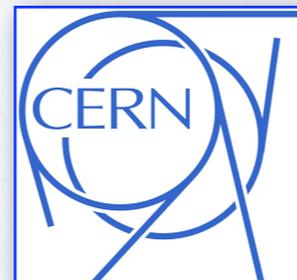


NON-ABELIAN INFRARED DIVERGENCES ON THE CELESTIAL SPHERE

Lorenzo Magnea

CERN TH - University of Torino - INFN Torino

CERN TH - 11/06/2021



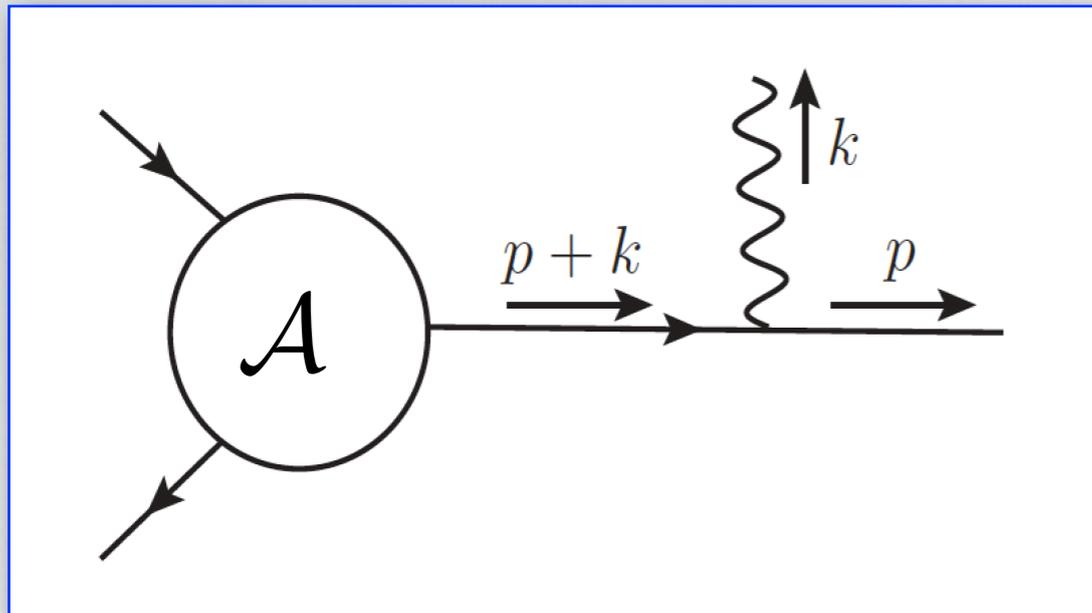
Outline

- Infrared factorisation of scattering amplitudes
- The celestial sphere
- Infrared factorisation on the celestial sphere
- A Lie-algebra conformal field theory
- Many open questions

INFRARED VISIONS



Textbook Infrared



Emission of a soft or collinear massless gauge boson

Singularities arise **only** when propagators go **on shell**

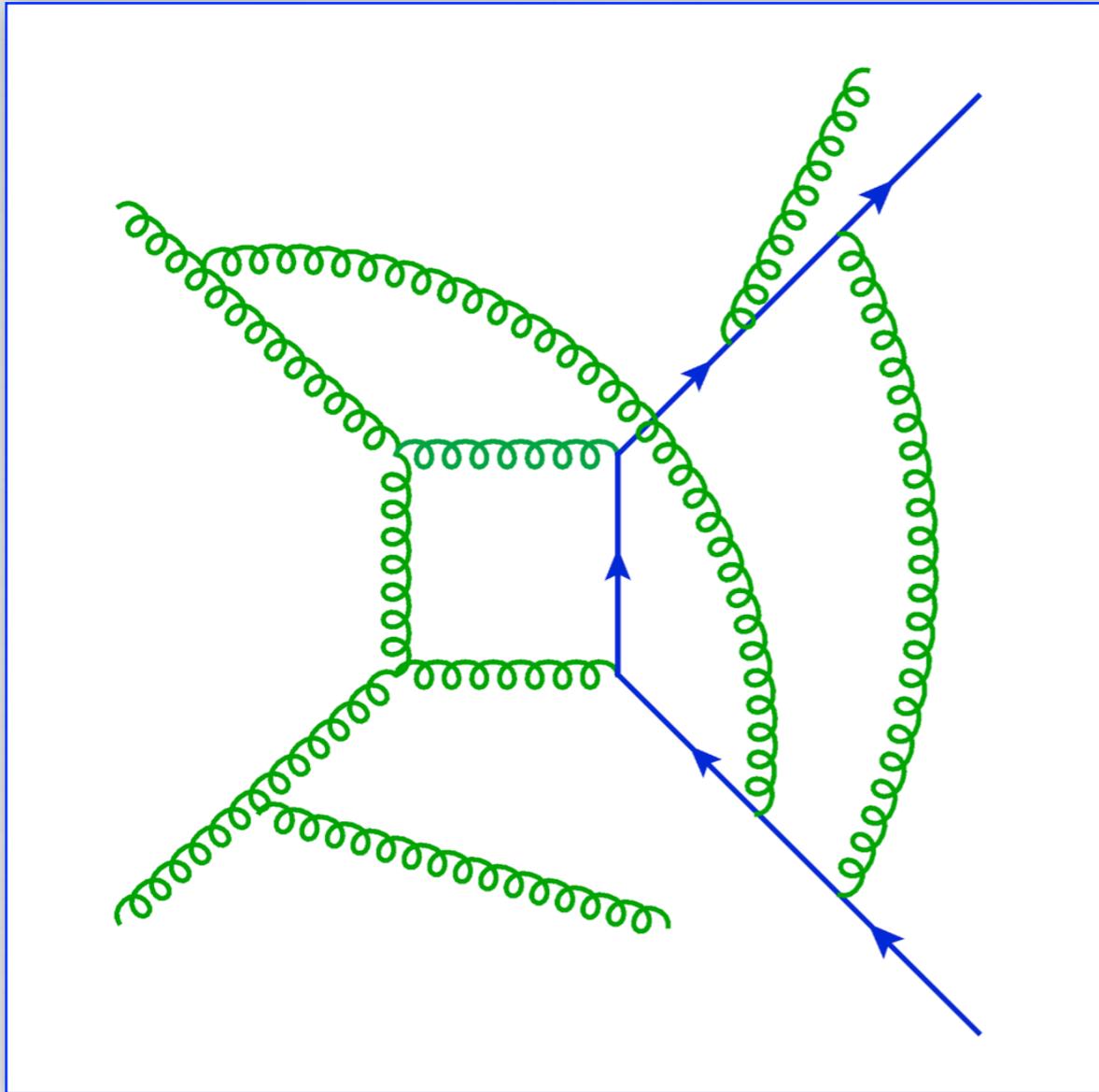
$$(p+k)^2 = 2p \cdot k = 2E_p w_k (1 - \cos \theta_{pk}) = 0 \\ \implies w_k = 0 \text{ (soft); } \cos \theta_{pk} = 1 \text{ (collinear)}$$

- ❖ Emission is **not suppressed** at long distances
- ❖ Isolated charged particles are **not true asymptotic states** of unbroken gauge theories

- ❖ A serious **problem**: the S matrix **does not exist** in the usual Fock space
- ❖ Possible **solutions**: construct finite transition probabilities (**KLN theorem**)
construct better asymptotic states (**coherent states**)
- ❖ Long-distance singularities obey a pattern of **exponentiation**

$$\mathcal{A} = \mathcal{A}_0 \left[1 - \kappa \frac{\alpha}{\pi} \frac{1}{\epsilon} + \dots \right] \implies \mathcal{A} = \mathcal{A}_0 \exp \left[-\kappa \frac{\alpha}{\pi} \frac{1}{\epsilon} + \dots \right]$$

Infrared factorisation



A gauge theory Feynman diagram with potential soft and collinear enhancements

- **Divergences** arise in scattering amplitudes from **leading regions** in loop momentum space.
- **Potential** singularities can be located using **Landau equations**.
- **Actual** singularities can be identified using infrared and collinear **power-counting** techniques.
- For **renormalised massless** theories only **soft** and **collinear** regions give divergences.
- **Soft** and **collinear** emissions have **universal** features, common to **all hard** processes.
- **Ward identities** can be used to prove **decoupling** of soft and collinear factors to **all orders**.
- A **soft-collinear factorisation** theorem for **multi-particle** matrix elements follows.
- **Similar** factorisation theorems hold for **inclusive** (soft and collinear safe) **cross sections**.

The factorised amplitude

Infrared divergences in **fixed-angle** multi-particle scattering amplitudes **factorise**

$$\mathcal{A}_n \left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = \mathcal{Z}_n \left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) \mathcal{F}_n \left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right),$$

The **infrared factor** is a colour **operator** determined by a **finite** anomalous dimension matrix

$$\mathcal{Z}_n \left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = \mathcal{P} \exp \left[\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \Gamma_n \left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2), \epsilon \right) \right],$$

All infrared poles arise from the **scale integration**, through the **d-dimensional** running **coupling**

$$\lambda \frac{\partial \alpha_s}{\partial \lambda} \equiv \beta(\alpha_s, \epsilon) = -2\epsilon \alpha_s - \frac{\alpha_s^2}{2\pi} \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{\pi} \right)^k b_k.$$

For **massless** theories, the **all-order** structure of the anomalous dimension is **known**, up to corrections due to **higher-order Casimir** operators of the gauge algebra

$$\Gamma_n \left(\frac{p_i}{\mu}, \alpha_s(\mu^2) \right) = \Gamma_n^{\text{dip}} \left(\frac{s_{ij}}{\mu^2}, \alpha_s(\mu^2) \right) + \Delta_n(\rho_{ijkl}, \alpha_s(\mu^2)),$$

$$\rho_{ijkl} = \frac{p_i \cdot p_j p_k \cdot p_l}{p_i \cdot p_l p_j \cdot p_k} = \frac{s_{ij} s_{kl}}{s_{il} s_{jk}}.$$

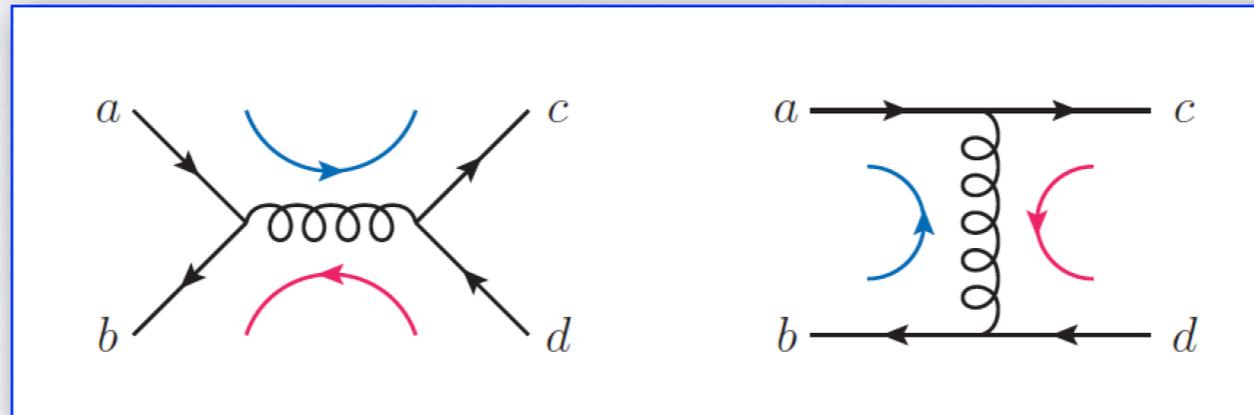
Color basis notation

The **amplitude** can be expressed in a **process-dependent** orthonormal **basis** of **colour tensors**

$$\mathcal{A}_n^{a_1 \dots a_n} \left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = \sum_L \mathcal{A}_n^L \left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) c_L^{a_1 \dots a_n}.$$

$$\sum_{\{a_i\}} c_L^{a_1 \dots a_n} (c_M^{a_1 \dots a_n})^* = \delta_{LM}.$$

A simple **example** is **quark-antiquark** scattering, where colour space is **two-dimensional**



Tree-level diagrams and color flows for quark-antiquark scattering

The amplitude is a **vector** in colour space, to **all** perturbative **orders**

$$\mathcal{A}_{abcd} = \mathcal{A}_1 c_{abcd}^{(1)} + \mathcal{A}_2 c_{abcd}^{(2)}, \quad c_{abcd}^{(1)} = \delta_{ac} \delta_{bd}, \quad c_{abcd}^{(2)} = \delta_{ab} \delta_{cd}.$$

The **exchange** of a **virtual gluon** will **shuffle** the colour **components**, even if the gluon is **soft**

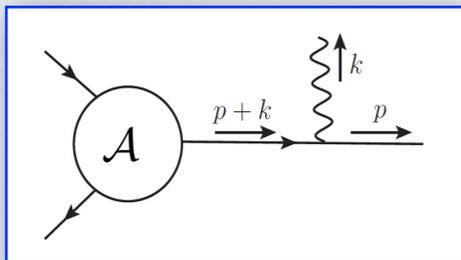
$$\text{QED : } \mathcal{A}_{\text{div}} = \mathcal{Z} \mathcal{A}_{\text{Born}} ; \quad \text{QCD : } [\mathcal{A}_{\text{div}}]_J = [\mathcal{Z}]_{JK} [\mathcal{A}_{\text{Born}}]_K.$$

Color operator notation

A powerful **basis-independent** notation uses **colour operators** 'inserting' soft gluons

$$\mathcal{A}_{n+1}^{a b_1 \dots b_n} \Big|_{\text{soft}} \propto \sum_{i=1}^n [\mathbf{T}_i^a]_{c_i}^{b_i} \mathcal{A}_n^{b_1 \dots c_i \dots b_n},$$

Soft gluon operators are **generators** of the algebra in the **representation** of the emitter



$$g\mu^\epsilon \bar{u}_{s_i}(p_i) \gamma_\alpha \frac{\not{p}_i + \not{k}}{2p_i \cdot k} (T^c)_{c_i d_i} \hat{\mathcal{A}}_{s_1 \dots s_n}^{c_1 \dots d_i \dots c_n}(\{p_j\}, k) \epsilon_\lambda^{*\alpha}(k),$$

At **leading power** in **k** :

$$g\mu^\epsilon \frac{\beta_i \cdot \epsilon_\lambda^*(k)}{\beta_i \cdot k} (T^c)_{c_i d_i} (\mathcal{A}_n)_{s_1 \dots s_n}^{c_1 \dots d_i \dots c_n}(\{p_j\}) \equiv g\mu^\epsilon \frac{\beta_i \cdot \epsilon_\lambda^*(k)}{\beta_i \cdot k} \mathbf{T}_i \mathcal{A}_n(\{p_j\}).$$

For different **emitters** :

$$\mathbf{T}_i \Big|_{q, \text{out}} \rightarrow T_{cd}^a, \quad \mathbf{T}_i \Big|_{\bar{q}, \text{out}} \rightarrow -T_{dc}^a, \quad \mathbf{T}_i \Big|_{g, \text{out}} \rightarrow -if_{cd}^a,$$

Colour operators **obey identities** inherited by the **algebra** and dictated by **gauge invariance**

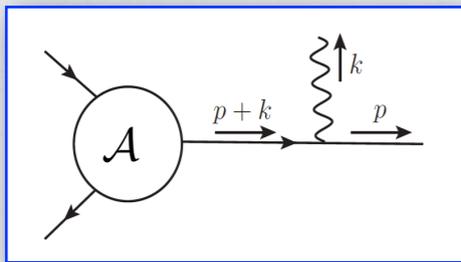
$$[\mathbf{T}_i^a, \mathbf{T}_i^b] = if_{cd}^a \mathbf{T}_i^c, \quad \mathbf{T}_i \cdot \mathbf{T}_i \equiv \mathbf{T}_i^a \mathbf{T}_i^b \delta_{ab} = C_i^{(2)}, \quad \sum_{i=1}^n \mathbf{T}_i = 0,$$

Color operator notation

A powerful **basis-independent** notation uses **colour operators** 'inserting' soft gluons

$$\mathcal{A}_{n+1}^{a b_1 \dots b_n} \Big|_{\text{soft}} \propto \sum_{i=1}^n [\mathbf{T}_i^a]_{c_i}^{b_i} \mathcal{A}_n^{b_1 \dots c_i \dots b_n},$$

Soft gluon operators are **generators** of the algebra in the **representation** of the emitter



$$g\mu^\epsilon \bar{u}_{s_i}(p_i) \gamma_\alpha \frac{\not{p}_i + \not{k}}{2p_i \cdot k} (T^c)_{c_i d_i} \hat{\mathcal{A}}_{s_1 \dots s_n}^{c_1 \dots d_i \dots c_n}(\{p_j\}, k) \epsilon_\lambda^{*\alpha}(k),$$

At **leading power** in **k** :

$$g\mu^\epsilon \frac{\beta_i \cdot \epsilon_\lambda^*(k)}{\beta_i \cdot k} (T^c)_{c_i d_i} (\mathcal{A}_n)_{s_1 \dots s_n}^{c_1 \dots d_i \dots c_n}(\{p_j\}) \equiv g\mu^\epsilon \frac{\beta_i \cdot \epsilon_\lambda^*(k)}{\beta_i \cdot k} \mathbf{T}_i \mathcal{A}_n(\{p_j\}).$$

For different **emitters** :

$$\mathbf{T}_i \Big|_{q, \text{out}} \rightarrow T_{cd}^a, \quad \mathbf{T}_i \Big|_{\bar{q}, \text{out}} \rightarrow -T_{dc}^a, \quad \mathbf{T}_i \Big|_{g, \text{out}} \rightarrow -if_{cd}^a,$$

Colour operators **obey identities** inherited by the **algebra** and dictated by **gauge invariance**

$$[\mathbf{T}_i^a, \mathbf{T}_i^b] = if_{cd}^a \mathbf{T}_i^c, \quad \mathbf{T}_i \cdot \mathbf{T}_i \equiv \mathbf{T}_i^a \mathbf{T}_i^b \delta_{ab} = C_i^{(2)}, \quad \sum_{i=1}^n \mathbf{T}_i = 0,$$

when acting on the amplitude

The dipole formula

Let's take a **closer look** at the **structure** of the infrared **anomalous dimension** matrix.

The **dipole** term :

$$\Gamma_n^{\text{dip}}\left(\frac{s_{ij}}{\mu^2}, \alpha_s(\mu^2)\right) = \frac{1}{2} \hat{\gamma}_K(\alpha_s(\mu^2)) \sum_{i=1}^n \sum_{j=i+1}^n \log\left(\frac{s_{ij} e^{i\pi\lambda_{ij}}}{\mu^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j + \sum_{i=1}^n \gamma_i(\alpha_s(\mu^2)) ,$$

The **cusp anomalous dimension** in the '**Casimir scaling**' limit:

$$\gamma_{K,r}(\alpha_s) = C_r^{(2)} \hat{\gamma}_K(\alpha_s) ,$$

Corrections start at **three** loops, with **quadrupoles**:

Ø. Almelid, C. Duhr, E. Gardi; J. Henn, B. Mistlberger.

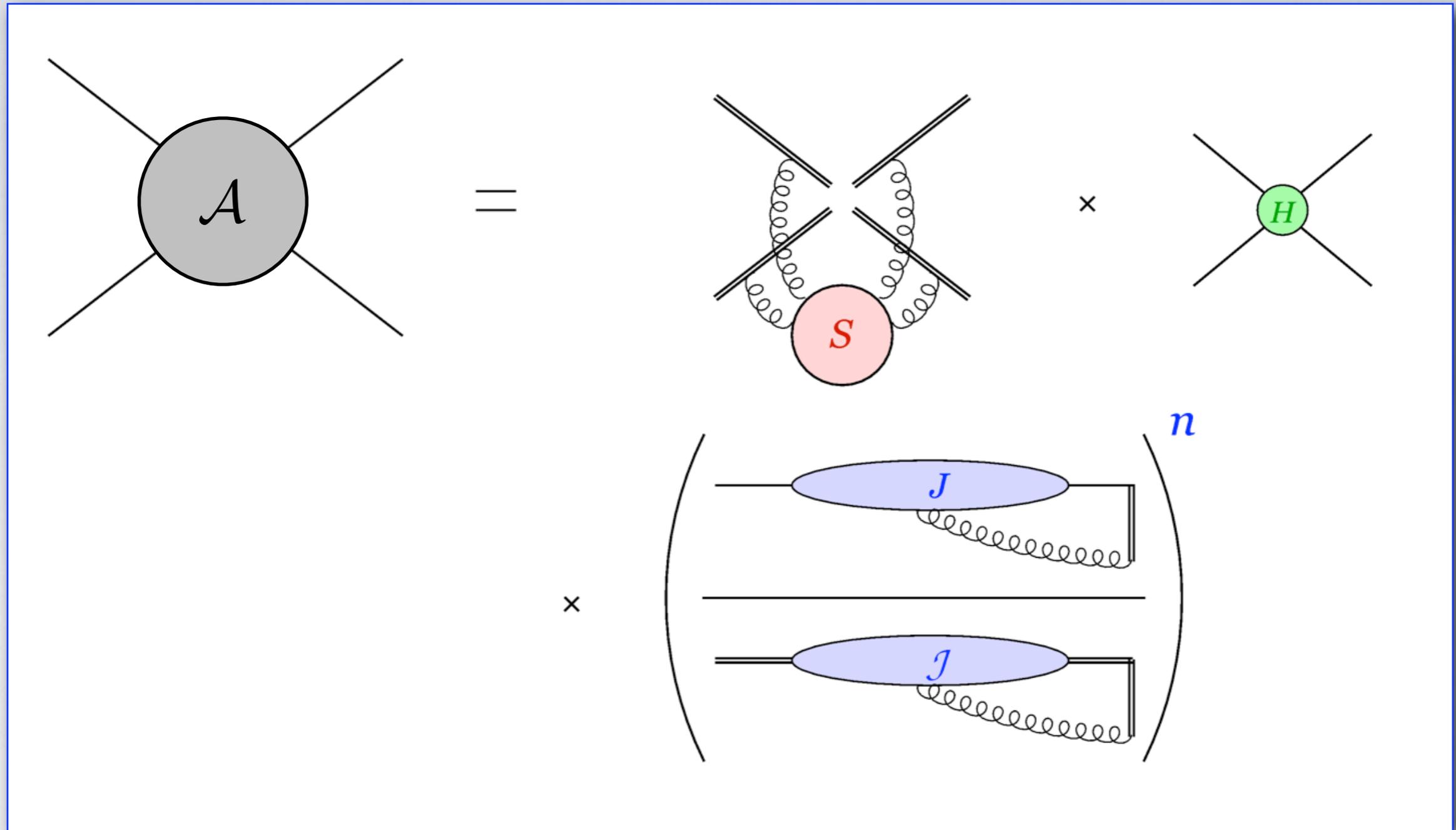
$$F_{ijkl}(\{\rho\}) f_{abe} f_{cd}^e \mathbf{T}_i^a \mathbf{T}_j^a \mathbf{T}_k^c \mathbf{T}_l^d ,$$

- The **colour dipole** is the **natural** structure arising at **one loop** from gluon exchange.
- The fact that it **survives at two loops** is a non-trivial consequence of **symmetries**.
- **Field anomalous dimensions** in color-**uncorrelated** terms govern **collinear** singularities.
- **Unitarity phases** contain crucial **analytic** information. For **final-state** pairs: $\lambda_{ij} = 1$.
- The **cusp anomalous dimension** plays a very special role: a **universal infrared coupling**.
- The structure **emerges** from the **constraints** of **scale invariance** in the soft limit.

DEEP INFRARED VISIONS



Infrared factorisation: pictorial



A pictorial representation of soft-collinear factorisation for fixed-angle scattering amplitudes

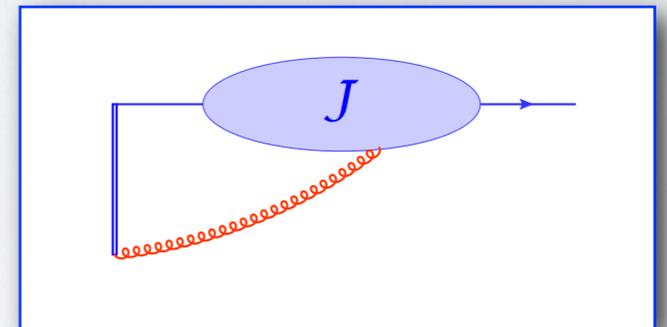
Operator Definitions

The precise **functional form** of this graphical factorisation is

$$\mathcal{A}_n \left(\frac{p_i}{\mu} \right) = \prod_{i=1}^n \left[\frac{\mathcal{J}_i \left((p_i \cdot n_i)^2 / (n_i^2 \mu^2) \right)}{\mathcal{J}_{E,i} \left((\beta_i \cdot n_i)^2 / n_i^2 \right)} \right] \mathcal{S}_n (\beta_i \cdot \beta_j) \mathcal{H}_n \left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2} \right)$$

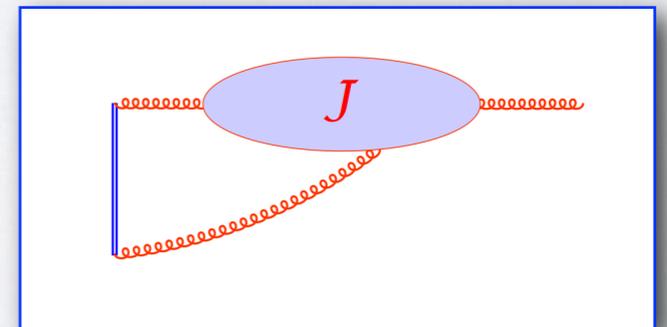
Here we introduced dimensionless **four-velocities** $\beta_i = p_i/Q$, and **factorisation vectors** n_i^μ , $n_i^2 \neq 0$ to define the jets in a **gauge-invariant** way. For **outgoing quarks**

$$\bar{u}_s(p) \mathcal{J}_q \left(\frac{(p \cdot n)^2}{n^2 \mu^2} \right) = \langle p, s | \bar{\psi}(0) \Phi_n(0, \infty) | 0 \rangle$$



where Φ_n is the **Wilson line** operator along the direction n . For **outgoing gluons**

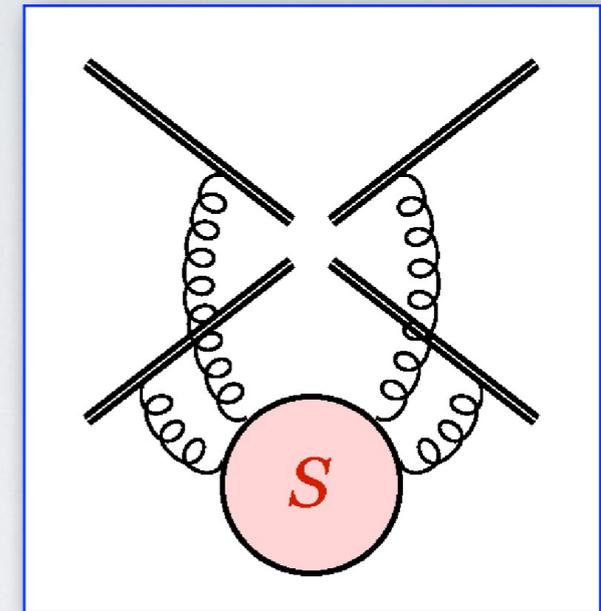
$$g_s \varepsilon_\mu^{*(\lambda)}(k) \mathcal{J}_g^{\mu\nu} \left(\frac{(k \cdot n)^2}{n^2 \mu^2} \right) \equiv \langle k, \lambda | \left[\Phi_n(\infty, 0) iD^\nu \Phi_n(0, \infty) \right] | 0 \rangle ,$$



Wilson line correlators

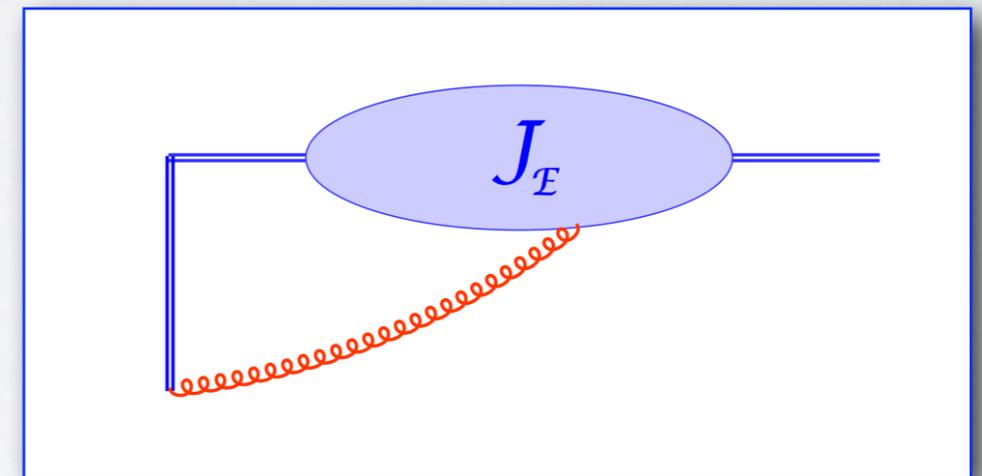
The **soft function** S is a **color operator**, mixing the available color tensors. It is defined by a correlator of **Wilson lines**.

$$\mathcal{S}_n(\beta_i \cdot \beta_j) = \langle 0 | \prod_{k=1}^n \Phi_{\beta_k}(\infty, 0) | 0 \rangle$$



The soft jet function J_E contains **soft-collinear** poles: it is defined by **replacing** the **field** in the ordinary jet J with a **Wilson line** in the appropriate **color representation**.

$$\mathcal{J}_E \left(\frac{(\beta \cdot n)^2}{n^2} \right) = \langle 0 | \Phi_{\beta}(\infty, 0) \Phi_n(0, \infty) | 0 \rangle$$



Wilson-line matrix elements **exponentiate** non-trivially and have **tightly constrained** functional **dependence** on their arguments. They are **known** to **three loops**.

On functional dependences

Straight **semi-infinite** Wilson lines are **scale-invariant**

$$\Phi_\beta(\infty, 0) \equiv P \exp \left[ig \int_0^\infty d\lambda \beta \cdot A(\lambda\beta) \right].$$

Correlators involving **light-like** Wilson lines **break** scale invariance due to **collinear poles**: a quantum **'anomaly'** proportional to the **cusp anomalous dimension**.

The **anomaly** must **cancel** in combination that are **free** from **collinear poles**

$$\widehat{\mathcal{S}}_{LK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon) \equiv \frac{\mathcal{S}_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon)}{\prod_{i=1}^n \mathcal{J}_{E,i} \left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right)}.$$

The **reduced function** depends only on **scale-invariant** combinations

$$\rho_{ij} \equiv \frac{(\beta_i \cdot \beta_j)^2 n_i^2 n_j^2}{(\beta_i \cdot n_i)^2 (\beta_j \cdot n_j)^2}.$$

At the level of **anomalous dimensions** the cancellation is particularly **striking**

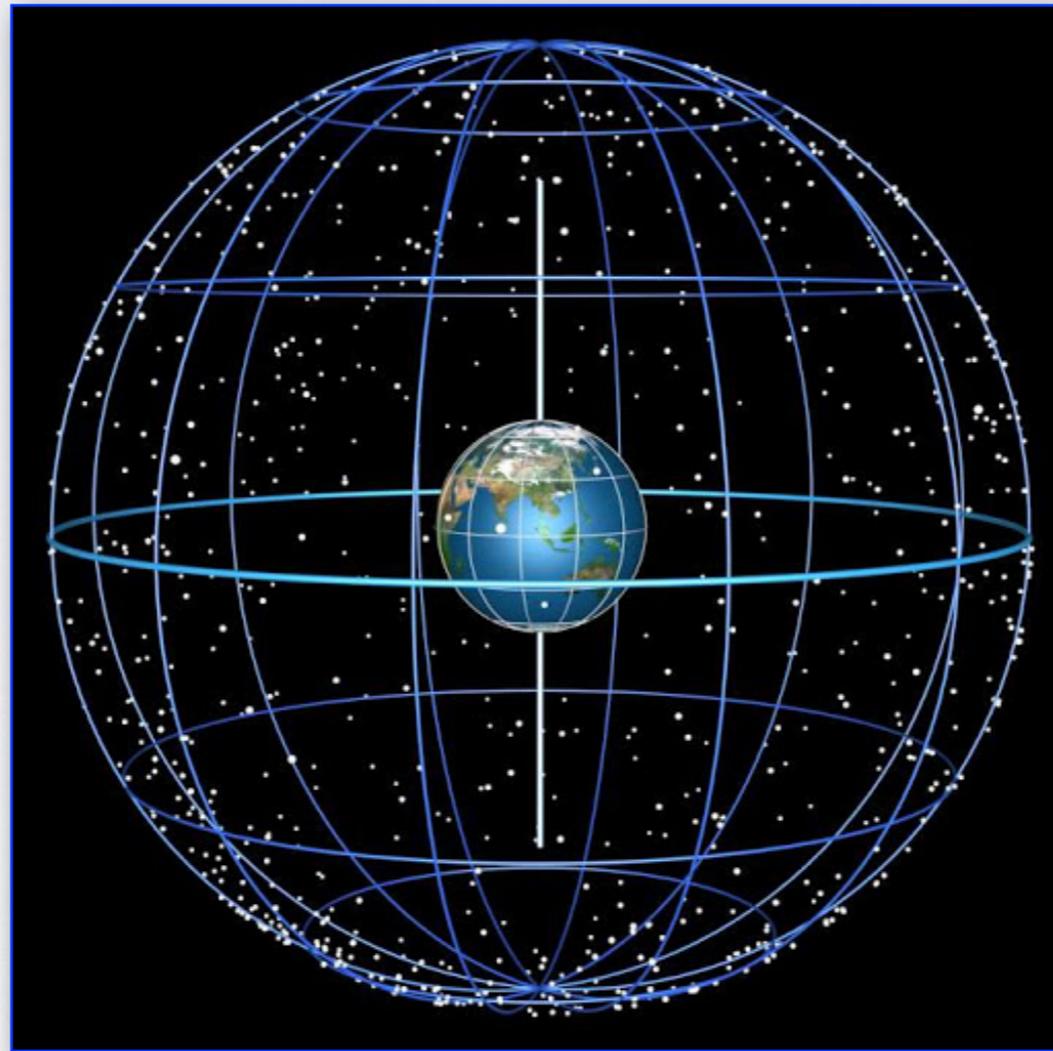
$$\Gamma_{KL}^{(\widehat{\mathcal{S}})}(\rho_{ij}, \alpha_s(\mu^2)) = \Gamma_{KL}^{(\mathcal{S})}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) - \delta_{KL} \sum_{i=1}^n \gamma_{\mathcal{J}_E} \left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right),$$

- 🔊 **Singular** terms in Γ_s must be diagonal.
- 🔊 **Finite diagonal** terms in Γ_s must form ρ_{ij} 's.
- 🔊 **Off-diagonal** terms in Γ_s must be **finite**, and must depend only on cross-ratios ρ_{ijkl}

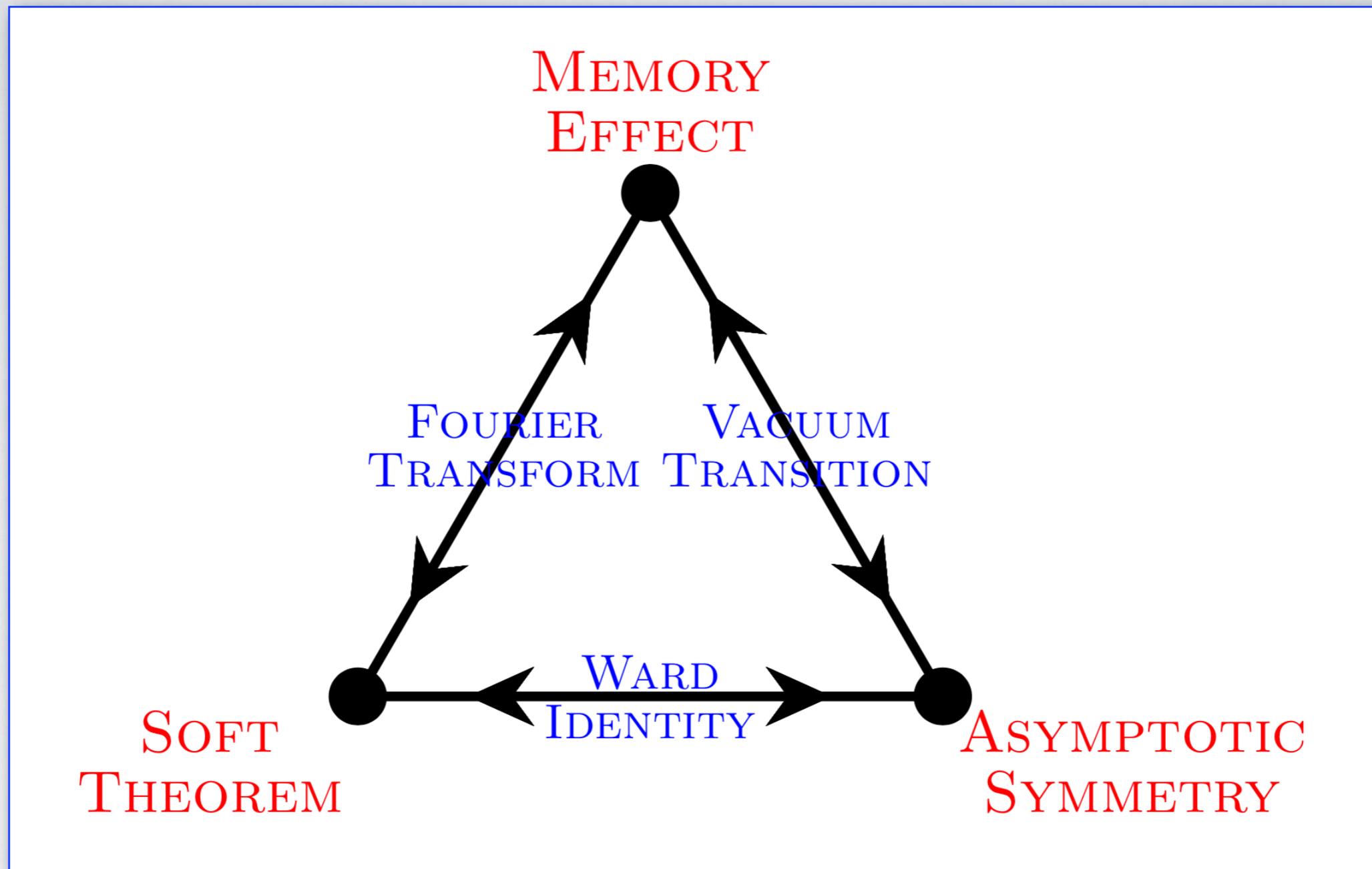
$$\sum_{i=1}^n \sum_{j \neq i} \frac{\partial}{\partial \ln \rho_{ij}} \Gamma_{KL}^{(\widehat{\mathcal{S}})}(\rho_{ij}, \alpha_s) = \frac{1}{4} \gamma_K(\alpha_s) \delta_{KL}.$$

An exact equation for the soft anomalous dimension

THE CELESTIAL SPHERE



The Strominger Triangle



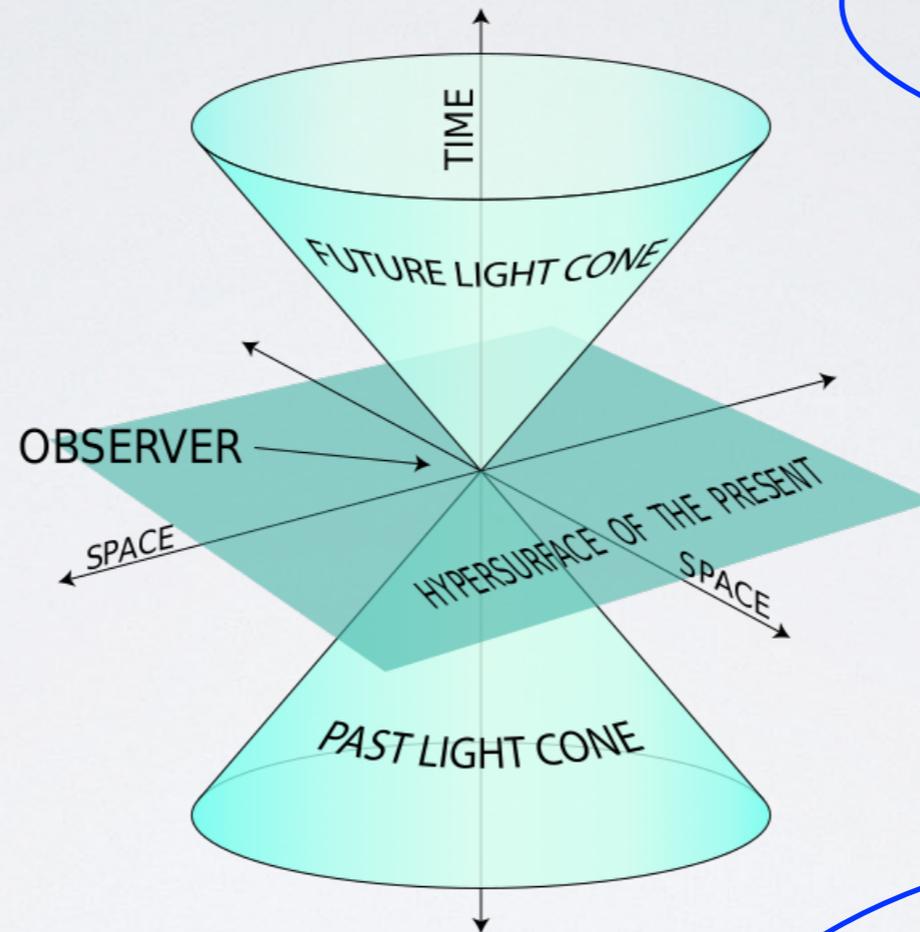
- A **new viewpoint** on infrared/long-distance phenomena in quantum field theory.
- A **lesson** from gravity: do **not trivialise** the behaviour and symmetries 'at infinity'.
- Does this **idea** lead to **new** calculational **techniques** for **non-abelian** theories?

Many directions

Electromagnetic, colour and gravitational memory effects

Asymptotically flat spacetimes and holography

Full conformal symmetry on the celestial sphere



Black hole soft hair and the information paradox

Soft, next-to soft, next-to-next-to soft

$$\mathcal{A}(\Delta_j, z_j) = \left(\prod_{i=1}^n \int_0^\infty \frac{d\omega_i}{\omega_i} \omega_i^{\Delta_i} \right) \mathbf{A}(\omega_j, z_j).$$

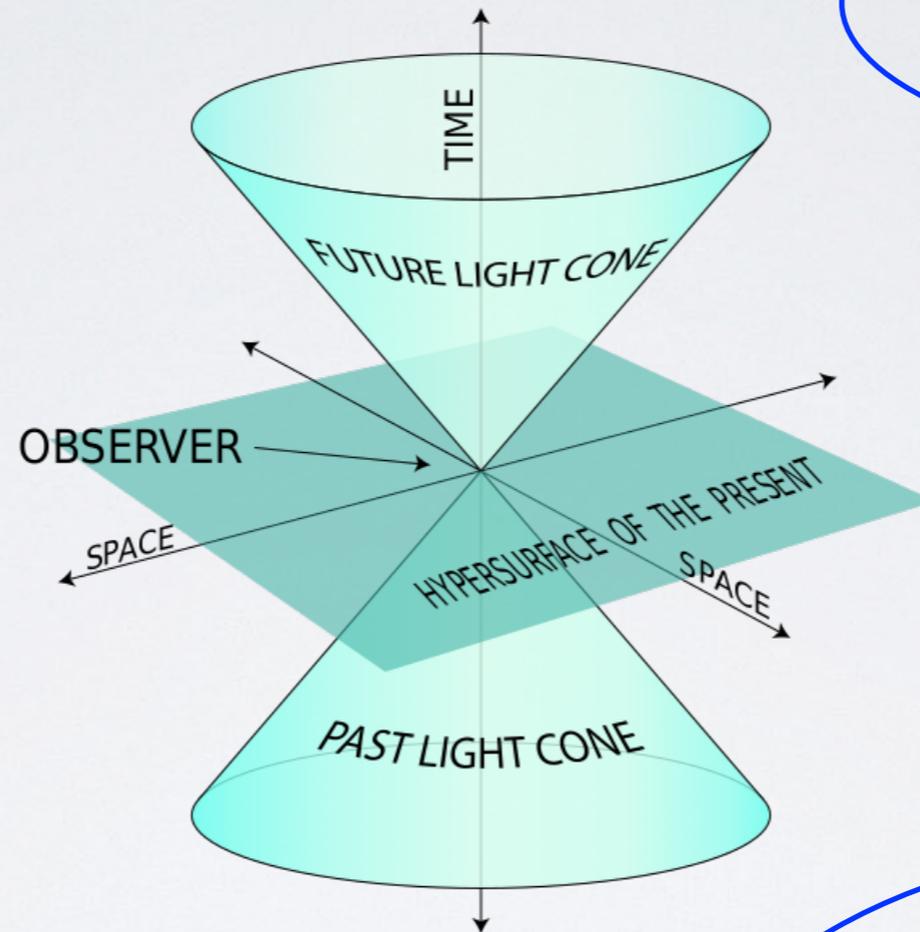
Celestial amplitudes

Many directions

Electromagnetic, colour and gravitational memory effects

Asymptotically flat spacetimes and holography

Full conformal symmetry on the celestial sphere



Black hole soft hair and the information paradox

Soft, next-to soft, next-to-next-to soft

$$\mathcal{A}(\Delta_j, z_j) = \left(\prod_{i=1}^n \int_0^\infty \frac{d\omega_i}{\omega_i} \omega_i^{\Delta_i} \right) \mathbf{A}(\omega_j, z_j).$$

Celestial amplitudes



INFRARED VISIONS ON THE CELESTIAL SPHERE



INFRARED VISIONS ON THE CELESTIAL SPHERE



On dipole correlations

Let us begin by **disentangling collinear** poles (which are **colour-singlets**) from **soft** poles (which are **colour-correlated**). We **replace** the **running** scale λ with the **fixed** scale μ in the logarithmic term, and **perform** the colour **sum** using **colour conservation**.

$$\begin{aligned} \Gamma_n^{\text{dipole}} \left(\frac{s_{ij}}{\lambda^2}, \alpha_s(\lambda, \epsilon) \right) &= \frac{1}{2} \widehat{\gamma}_K(\alpha_s(\lambda, \epsilon)) \sum_{i=1}^n \sum_{j=i+1}^n \ln \left(\frac{-s_{ij} + i\eta}{\mu^2} \right) \mathbf{T}_i \cdot \mathbf{T}_j \\ &\quad - \sum_{i=1}^n \gamma_i(\alpha_s(\lambda, \epsilon)) - \frac{1}{4} \widehat{\gamma}_K(\alpha_s(\lambda, \epsilon)) \ln \left(\frac{\mu^2}{\lambda^2} \right) \sum_{i=1}^n C_i^{(2)} \\ &\equiv \Gamma_n^{\text{corr.}} \left(\frac{s_{ij}}{\mu^2}, \alpha_s(\lambda, \epsilon) \right) + \Gamma_n^{\text{singl.}} \left(\frac{\mu^2}{\lambda^2}, \alpha_s(\lambda, \epsilon) \right), \end{aligned}$$

At **one loop**, integrating the **colour-correlated** term yields **single soft poles**, while the **singlet** term yields **single collinear** and **double soft-collinear** poles

$$\alpha_s(\lambda, \epsilon) = \alpha_s(\mu) \left(\frac{\lambda^2}{\mu^2} \right)^{-\epsilon},$$

$$\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \alpha_s(\lambda, \epsilon) = -\frac{1}{\epsilon} \alpha_s(\mu), \quad \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \ln \left(\frac{\lambda^2}{\mu^2} \right) \alpha_s(\lambda, \epsilon) = -\frac{1}{\epsilon^2} \alpha_s(\mu), \quad (\epsilon < 0).$$

At **h loops**, **multiple** poles (up to order **h+1**) are generated by the β function. For **conformal gauge theories** the logarithm of the infrared factor has **only single and double poles**.

Celestial dipoles

Crucially, we now parametrise the light-cone momenta in celestial coordinates

$$p_i^\mu = \omega_i \left\{ 1 + z_i \bar{z}_i, z_i + \bar{z}_i, -i(z_i - \bar{z}_i), 1 - z_i \bar{z}_i \right\},$$

where the energy ω_i and the sphere coordinates z_i have simple transformation properties under the Lorentz group acting as $SL(2, \mathbb{C})$:

$$\omega' = |cz + d|^2 \omega, \quad z' = \frac{az + b}{cz + d},$$

Mandelstam invariants are distances on the sphere

$$s_{ij} = 2p_i \cdot p_j = 4\omega_i \omega_j |z_i - z_j|^2,$$

which unpacks the logarithms

$$\log(-s_{ij} + i\eta) = \log(|z_i - z_j|^2) + \log \omega_i + \log \omega_j + 2 \log 2 + i\pi,$$

Energies give new singlet terms

$$\Gamma_n^{\text{dipole}} \left(\frac{s_{ij}}{\lambda^2}, \alpha_s(\lambda, \epsilon) \right) \equiv \hat{\Gamma}_n^{\text{corr.}} \left(z_{ij}, \alpha_s(\lambda, \epsilon) \right) + \hat{\Gamma}_n^{\text{singl.}} \left(\frac{\omega_i}{\lambda}, \alpha_s(\lambda, \epsilon) \right),$$

which take the form

$$\hat{\Gamma}_n^{\text{singl.}} \left(\frac{\omega_i}{\lambda}, \alpha_s(\lambda, \epsilon) \right) = - \sum_{i=1}^n \gamma_i(\alpha_s(\lambda, \epsilon)) - \frac{1}{4} \hat{\gamma}_K(\alpha_s(\lambda, \epsilon)) \sum_{i=1}^n \ln \left(\frac{-4\omega_i^2 + i\eta}{\lambda^2} \right) C_i^{(2)},$$

Celestial dipoles

The **colour-correlated** term, responsible for **all soft poles**, is **remarkably simple**

$$\widehat{\Gamma}_n^{\text{corr.}}(z_{ij}, \alpha_s(\lambda, \epsilon)) = \frac{1}{2} \widehat{\gamma}_K(\alpha_s(\lambda, \epsilon)) \sum_{i=1}^n \sum_{j=i+1}^n \ln(|z_{ij}|^2) \mathbf{T}_i \cdot \mathbf{T}_j.$$

Scale and **coupling** dependence are **completely factored** from **colour** and **kinematics**, and equal for all dipoles. The **scale integral** can this be **performed** in full generality, yielding

$$\begin{aligned} \mathcal{Z}_n^{\text{corr.}}(z_{ij}, \alpha_s(\mu), \epsilon) &\equiv \exp \left[\int_0^\mu \frac{d\lambda}{\lambda} \widehat{\Gamma}_n^{\text{corr.}}(z_{ij}, \alpha_s(\lambda, \epsilon)) \right] \\ &= \exp \left[-K(\alpha_s(\mu), \epsilon) \sum_{i=1}^n \sum_{j=i+1}^n \ln(|z_{ij}|^2) \mathbf{T}_i \cdot \mathbf{T}_j \right], \end{aligned}$$

The scale factor **K** is **well-known** in **QCD** from **form-factor** calculations, and gives the perturbative **Regge trajectory** in the **high-energy** limit of **four-point** amplitudes. It is

G. Korchemsky, I.A. Korchemskaya; V. Del Duca, C. Duhr,
E. Gardi, LM, C. White; G. Falcioni, L. Vernazza, ...

$$K(\alpha_s(\mu), \epsilon) = -\frac{1}{2} \int_0^\mu \frac{d\lambda}{\lambda} \widehat{\gamma}_K(\alpha_s(\lambda, \epsilon)).$$

The function **K** can be **computed** order by order in terms of the **cusp** and the **β function**

$$\begin{aligned} K(\alpha_s, \epsilon) &= \frac{\alpha_s}{\pi} \frac{\widehat{\gamma}_K^{(1)}}{4\epsilon} + \left(\frac{\alpha_s}{\pi}\right)^2 \left(\frac{\widehat{\gamma}_K^{(2)}}{8\epsilon} + \frac{b_0 \widehat{\gamma}_K^{(1)}}{32\epsilon^2} \right) \\ &+ \left(\frac{\alpha_s}{\pi}\right)^3 \left(\frac{\widehat{\gamma}_K^{(3)}}{12\epsilon} + \frac{b_0 \widehat{\gamma}_K^{(2)} + b_1 \widehat{\gamma}_K^{(1)}}{48\epsilon^2} + \frac{b_0^2 \widehat{\gamma}_K^{(1)}}{192\epsilon^3} \right) + \mathcal{O}(\alpha_s^4), \end{aligned}$$

$\beta \rightarrow 0$

$$K(\alpha_s, \epsilon) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \frac{\widehat{\gamma}_K^{(n)}}{4n\epsilon},$$

COLOUR
ON THE CELESTIAL SPHERE



Hints of a celestial theory

The **colour-correlated** term in the anomalous dimension matrix is **strongly reminiscent** of **conformal field theory** results. One needs only go so far as **Joe Polchinski's book** to find

2.3 The expectation value of a product of exponential operators on the plane is

$$\left\langle \prod_{i=1}^n :e^{ik_i X(z_i, \bar{z}_i)} : \right\rangle = iC^X (2\pi)^D \delta^D(\sum_{i=1}^n k_i) \prod_{\substack{i,j=1 \\ i < j}}^n |z_{ij}|^{\alpha' k_i \cdot k_j},$$

with C^X a constant. This can be obtained as a limit of the expectation value (6.2.17) on the sphere, which we will obtain by several methods in chapter 6.

A **correlator** of **vertex operators** in a **free-boson** theory (such as the **bosonic string**) has the **correct form**, up to the **substitution** of **momenta** with **colour matrices**.

This was noticed by **N. Kalyanapuram** in **2011.11412**, for the simple case of **QED**. He writes

$$\ln \left(\mathcal{A}_{n,s=1}^{soft} |_{vir} \right) = -\frac{1}{8\pi^2 \epsilon} \sum_{i \neq j} e_i e_j \ln |z_i - z_j|^2.$$

The result is **formally reproduced** by introducing **vertex operators** with **electric charges**

$$V_j(z_j, \bar{z}_j) =: e^{ie_j \varphi(z_j, \bar{z}_j)} :$$



$$\langle V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n) \rangle = A_n^{soft} |_{vir, s=1}.$$

Lie-algebra-valued free bosons

It is natural to **mimic** the **bosonic string**, considering **free bosons** spanning the **gauge algebra**.

$$S(\phi) = \frac{1}{2\pi} \int d^2z \partial_z \phi^a(z, \bar{z}) \partial_{\bar{z}} \phi_a(z, \bar{z}),$$

The free bosons **could be organised** in a **matrix field** :

gauge **generators** at **different points** must then be taken to **commute**

$$\Phi_r(z, \bar{z}) \equiv \phi_a(z, \bar{z}) T_{r,z}^a,$$

The **well-known** results for free bosons in **d=2** can be directly **transcribed**.

The **equations of motions** are:

$$\partial_z \partial_{\bar{z}} \phi^a(z, \bar{z}) = 0,$$

implying that the **derivatives** of the fields are **(anti)holomorphic**

A **normal-ordered product** can be defined, obeying the **classical** equation of motion

$$:\phi^a(z, \bar{z}) \phi^b(w, \bar{w}): = \phi^a(z, \bar{z}) \phi^b(w, \bar{w}) + \frac{1}{2} \delta^{ab} \log |z - w|^2,$$

There is a **traceless** conserved **energy-momentum tensor**, and a conserved **Noether current**

$$T(z) = - : \partial_z \phi^a(z, \bar{z}) \partial_z \phi_a(z, \bar{z}) :,$$

$$j^a(z) = \partial_z \phi^a(z, \bar{z}),$$

Matrix vertex operators

Guided by the QED example, we can tentatively define a matrix-valued vertex operator

$$V(z, \bar{z}) \equiv : e^{i\kappa \mathbf{T}_z \cdot \phi(z, \bar{z})} : = : e^{i\kappa \Phi(z, \bar{z})} :,$$

A 'single-copy' of the string vertex operator!

In colour space, this is a matrix in the representation of \mathbf{T}_z , defined on the boundary sphere and acting on the bulk colour degrees of freedom. But is it a conformal primary field?

For conventional vertex operators (as for example for bosonic strings)

$$V_{\text{c.s.}}(z, \bar{z}) \equiv : e^{ik^\mu X_\mu(z, \bar{z})} : \longrightarrow h = \frac{1}{4} k^\mu k^\nu \eta_{\mu\nu} = \frac{k^2}{4},$$

The same calculation yields

$$V(z, \bar{z}) \equiv : e^{i\kappa \mathbf{T}_z \cdot \phi(z, \bar{z})} : \longrightarrow h = \frac{\kappa^2}{4} \mathbf{T}_z \cdot \mathbf{T}_z = \frac{\kappa^2}{4} C_r^{(2)},$$

Crucially, this is a positive real number and not a matrix. For consistency, two-point functions must evaluate to a power of the distance given by the conformal weight $\Delta = h + \bar{h}$. Indeed

$$\langle V(z_1, \bar{z}_1) V(z_2, \bar{z}_2) \rangle \sim |z_{12}|^{-2\Delta},$$

by colour conservation $\mathbf{T}_1 + \mathbf{T}_2 = 0$

Note analogies with other constructions.

Vertex operator construction of Kac-Moody algebras:

$$U^\alpha(z) = z^{\alpha^2/2} : e^{i\alpha \cdot Q(z)} :.$$

not the same

Reggeon fields for high-energy scattering:

(Caron-Huot 2013)

$$U(z) = e^{ig_s T^a W^a(z)}.$$

closely related

A conformal correlator

Our **construction** from the beginning **targeted** the **n-point correlator**

$$\mathcal{C}_n(\{z_i\}, \kappa) \equiv \left\langle \prod_{i=1}^n V(z_i, \bar{z}_i) \right\rangle.$$

The calculation is a **textbook exercise**: it can be done with **oscillators**, after expanding the **free fields** in **modes** on the sphere, or computing the **path integral** (Polchinski). The result is

$$\mathcal{C}_n(\{z_i\}, \kappa) = C(N_c) \exp \left[\frac{\kappa^2}{2} \sum_{i=1}^n \sum_{j=i+1}^n \ln(|z_{ij}|^2) \mathbf{T}_i \cdot \mathbf{T}_j \right],$$

reproducing the structure of the gauge theory **infrared operator**. **Note that**

$$\sum_{i=1}^n \mathbf{T}_i = 0,$$

- The correlator has **support** only on **colour conserving configurations**
- The **field normalisation** κ maps to the **integral** K , carrying **scale** and **regulator** dependence.
- In a **path integral** evaluation on a **curved** surface (say, a **finite sphere** with radius R) the correlator acquires a **scale-dependent** 'Weyl' **factor**, which in this setting maps to an (undetermined) colour-singlet **collinear contribution**.

$$\mathcal{W}_n(\{z_i\}, \kappa) = \exp \left[-\frac{1}{2} \sum_{i=1}^n C_i^{(2)} g(z_i, \bar{z}_i) \right],$$

A tree-level soft theorem

Real emission of a soft massless gauge boson from a fixed angle hard amplitude factorises in any non-abelian theory in the form

$$\langle c | \otimes \langle \lambda | \mathcal{A}_{g, f_1 \dots f_n}(k, p_1, \dots, p_n) \rangle_{\text{soft}} = \epsilon_\lambda(k) \cdot J^c(k) | \mathcal{A}_{f_1 \dots f_n}(p_1, \dots, p_n) \rangle ,$$

The tree-level soft-gluon current has the classic eikonal form and is gauge-invariant

$$\mathbf{J}^\mu(k) = g \sum_{i=1}^n \mathbf{T}_i \frac{\beta_i^\mu}{\beta_i \cdot k} ,$$

$$k \cdot \mathbf{J}^\mu(k) = g \sum_{i=1}^n \mathbf{T}_i = 0 ,$$

The tree-level soft theorem is reproduced by the Ward identity for the Noether current associated with invariance under field translations in the Lie algebra. Using the conformal operator product expansion one finds

A. Strominger, T. He, P. Mitra, A. Nande, M. Pate, W. Fan, A. Fotopoulos, T.R. Taylor, ...

$$\left\langle \partial_z \phi^a(z, \bar{z}) \prod_{i=1}^n V(z_i, \bar{z}_i) \right\rangle \simeq -\frac{i}{2} \sum_{i=1}^n \frac{\mathbf{T}_i^a}{z - z_i} \mathcal{C}_n(\{z_i\}, \kappa) .$$

where the poles as $z \rightarrow z_i$ are collinear poles, since the celestial theory is energy-independent.

Celestial Sudakov parametrisation

In order to study **collinear limits**, it is useful to build a **Sudakov parametrisation** on the sphere

$$p_1^\mu = xp^\mu + p_\perp^\mu - \frac{p_\perp^2}{2xp \cdot n} n^\mu,$$
$$p_2^\mu = (1-x)p^\mu - p_\perp^\mu - \frac{p_\perp^2}{2(1-x)p \cdot n} n^\mu,$$

One can **fix** the **light-like** Sudakov vector n^μ , and then **compute** the **collinear** momentum p^μ

$$n^\mu = \frac{1}{2}\{1, 0, 0, -1\} \rightarrow n \cdot p_i = \omega_i, \quad n \cdot p = \omega_1 + \omega_2 \equiv \omega,$$

$$p^\mu = \omega \left\{ 1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z} \right\},$$

The **point** z is on the line **joining** z_1 and z_2 :

$$z = xz_1 + (1-x)z_2, \quad \omega_1 = x\omega, \quad \omega_2 = (1-x)\omega.$$

Once the **collinear** direction is **fixed**, the **transverse momentum** vector can be **computed**

$$p_\perp^\mu = \omega x(1-x) \left\{ (1-2x)(z_1\bar{z}_2 + \bar{z}_1z_2) + 2xz_1\bar{z}_1 - 2(1-x)z_2\bar{z}_2, \right. \\ \left. z_{12} + \bar{z}_{12}, -i(z_{12} - \bar{z}_{12}), -(1-2x)(z_1\bar{z}_2 + \bar{z}_1z_2) - 2xz_1\bar{z}_1 + 2(1-x)z_2\bar{z}_2 \right\}.$$

It is **antisymmetric**, and satisfies

$$p_\perp^2 = -x(1-x)s_{12} = -4x(1-x)\omega_1\omega_2|z_{12}|^2,$$

Collinear limits

The **operator product expansion** governs the **collinear limit** on the sphere. One can **transcribe** the **textbook result** substituting **colour** operators for **momenta**.

$$: e^{i\kappa \mathbf{T}_1 \cdot \phi(z_1, \bar{z}_1)} : : e^{i\kappa \mathbf{T}_2 \cdot \phi(z_2, \bar{z}_2)} : \sim |z_{12}|^{\kappa^2 \mathbf{T}_1 \cdot \mathbf{T}_2} : e^{i\kappa (\mathbf{T}_1 + \mathbf{T}_2) \cdot \phi(z, \bar{z})} : ,$$

Note that the **exact** collinear limit is **outside** the validity of the **original factorisation**. But it can be **approached**: on the gauge theory side, one defines a **splitting anomalous dimension**

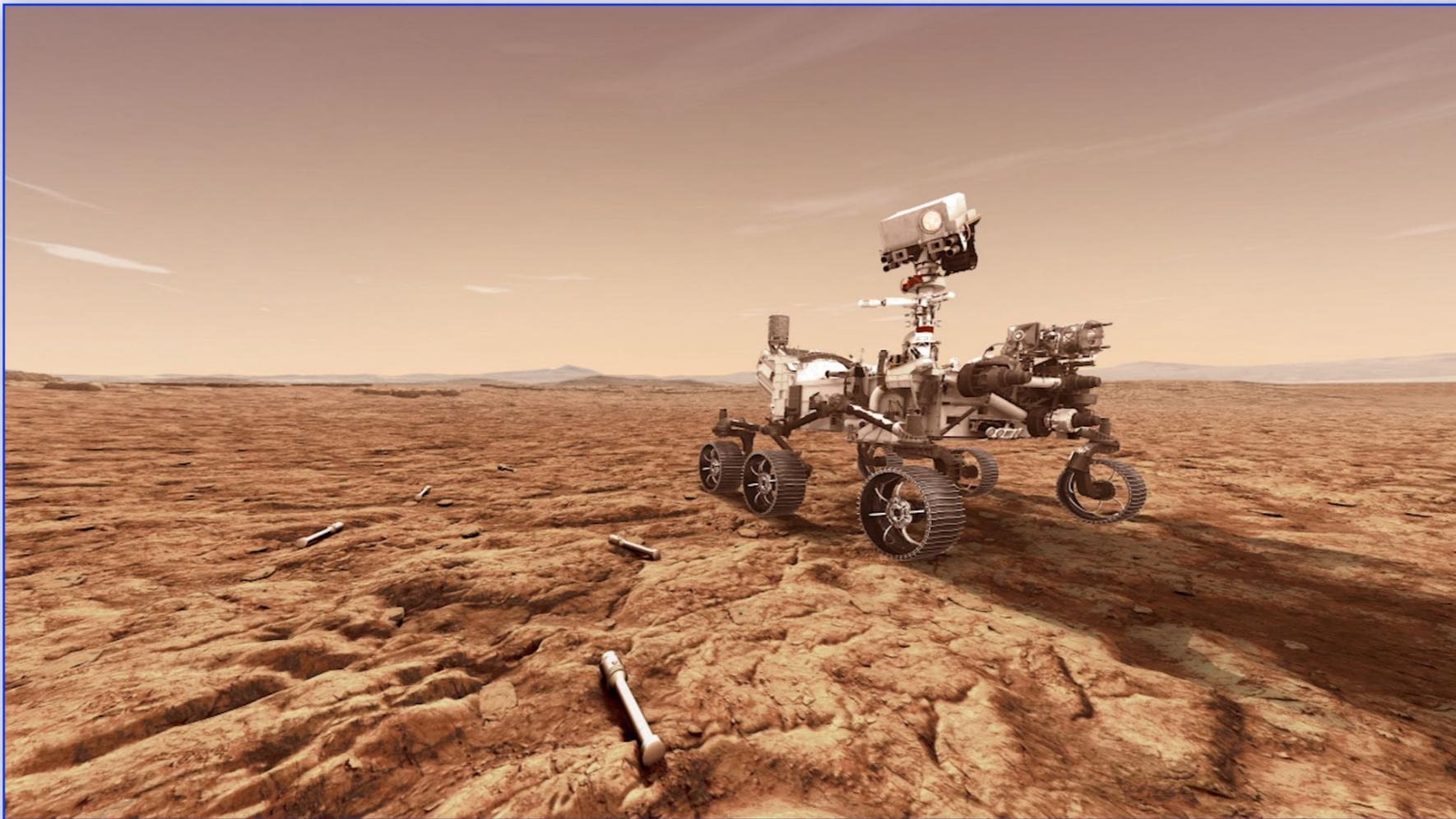
$$\Gamma_{\text{Sp.}}(p_1, p_2) \equiv \Gamma_n(p_1, p_2, \dots, p_n) - \Gamma_{n-1}(p, p_3, \dots, p_n) \Big|_{\mathbf{T}_p \rightarrow \mathbf{T}_1 + \mathbf{T}_2} . \quad (\text{Becher-Neubert 2009})$$

The **OPE encodes collinear factorisation**: the **n-point** correlator reduces to **(n-1)-points**, with the 'merged' point carrying the **sum of the colours** of (**only!**) the two collinear particles.

The calculation of the **splitting function** is then **the same** as in the **gauge theory**, but requires **reinstating** the **energy dependence**, which is **not encoded** by the conformal correlator.

$$\Gamma_{\text{Sp.}}(p_1, p_2) = \frac{1}{2} \hat{\gamma}_K(\alpha_s) \left[\ln \left(\frac{-s_{12} + i\eta}{\mu^2} \right) \mathbf{T}_1 \cdot \mathbf{T}_2 - \ln x \mathbf{T}_1 \cdot (\mathbf{T}_1 + \mathbf{T}_2) - \ln(1-x) \mathbf{T}_2 \cdot (\mathbf{T}_1 + \mathbf{T}_2) \right] ,$$

MANY QUESTIONS



Many Questions

 The **choice** of the **gauge coupling**.

Our construction **lends support** to the idea that the **cusp anomalous dimension** should be taken as the **definition** of the **strong coupling** in the **infrared**.

How far can one take this definition?

S. Catani, B. Webber, G. Marchesini; A. Grozin et al.;
A. Banfi et al.; O. Erdogan, G. Sterman;
S. Catani, D. DeFlorian, M. Grazzini.

 **Scale** and **regulator** dependence.

It is **remarkable**, and **necessary**, that infrared singularities be hidden in the **matching condition** between the **gauge** theory and the **conformal** theory.

How can one make this correspondence more precise?

 **Beyond** the **free** theory.

The celestial conformal theory **certainly has corrections** involving **structure constants** (as **confirmed** by the structure of Δ). The **deformed** theory is still **conformal**.

What drives the deformation?

 **Constraints** from vast **field theory data**.

Soft and collinear **factorisation kernels** are known to **three loops**, and in the **massive** case to **two loops**. In most cases their **remarkable simplicity** is only partly explained.

How can we harness these data to constrain the celestial theory?

The exploration has just begun!

