c.f. harmonic forms on $\mathbb{P}^4$)

$$\{\text{degree } k_\phi \ (0,0)\text{-form}\} = \text{degree } k_\phi \text{ polynomials}$$
$$\{\text{degree } 2 \ (1,0)\text{-form}\} = \{z_0 \mathrm{d}z_1 - z_1 \mathrm{d}z_0, z_0 \mathrm{d}z_2 - z_2 \mathrm{d}z_0, \ldots\}$$

1. Compute matrices Δ_{AB} and O_{AB} numerically for independent choices of (p, q)
2. Find eigenvalues and eigenvectors

Toy example: $\mathbb{P}^3$ with FS metric

Spectrum on $\mathbb{P}^3$ at $k_\phi = 3$ with exact metric and $N = 10^6$ points



- Eigenvalues and multiplicities determined by SU(4) irreps
- SU(4) symmetry recovered as number of integration points $\to \infty$

Fermat quintic

Fermat quintic with $N = 3 \times 10^6$ and $k_\phi = 3$

$$Q_{\psi=0}(z) \equiv z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$$

(p,q)	$(0,0)$		$(1,0)$		$(2,0)$		$(3,0)$		$(1,1)$		$(2,1)$	
$\dim\{\alpha_{p,q}\}$	1225		1400		350		350		1600		400	
n	λ_n	μ_n	λ_n	μ_n	λ_n	μ_n	λ_n	μ_n	λ_n	μ_n	λ_n	μ_n
0	0.00	1	43.2 ± 0.1	20	76.2 ± 0.2	30	45.3	1	7.0	1	56.4 ± 0.1	20
1	41.1 ± 0.2	20	67.0 ± 0.2	30	78.1 ± 0.2	30	97.9 ± 0.3	20	50.4 ± 0.2	50*	59.2 ± 0.2	20
2	78.7 ± 0.3	20	73.3 ± 0.2	30	82.0 ± 0.1	20	102 ± 0.3	20	56.2 ± 0.2	20	70.5 ± 0.2	30
3	84.5 ± 0.1	4	84.6 ± 0.2	34*	94.5 ± 0.2	20	116 ± 0.1	4	82.5 ± 0.2	60	92.3 ± 0.4	60
4	94.6 ± 0.6	60	96.0 ± 0.2	20	115 ± 0.3	40	127 ± 0.5	30	84.2 ± 0.3	120	99.7 ± 0.2	4
5	101 ± 1	30	99.6 ± 0.3	60	122 ± 0.3	30	142 ± 0.6	30	85.2 ± 0.2	60	111 ± 0.4	40

- Multiplicities = dimension of irreps of $(S_5 \times \mathbb{Z}_2) \ltimes (\mathbb{Z}_5)^4$
- Ω and J should give zero modes for $(3,0)$ and $(1,1)$ – improves as size of approximate basis is increased

“Typical” properties

What are the “typical” or average properties of a Calabi–Yau as a function of moduli?

- Spectral gap λ_1?
- Statistics of full spectrum?

Important for understanding landscape of string compactifications and properties of CFTs!

- Without fixing moduli, what kind of physics is possible
- Low-energy modes of CFT captured by spectrum of Δ on target space

Ensembles of CYs [Afkhami-Jeddi, AA, Córdova]

Generic quintic threefold given by quintic equation in $\mathbb{P}^4$

$$Q \equiv \sum_{m,n,p,q,r} c_{mnpqr} z_m z_n z_p z_q z_r = 0$$

101 complex structure parameters

Choose the c_{mnpqr} randomly from disk in complex plane

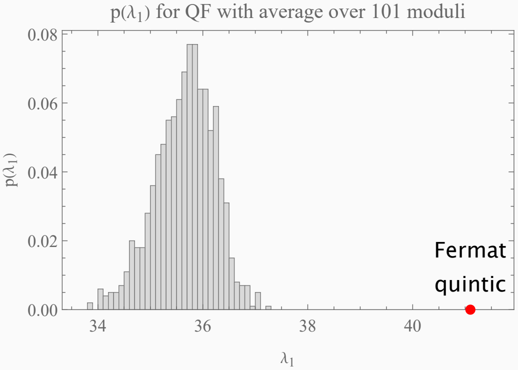
$$c_{mnpqr} \in \mathbb{C}, \qquad |c_{mnpqr}| < 1$$

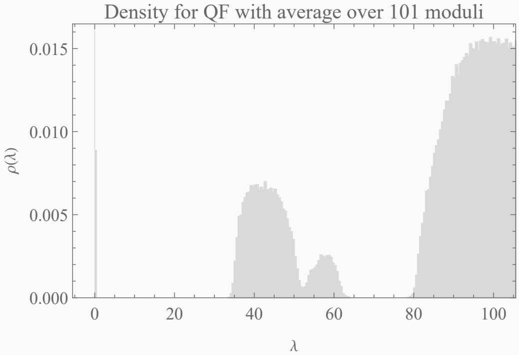
Then compute the approximate CY metric and the spectrum of the scalar Laplacian for each instance

Distribution of eigenvalues

Compute metric and scalar spectrum for 1000 samples

- Gives statistical non-BPS CY "data"
- Mean gap $\langle \lambda_1 \rangle = 35.7$





Application: the finite SDC

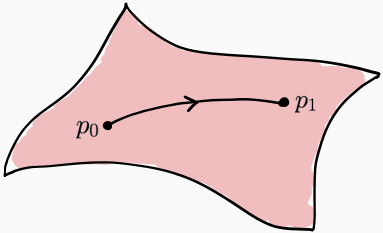
The Swampland Distance Conjecture [Ooguri, Vafa '06]

For a theory coupled to gravity with a moduli space parametrised by vevs of some fields, moving an infinite distance in moduli space brings down a tower of massless states that spoil the initial effective theory.

Finite version: there is a relation between the mass of a tower of states and the distance you move in moduli space

- Expectation: trans-Planckian distance $\rightarrow$ tower of exponentially light states

Compare the effective theory at two points p_0 and p_1 in moduli space which are a geodesic distance $d(p_0, p_1)$ apart



Geodesic γ with $\gamma(\tau_i) = p_i$ with distance along curve given by

$$d(p_0, p_1) = \int_{\tau_1}^{\tau_2} \sqrt{G_{ab}\dot{\gamma}^a\dot{\gamma}^b}$$

where G_{ab} is metric on moduli space

Conjecture implies that a tower of states becomes light on moving from p_0 to p_1 with masses of the order

$$m(p_1) \sim m(p_0)\mathrm{e}^{-\alpha\, d(p_0,p_1)}$$

where α is order one

A check of the SDC for complex structure [AA, Ruehle '21]

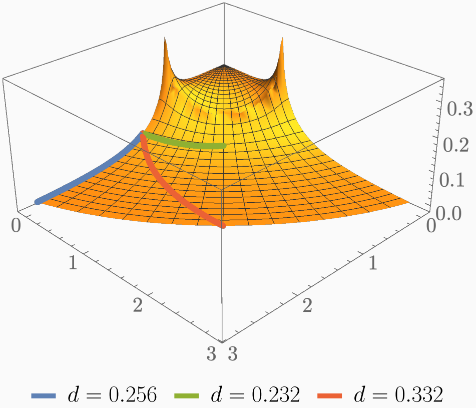
We can compute the spectrum of massive KK modes numerically and check the conjecture for this tower of states on the quintic Q_ψ

Focus on complex structure moduli space $\psi \in [2, 1000]$

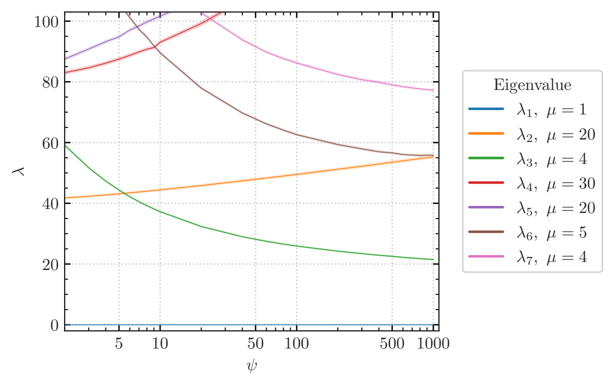
1. Compute the moduli space metric
   [Candelas et al. '98; Keller, Lukic '09]

$$e^{-K_{cs}} = i \int_{Q_\psi} \Omega \wedge \bar{\Omega}, \quad G_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K_{cs}$$

2. Compute geodesics and distances in moduli space
3. Compute the numerical CY metric and the spectrum of the Laplacian – focus on scalar spectrum

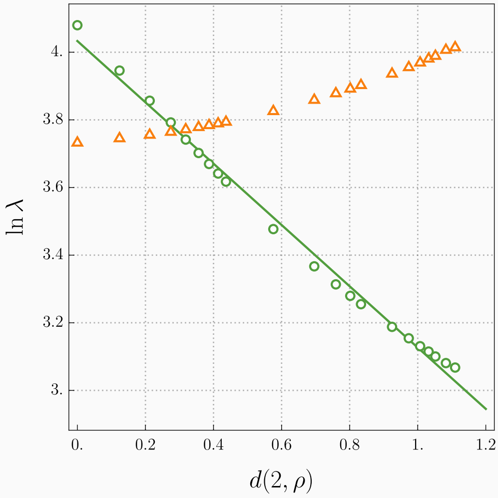


Scalar spectrum with varying ψ



1. Zero mode always present; massive modes appear with multiplicities given by irreps of $(S_5 \times \mathbb{Z}_2) \ltimes (\mathbb{Z}_5)^3$
2. Eigenvalues with small degeneracy become lighter
3. Local maximum in spectral gap for $\psi \approx 4$

A check of the SDC



1. λ_2 falls exponentially as

$$\lambda_2 = 56.4\, \mathrm{e}^{-(0.906 \pm 0.034)\, d(2,\rho)}$$
$$\sim \mathrm{e}^{-2\alpha\, d(p_0, p_1)}$$

2. We see $\alpha \approx 0.45$ which is indeed order one
3. Almost saturates lower bound of $1/\sqrt{6} \approx 0.41$ of [Andriot et al. '20]

Summary and outlook

Calabi–Yau metrics are accessible with numerical methods

Spectrum is source of interesting non-BPS "data" with uses in CFTs, swampland, etc.

- Include gauge fields
  - Works for bundles too and can diagnose stability [Douglas et al. '06; Anderson et al. '10; Anderson et al. '11]
  - Yukawa couplings?
- CY threefolds appear as target spaces for $(2,2)$ SCFTs
  - Spectrum of CY $\subset$ spectrum of CFT operators
  - Input for conformal bootstrap? [Lin et al. '15; Lin et al. '16]
- Typical compactifications? Distribution of Yukawa couplings? Crossing in eigenvalues? SYZ conjecture?

Thank you!