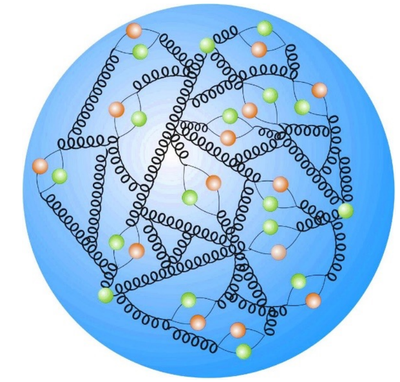
Scaling dimensions of fixed charge operators

Oleg Antipin

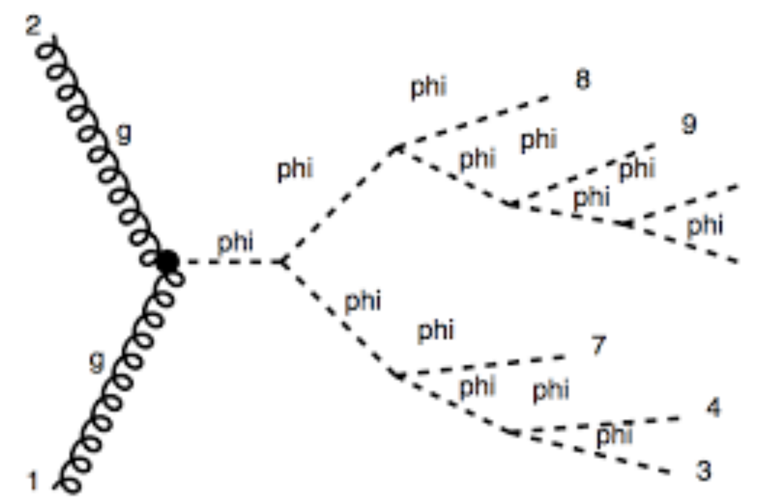
Portoroz 2021

Towards Solving QFT



Strongly-coupled QFT
Non-perturbative physics

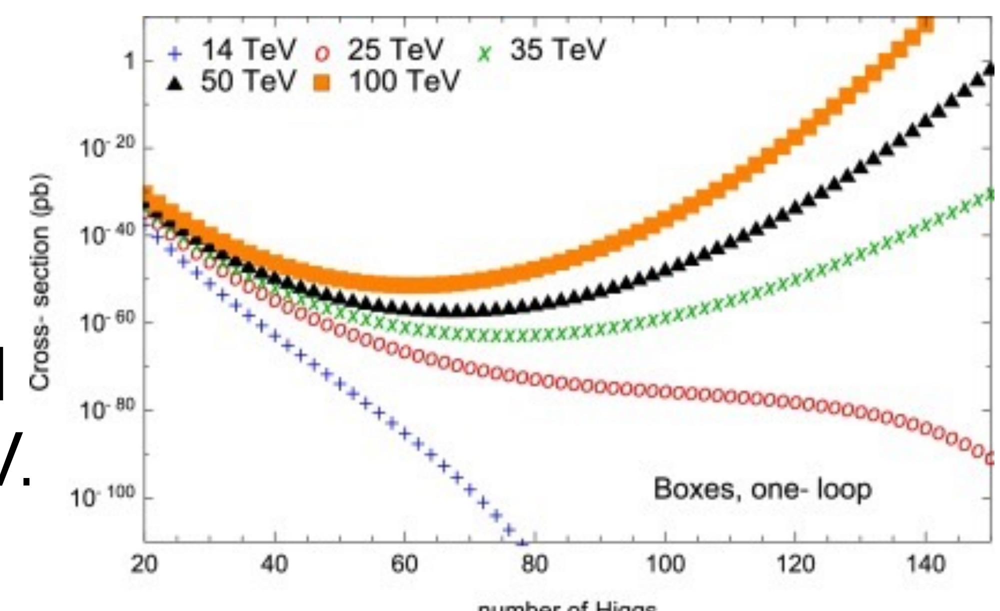
Weakly-coupled QFT
Perturbative expansion: { **Many-loops** / **Many-legs** }



Rapid growth of the number of Feynman diagrams, with the number of loops/ external legs

The perturbative expansion diverges factorially.

Possible violation of perturbative unitarity in SM multi-boson production processes at E≈100 TeV.



[Degrande, Khoze, Mattelaer, 2016]

→ Semiclassical approach is useful

Perturbative loop expansion: semiclassical approach

Consider the two-point function in the $U(1)$ complex scalar model

$$\mathcal{S} = \int d^4x \left[\partial\bar{\phi}\partial\phi + \frac{\lambda_0}{4}\left(\bar{\phi}\phi\right)^2\right]$$

Rescale the field as $\phi \to \phi/\sqrt{\lambda_0}$:

$$\langle\bar{\phi}(x_f)\phi(x_i)\rangle \equiv \frac{\int D\phi D\bar{\phi}\,\bar{\phi}(x_f)\phi(x_i)e^{-S}}{\int D\phi D\bar{\phi}\,e^{-S}} = \frac{1}{\lambda_0}\frac{\int D\phi D\bar{\phi}\,\bar{\phi}(x_f)\phi(x_i)e^{-\frac{S}{\lambda_0}}}{\int D\phi D\bar{\phi}\,e^{-\frac{S}{\lambda_0}}}$$

Ordinary loop expansion with λ_0 the loop counting parameter. For $\lambda_0 \ll 1$ the path integral is dominated by the extrema of S.

Evaluate via a saddle point expansion by expanding the action around the stationary configuration $\phi_0 = 0$

$$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \ldots$$

ϕ_0 is the solution of the classical EOM

Large charge expansion: Semiclassical approach

Rattazzi et al '19

The operator ϕ ($\bar{\phi}$) carries $U(1)$ charge $+1$ (-1).

Then ϕ^n ($\bar{\phi}^n$) carries $U(1)$ charge $+n$ ($-n$)

Consider the two-point function $\langle \bar{\phi}^n \phi^n \rangle$

$$\langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle \equiv \frac{1}{\lambda_0^n}\frac{\int D\phi D\bar{\phi}\, \bar{\phi}^n(x_f)\phi^n(x_i)e^{-\frac{S}{\lambda_0}}}{\int D\phi D\bar{\phi}\, e^{-\frac{S}{\lambda_0}}}$$

ϕ^n and $\bar{\phi}^n$ can be brought up in the exponent, obtaining $\quad \phi \to \phi\sqrt{Q}$

$$\lambda_0^n \langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle = \frac{\int D\phi D\bar{\phi}\; e^{-\frac{1}{\lambda_0}\left[\int \partial\bar{\phi}\partial\phi + \frac{1}{4}(\bar{\phi}\phi)^2 - \lambda_0 n\left(\ln\bar{\phi}(x_f) + \ln\phi(x_i)\right)\right]}}{\int D\phi D\bar{\phi}\; e^{-\frac{1}{\lambda_0}\left[\int \partial\bar{\phi}\partial\phi + \frac{1}{4}(\bar{\phi}\phi)^2\right]}}.$$

- n counts number of the external legs and 1/n is our expansion parameter.

Goal to compute: $\Delta_{\phi^n} = n(d/2-1) + \gamma_{\phi_n}(\lambda n)$

Large charge expansion: Semiclassical approach

$$\lambda_0^n \langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle = \frac{\int D\phi D\bar{\phi}\; e^{-\frac{1}{\lambda_0}\left[\int \partial\bar{\phi}\partial\phi + \frac{1}{4}(\bar{\phi}\phi)^2 - \lambda_0 n\left(\ln\bar{\phi}(x_f) + \ln\phi(x_i)\right)\right]}}{\int D\phi D\bar{\phi}\; e^{-\frac{1}{\lambda_0}\left[\int \partial\bar{\phi}\partial\phi + \frac{1}{4}(\bar{\phi}\phi)^2\right]}}\,.$$

The dependence on λ_0 and n shows that **we can perform the path integral via a saddle point expansion** around the stationary points of

$$S_{\text{eff}} \equiv \int d^d x[\partial\bar{\phi}\partial\phi + \frac{1}{4}(\bar{\phi}\phi)^2] - n\lambda_0\left(\log\bar{\phi}(x_f) + \log\phi(x_i)\right).$$

in the limit of small λ_0, while keeping $\lambda_0 n$ fixed.
The result is organized as a 't Hooft expansion in coupling λn.
The conformal dimension of ϕ^n takes the form

$$\Delta_{\phi^n} = \sum_{k=-1} \frac{1}{n^k}\Delta_k(\lambda n)$$

where Δ_k is the $(k+1)$-loop correction in the saddle point expansion.

Reorganizing perturbative expansion

For a well-defined limit need to introduce 't Hooft coupling $\mathcal{A}$

- Large-N_c : Planar limit : $A_c \equiv g^2 N_c = fixed$
- Large-N_f : Bubble diagrams : $A_f \equiv g^2 N_f = fixed$
- Large-charge expansion : $A_Q \equiv gQ = fixed$ **this talk**

Then we have

$$\boxed{observable \sim \sum_{l=loops} g^l P_l(N) = \sum_k \frac{1}{N^k} F_k(\mathcal{A})}$$

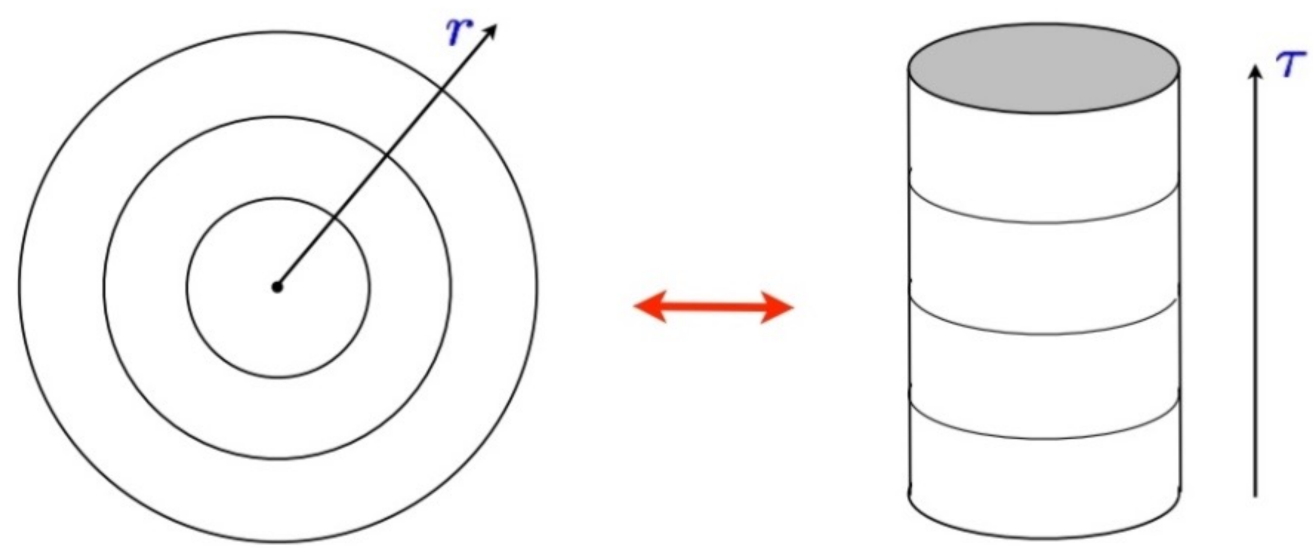
$$N = \{N_c, N_f, Q\}$$

Weyl map to the cylinder

$$\mathbb{R}^d \to \mathbb{R} \times S^{d-1}$$

$$r = R e^{\tau/R}$$

R is the radius of the sphere.



The eigenvalues of the dilation charge operator (the scaling dimensions) become the energy spectrum on the cylinder (eigenvalues of Hamiltonian).

$$\mathbf{E} = \mathbf{\Delta}/\mathbf{R}$$

State-operator correspondence:
States and operators are in 1-to-1 correspondence.

Works at the conformal fixed point

Minimal scaling dimension

Conformal invariance: $\langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle_{flat} = \frac{1}{|x_f - x_i|^{2\Delta_{\phi^n}}}$

$$\langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle_{cylinder} \overset{\tau_i \to -\infty}{=} e^{-E_{\phi^n}(\tau_f - \tau_i)}$$

$$E_{\phi^n} = \Delta_{\phi^n}/R,$$

- By computing the ground state energy on the cylinder we compute the minimal scaling dimension operators carrying the charge Q

Effective action

We introduce polar coordinates for the field $\phi = \frac{\rho}{\sqrt{2}}\, e^{i\chi}$, $\bar{\phi} = \frac{\rho}{\sqrt{2}}\, e^{-i\chi}$

$$\boxed{\langle\psi_n|\, e^{-HT}\, |\psi_n\rangle = \mathcal{Z}^{-1}\int_{\rho=f}^{\rho=f} D\rho D\chi e^{-S_{\text{eff}}}}$$

$$S_{\text{eff}} = \int d\tau \int d\Omega \left[\frac{1}{2}(\partial\rho)^2 + \frac{1}{2}\rho^2(\partial\chi)^2 + \frac{m^2}{2}\rho^2 + \frac{\lambda}{16}\rho^4 + i\frac{n}{R^{d-1}\Omega}\dot{\chi}\right].$$

- The term in red fixes the charge of initial and final states to n.
- A mass term appears $m^2 = \left(\frac{d-2}{2R}\right)^2$, stemming from R the radius of the sphere.
- Ω is the solid angle in $d-1$ dimensions.

Leading order: $S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \ldots$

$$\phi = \frac{\rho}{\sqrt{2}}\, e^{i\chi} \qquad \bar{\phi} = \frac{\rho}{\sqrt{2}}\, e^{-i\chi}$$

The classical solution of the EOM with minimal energy is spatially homogeneous and reads

$$\rho = f = \text{const.}\,, \qquad \chi = -i\mu\tau$$

Superfluid phase with homogeneous charge density

where

$$(\mu^2 - m^2) = \frac{\lambda}{4} f^2 \qquad \text{EOM}$$

$$\mu f^2 R^{d-1} \Omega_{d-1} = n \quad \left(\frac{n}{\text{vol.}} = \mu f^2\right) \qquad \text{Noether charge}$$

Action evaluated on this classical trajectory gives Δ_{-1} term in

$$\Delta_{\phi^n} = \sum_{k=-1} \frac{1}{n^k} \Delta_k(\lambda n)$$

Leading order: Δ_{-1}

Δ_{-1} is given by the effective action evaluated on the classical trajectory at the fixed point

$$\frac{4\Delta_{-1}}{\lambda_* n} = \frac{3^{\frac{2}{3}}\left(x+\sqrt{-3+x^2}\right)^{\frac{1}{3}}}{3^{\frac{1}{3}}+\left(x+\sqrt{-3+x^2}\right)^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}\left(3^{\frac{1}{3}}+\left(x+\sqrt{-3+x^2}\right)^{\frac{2}{3}}\right)}{\left(x+\sqrt{-3+x^2}\right)^{\frac{1}{3}}}$$

where $x \equiv \frac{6\lambda_* n}{16\pi^2}$.

$$\frac{\Delta_{-1}}{\lambda_*} = \begin{cases} n\left[1+\frac{1}{2}\left(\frac{\lambda_* n}{16\pi^2}\right)-\frac{1}{2}\left(\frac{\lambda_* n}{16\pi^2}\right)^2+\mathcal{O}\left(\frac{(\lambda_* n)^3}{(4\pi)^6}\right)\right] & \lambda_* n \ll (4\pi)^2 \\ \frac{8\pi^2}{\lambda_*}\left[\frac{3}{4}\left(\frac{\lambda_* n}{8\pi^2}\right)^{4/3}+\frac{1}{2}\left(\frac{\lambda_* n}{8\pi^2}\right)^{2/3}+\mathcal{O}(1)\right] & \lambda_* n \gg (4\pi)^2 \end{cases}$$

Quantum physics *“classicalizes”* in the presence of large quantum numbers.

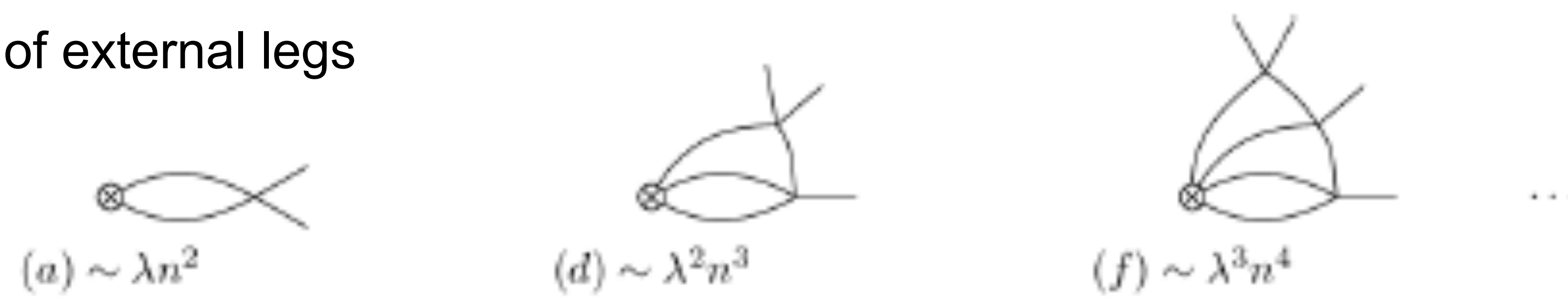
Large charge expansion: $U(1)$ complex scalar model

For small $\lambda_* n$:

$$\frac{\Delta_{-1}}{\lambda_*} = n\left[1 + \frac{1}{2}\left(\frac{\lambda_* n}{16\pi^2}\right) - \frac{1}{2}\left(\frac{\lambda_* n}{16\pi^2}\right)^2 + \mathcal{O}\left(\frac{(\lambda_* n)^3}{(4\pi)^6}\right)\right]$$

result resums an infinite number of Feynman diagrams!

- n counts the number of external legs



For large $\lambda_* n$:

$$\frac{\Delta_{-1}}{\lambda_*} = \frac{8\pi^2}{\lambda_*}\left[\frac{3}{4}\left(\frac{\lambda_* n}{8\pi^2}\right)^{4/3} + \frac{1}{2}\left(\frac{\lambda_* n}{8\pi^2}\right)^{2/3} + \mathcal{O}(1)\right]$$

This is large charge expansion. System in the superfluid phase

Next-to-leading order: Δ_0

$$\Delta_{\phi^n} = \sum_{k=-1} \frac{1}{n^k} \Delta_k(\lambda n) \qquad \mathcal{S} = \mathcal{S}(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 \mathcal{S}''(\phi_0) + \dots$$

Δ_0 is given by the fluctuation determinant around the classical trajectory

$$\rho(x) = f + r(x), \qquad \chi(x) = -i\mu\tau + \frac{1}{f\sqrt{2}}\pi(x)$$

Goldstone boson of spontaneously broken U(1) and the radial mode

$$S^{(2)} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{d-1} \left[\frac{1}{2}(\partial r)^2 + \frac{1}{2}(\partial\pi)^2 - 2i\mu\, r\partial_\tau\pi + (\mu^2 - m^2)r^2\right]$$

$$\omega_{\pm}^2(\ell) = J_\ell^2 + 3\mu^2 - m^2 \pm \sqrt{4J_\ell^2\mu^2 + (3\mu^2 - m^2)^2}$$

Dispersion relations of the spectrum:

$$J_\ell^2 = \ell(\ell + d - 2)/R^2$$

$$n_\ell = \frac{(2\ell + d - 2)\Gamma(\ell + d - 2)}{\Gamma(\ell + 1)\Gamma(d - 1)}$$

$$\Delta_0 = \frac{R}{2}\sum_{\ell=0}^{\infty} n_\ell \left[\omega_+(\ell) + \omega_-(\ell)\right]$$

ℓ labels the eigenvalues of the momentum which have degeneracy n_ℓ

Generalization: O(N) model

Phys.Rev.D 102, 045011.

- O(2)=U(1) is the Abelian complex scalar model before

- O(N) scalar theory in d= 4-ε dimensions where it features an infrared Wilson-Fisher fixed point

$$\mathcal{S} = \int d^d x \left(\frac{(\partial \phi_i)^2}{2} + \frac{(4\pi)^2 g_0}{4!} (\phi_i \phi_i)^2 \right) \qquad g^*(\epsilon) = \frac{3\epsilon}{8+N} + \mathcal{O}\left(\epsilon^2\right)$$

$\epsilon = \mathbf{0}$ and $\mathbf{N} = \mathbf{4}$

Standard Model Higgs

$\epsilon \to \mathbf{1}$

Superfluid He^4, Magnets, Superconductors, ..

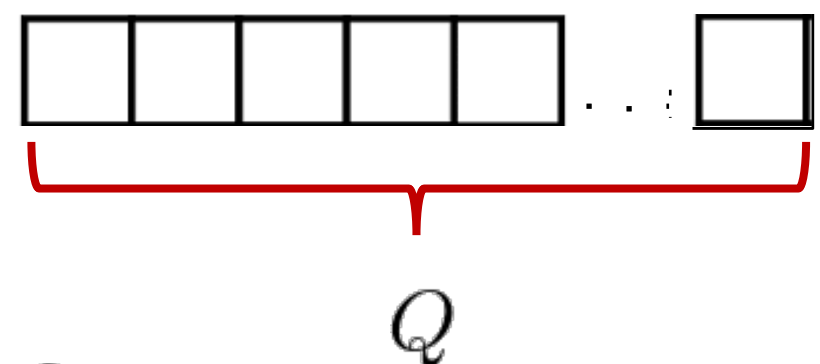
- Consider even N: we can fix up to N/2 charges, which is the rank of the O(N) group

Scaling dimension

- We compute the scaling dimension $\Delta_{\vec{Q}}$ of the operators carrying the charges Q_i and having the minimal scaling dimension.

  These operators have classical scaling dimension Q and transform in the Q-indices traceless symmetric O(N) representations where

$$Q \equiv \sum_{i=1}^{N/2} Q_i$$



$$Q = 1 \rightarrow \phi_i \qquad Q = 2 \rightarrow \phi_i \phi_j - \frac{\delta_{ij}}{N}\phi^2$$

$$Q = 3 \longrightarrow \phi^a\phi^b\phi^c - \frac{\phi^2}{N+2}\left(\phi^a\delta^{bc} + \phi^b\delta^{ac} + \phi^c\delta^{ab}\right)$$

These operators represent anisotropic perturbations in O(N)-invariant systems. $\Delta_{\vec{Q}}$ define a set of crossover (critical) exponents measuring the stability of the system (e.g. magnets) against anisotropic perturbations (e.g. crystal structure).

Boosting perturbation theory

By expanding the Δ_k's in the limit of small 't Hooft coupling A=gQ, we obtain the conventional perturbative expansion

Red terms: Δ_{-1}

Blue terms: Δ_0

Feynman diagrams crosscheck: Jack and Jones '20 '21

$$
\begin{aligned}
\Delta_Q = {} & Q + \left(\frac{Q^2}{8+N} - \frac{(N+10)}{2(8+N)}Q\right)\epsilon \\
& - \left[\frac{2}{(8+N)^2}Q^3 + \frac{(N-22)(N+6)}{2(8+N)^3}Q^2 + \frac{184+N(14-3N)}{4(8+N)^3}Q\right]\epsilon^2 \\
& + \left[\frac{8}{(8+N)^3}Q^4 + \frac{-456-64N+N^2+2(8+N)(14+N)\zeta(3)}{(8+N)^4}Q^3\right. \\
& - \frac{-31136-8272N-276N^2+56N^3+N^4+24(N+6)(N+8)(N+26)\zeta(3)}{4(N+8)^5}Q^2 \\
& \left. + \frac{-65664-8064N+4912N^2+1116N^3+48N^4-N^5+64(N+8)(178+N(37+N))\zeta(3)}{16(N+8)^5}Q\right]\epsilon^3 \\
& + \left[c_5Q^5 + c_4Q^4 + c_3Q^3 + c_2Q^2 + c_1Q\right]\epsilon^4 + \left[e_6Q^6 + e_5Q^5 + e_4Q^4 + e_3Q^3 + e_2Q^2 + e_1Q\right]\epsilon^5 + \ \dots
\end{aligned}
$$

We are computing RG functions (anomalous dimensions) of the infinite tower of Higgs-like operators. For N=4 and Q=1 we have the SM Higgs field itself.

Large charge EFT

JHEP12(2015)071

In the large ‘t Hooft coupling limit we obtain the form predicted by the EFT approach

$$\Delta_{\phi_Q} = Q^{\frac{d}{d-1}}\left[\alpha_1 + \alpha_2 Q^{\frac{-2}{d-1}} + \alpha_3 \bar{Q}^{\frac{-4}{d-1}} + \ldots\right] + Q^0\left[\beta_0 + \beta_1 Q^{\frac{-2}{d-1}} + \ldots\right] + \ \ldots$$

d=4: $\frac{3}{4}\left(\frac{\lambda_* n}{8\pi^2}\right)^{4/3} + \ \ldots$

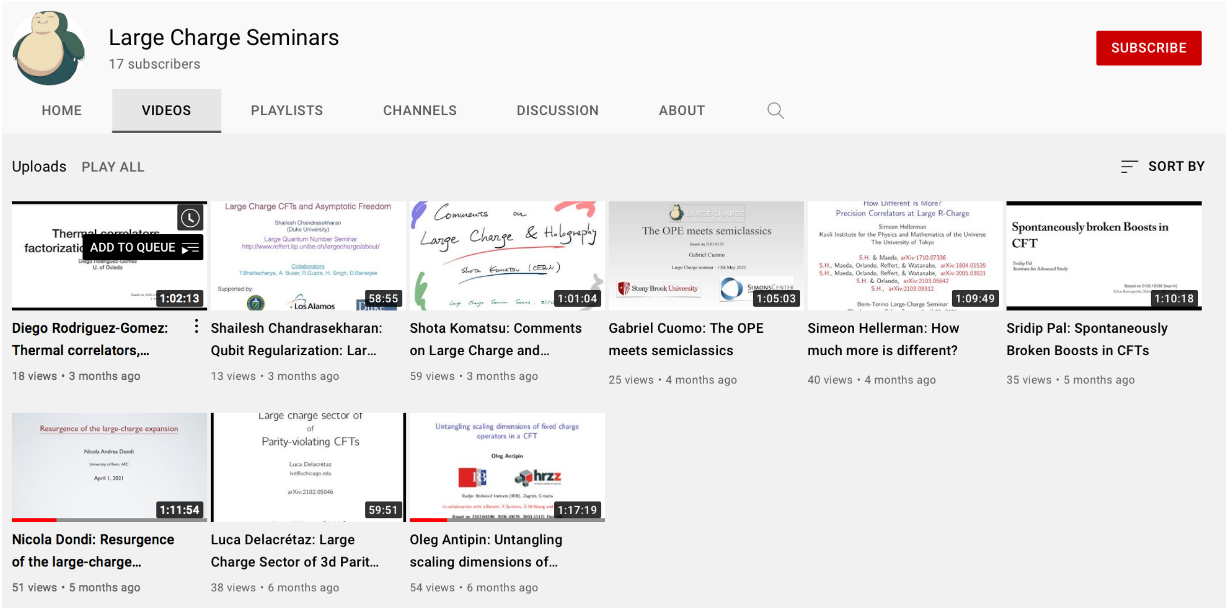
First-principle computation of α_1

For $\epsilon = 1$

	ϵ-expansion	$1/N$-expansion	Monte Carlo
$N=2$	0.424	0.471	0.337
$N=3$	0.39	0.39	0.32
$N=4$	0.368	0.333	0.301
$N=5$	0.35	0.30	0.29

Conclusions

- *Improve and check diagrammatic calculations.*
- *Access the large order behaviour of perturbation theory (resurgence)*
- *Applications: Higgsplosion, different global symmetries, AdS/CFT, CFT data, condensed matter systems, …(“Large charge seminars” series on youtube)*



Thank you!

Back up

Applications: Multi-boson production

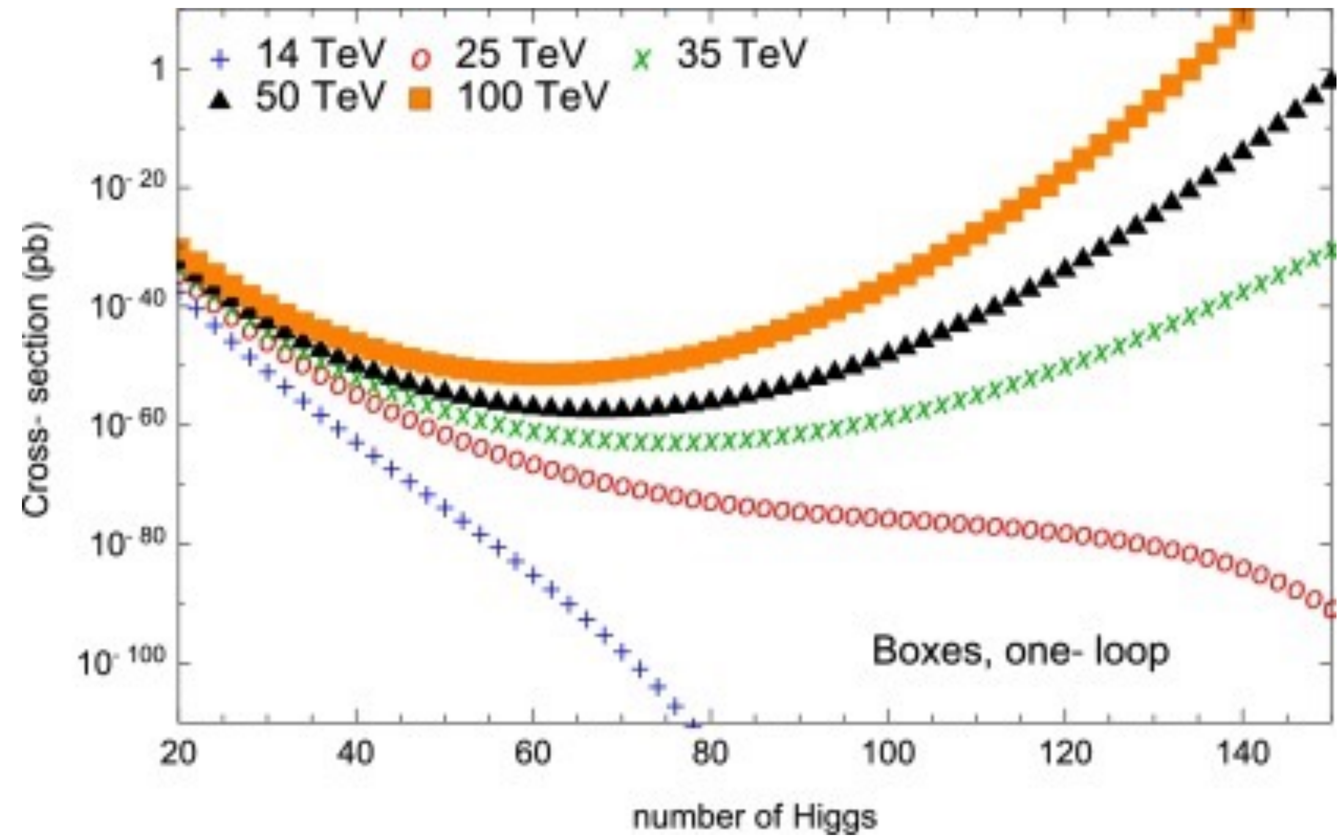
$$\lambda\phi^4$$

Consider the $1 \to n$ amplitude

$$A^{tree} = n!\lambda^{\frac{n-1}{2}} e^{-\frac{5}{6}En}$$

$$A = A^{tree} e^{B\lambda n}$$

$$\sigma(1 \to n) = e^{F(\lambda n, E)}$$



[Degrande, Khoze, Mattelaer, 2016]

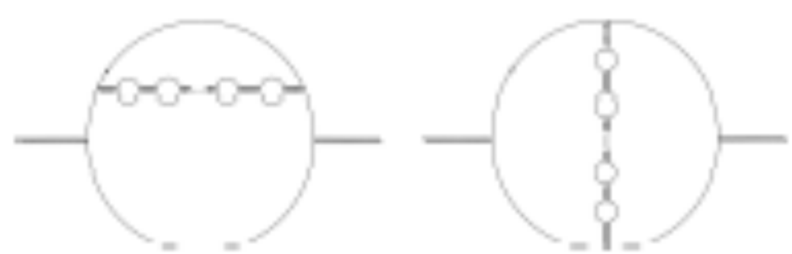
Examples

- Perturbative loop expansion in small coupling (Feynman diagrams)
- Large-N_c in $SU(N_c)$ gauge theories: Planar limit ($1/N_c$ expansion)

Planar diagram, $\sim \lambda^2$

Non-planar diagram, $\sim \lambda^2/N_c$
Suppressed by $1/N_c$

- Large-N_f (topic of this talk) : Bubble diagrams ($1/N_f$ expansion)



- Large-charge expansion (topic of this talk) (1/Q expansion)

$(a) \sim \lambda n^2$ $(d) \sim \lambda^2 n^3$ $(f) \sim \lambda^3 n^4$ . . .