THE FACETS OF PRODUCT MOMENTS: FROM EFT-HEDRON TO MODULAR-HEDRON

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[A] "The EFT hedron" N. Arkani-Hamed (IAS), Tzu-Chen Huang (Caltech), Y-T H 2012.15849

[B] "Into the EFThedron and UV constraints from IR consistency" Li-Yuan Chiang, Wei Li, He-Chen Wen, Laurentiu Rodina, Y-T H 2105.02862

[C] "The modular-hedron" Wei Li, Y-T H, and Tzu-Chen Huang Laurentiu Rodina

Let's begin with the "gist"

$$[y]_{d+1\times d+1} = \begin{pmatrix} y^{(0,0)} & y^{(0,1)} & \dots & y^{(0,d)} \\ y^{(1,0)} & y^{(1,1)} & \dots & y^{(1,d)} \\ \vdots & \vdots & \vdots & \vdots \\ y^{(d,0)} & y^{(d,1)} & \dots & y^{(d,d)} \end{pmatrix} \in \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \\ x_i^d \end{pmatrix} \begin{pmatrix} 1 \\ \tilde{x}_i \\ \tilde{x}_i^2 \\ \vdots \\ \tilde{x}_i^d \end{pmatrix}^T \quad , \quad \forall c_i > 0$$

The convex hull of product moment curve

We would like to find the boundary of this hull

$$f[y^{(m,n)}] \geq 0$$

carves out the image of the hull in the space of $[y]_{d+1\times d+1}$

The multivariate moment problem: what are the sufficient conditions on $[y]_{d+1\times d+1}$ such that a solution in the form of the RHS exists.

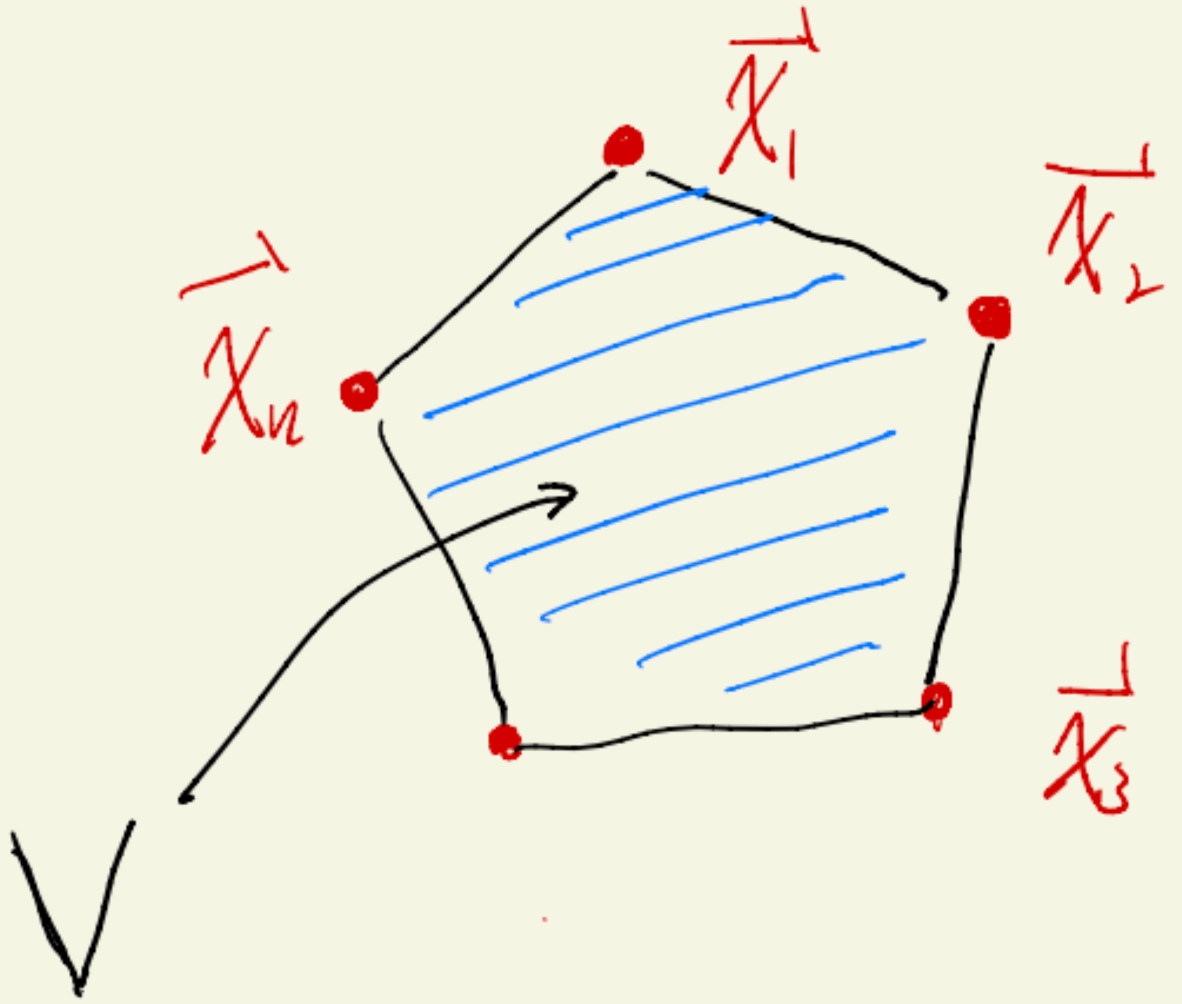
The geometry of spaces

Vertex center

$$V \in \frac{\sum_j m_i \vec{X}_i}{M} = \sum_j \hat{m}_i \vec{X}_i$$

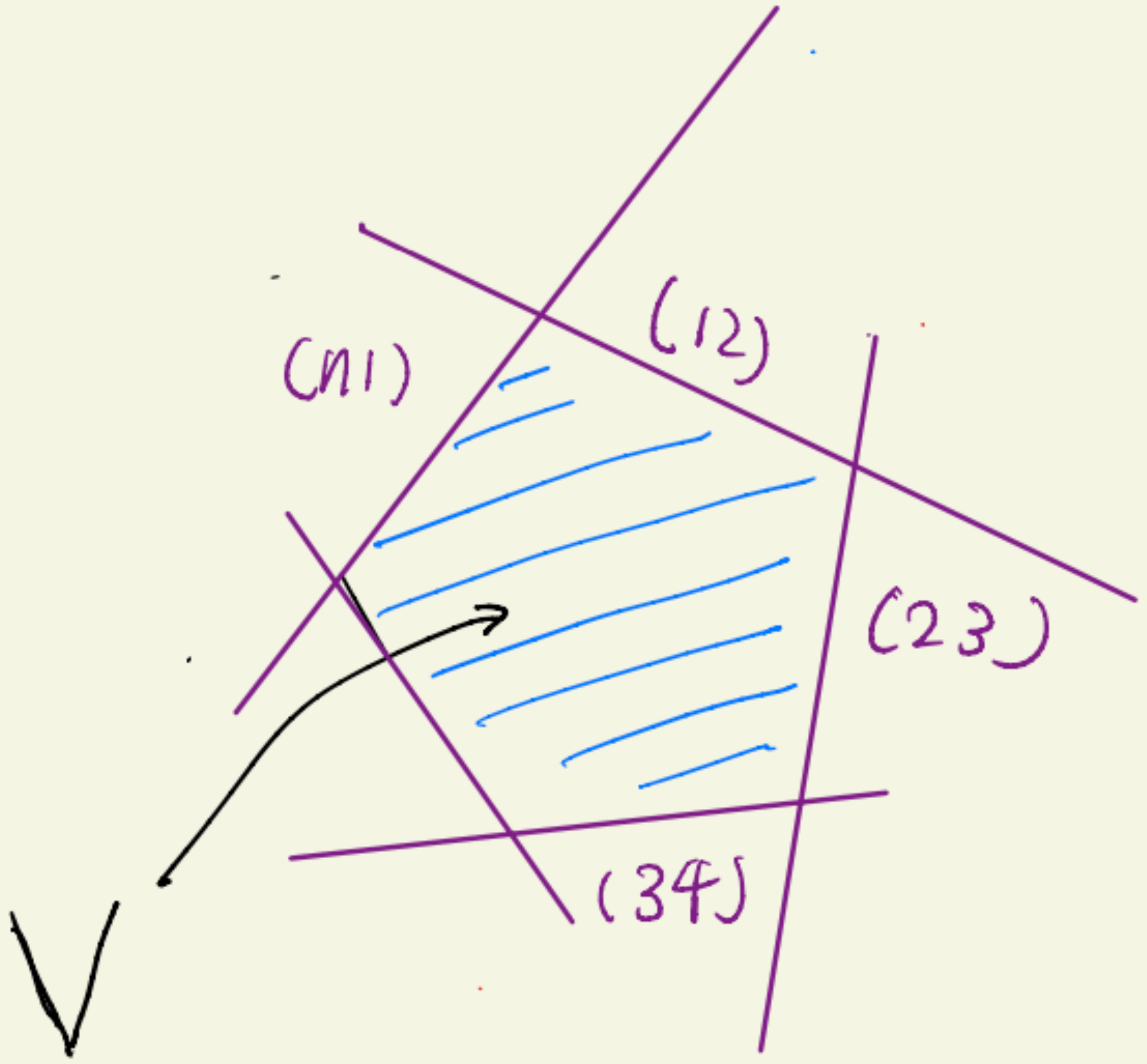
↓

$$\sum_j \hat{m}_i = 1$$



Face center

$$W_{(ij)} \cdot V \geq 0$$



The Vertex centered view represent the space through a convex hull

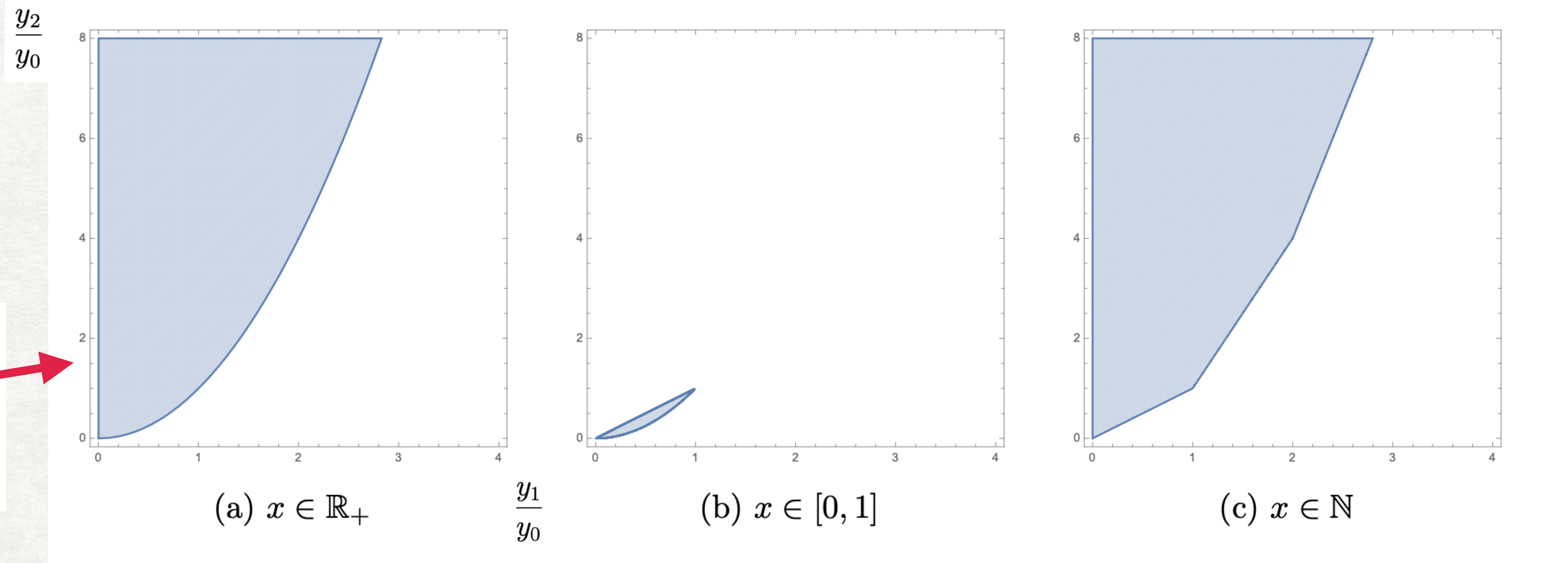
The Face centered view carves out the space (boundary) through a set of inequalities

Begin with a single moment

$$\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix} \in \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \\ x_i^d \end{pmatrix}, \quad \forall\, c_i > 0$$

Start with **P**^{2}

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \in \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ (x_i)^2 \end{pmatrix} \qquad c_i,\ x_i \geq 0.$$



(a) $x \in \mathbb{R}_+$

(b) $x \in [0,1]$

(c) $x \in \mathbb{N}$

$y_0,\ y_1,$

$$\det\begin{pmatrix} y_0 & y_1 \\ y_1 & y_2 \end{pmatrix} > 0$$

Begin with a single moment

$$\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix} \in \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \\ x_i^d \end{pmatrix}, \quad \forall\, c_i > 0$$

Start with P^2 the space is carved out by the positivity of the leading principle minors of the Hankel and (shifted) Hankel matrix

$$\color{red}{\left(\begin{array}{c|c|} y_0 & y_1 \\ \cline{1-1} y_1 & y_2 \\ \hline \end{array}\right)}, \qquad \left(\begin{array}{c|c} y_1 & y_2 \\ \cline{1-1} y_2 & y_3 \end{array}\right)$$

Generalize to arbitrary dimensions

$$K_n[\vec{y}] \equiv \begin{pmatrix} y_0 & y_1 & \cdots & y_n \\ y_1 & y_2 & \cdots & y_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ y_n & y_{n+1} & \cdots & y_{2n} \end{pmatrix}$$

→

$$\text{Det}\ (K_n[\vec{y}]) \geq 0, \quad \text{Det}\ \left(K_n^{\text{shift}}[\vec{y}]\right) \equiv \text{Det}\ (K_n[\vec{y}])\,|_{y_i \to y_{i+1}} \geq 0\,.$$

$$K_n[\vec{y}] \equiv \begin{pmatrix} y_0 & y_1 & \cdots & y_n \\ y_1 & y_2 & \cdots & y_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ y_n & y_{n+1} & \cdots & y_{2n} \end{pmatrix} \longrightarrow \text{Det}\ (K_n[\vec{y}]) \geq 0, \quad \text{Det}\ \left(K_n^{\text{shift}}[\vec{y}]\right) \equiv \text{Det}\ (K_n[\vec{y}])\,|_{y_i \to y_{i+1}} \geq 0.$$

From the vertex point of view these boundaries form a hierarchal complex

$$\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix} \in \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \\ x_i^d \end{pmatrix} \longrightarrow$$

$$\text{Det}\ (K_n[\vec{y}]) = \sum_{\{i_1,i_2,\cdots,i_{n+1}\}} \left[\prod_{a=1}^{n+1} c_{i_a}\right] \prod_{1\leq a<b<n+1} (x_{i_a} - x_{i_b})^2$$

$$\text{Det}\ \left(K_n^{\text{shift}}[\vec{y}]\right) = \sum_{\{i_1,i_2,\cdots,i_{n+1}\}} \left[\prod_{a=1}^{n+1} c_{i_a} x_{i_a}\right] \prod_{1\leq a<b<n+1} (x_{i_a} - x_{i_b})^2$$

vanishes if there are less than n+1 elements

furthermore $\text{Det}\ K_n^{\text{shift}} = 0, \quad \text{Det}\ K_n \neq 0$ if there is at most n elements + origin

Thus

$$\text{Det}\ K_0 \subset \text{Det}\ K_0^{\text{shift}} \subset \ \cdots \subset \text{Det}\ K_{n-1} \subset \text{Det}\ K_{n-1}^{\text{shift}} \subset \text{Det}\ K_n \subset \text{Det} K_n^{\text{shift}}$$

$$\longleftarrow$$

successive vanishing of the Hankel determinant represents the reduction of rank

The positivity conditions on the Hankel matrices can be easily understood as follows: writing

$$K(y)=\sum_i c_i\mathbf{x}_i\mathbf{x}_i^T=\sum_i c_i\begin{pmatrix}1\\ x_i\\ x_i^2\\ \vdots\end{pmatrix}\begin{pmatrix}1 & x_i & x_i^2 & \cdots\end{pmatrix}, \longrightarrow \quad \mathbf{v}^TK\mathbf{v}=\sum_i c_i(\mathbf{v}^T\mathbf{x}_i)^2\geq 0\,.$$

Positive for any vector v implies the principle minors are positive

For bounded moments we further have, say $a<x_i<b$ then

$$\mathbf{v}^T\left(\sum_i c_i(x_i-a)(b-x_i)\mathbf{x}_i\mathbf{x}_i^T\right)\mathbf{v}\geq 0$$

For example $a=0, b\rightarrow\infty.$

$$\mathbf{v}^T\left(\sum_i c_i(x_i)\mathbf{x}_i\mathbf{x}_i^T\right)\mathbf{v}\Rightarrow\begin{pmatrix}y_1 & y_2 & y_3 & \cdots\\ y_2 & y_3 & y_4 & \cdots\\ y_3 & y_4 & y_5 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}\geq 0,$$

Positive for any vector v implies the principle minors of the shifted Hankel is positive

This will guild us to the boundary of the double moment

$$[y]_{d+1\times d+1} = \begin{pmatrix} y^{(0,0)} & y^{(0,1)} & \dots & y^{(0,d)} \\ y^{(1,0)} & y^{(1,1)} & \dots & y^{(1,d)} \\ \vdots & \vdots & \vdots & \vdots \\ y^{(d,0)} & y^{(d,1)} & \dots & y^{(d,d)} \end{pmatrix} \in \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \\ x_i^d \end{pmatrix} \begin{pmatrix} 1 \\ \tilde{x}_i \\ \tilde{x}_i^2 \\ \vdots \\ \tilde{x}_i^d \end{pmatrix}^T \quad , \quad \forall c_i > 0$$

This suggests generalized Hankel matrix

$$K(y) = \sum_i p_i \begin{pmatrix} 1 \\ x_i \\ \tilde{x}_i \\ x_i^2 \\ x_i\tilde{x}_i \\ \tilde{x}_i^2 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 & x_i & \tilde{x}_i & x_i^2 & x_i\tilde{x}_i & \tilde{x}_i^2 & \cdots \end{pmatrix} = \begin{pmatrix} y^{(0,0)} & y^{(1,0)} & y^{(0,1)} & \dots \\ y^{(1,0)} & y^{(2,0)} & y^{(1,1)} & \dots \\ y^{(0,1)} & y^{(1,1)} & y^{(0,2)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

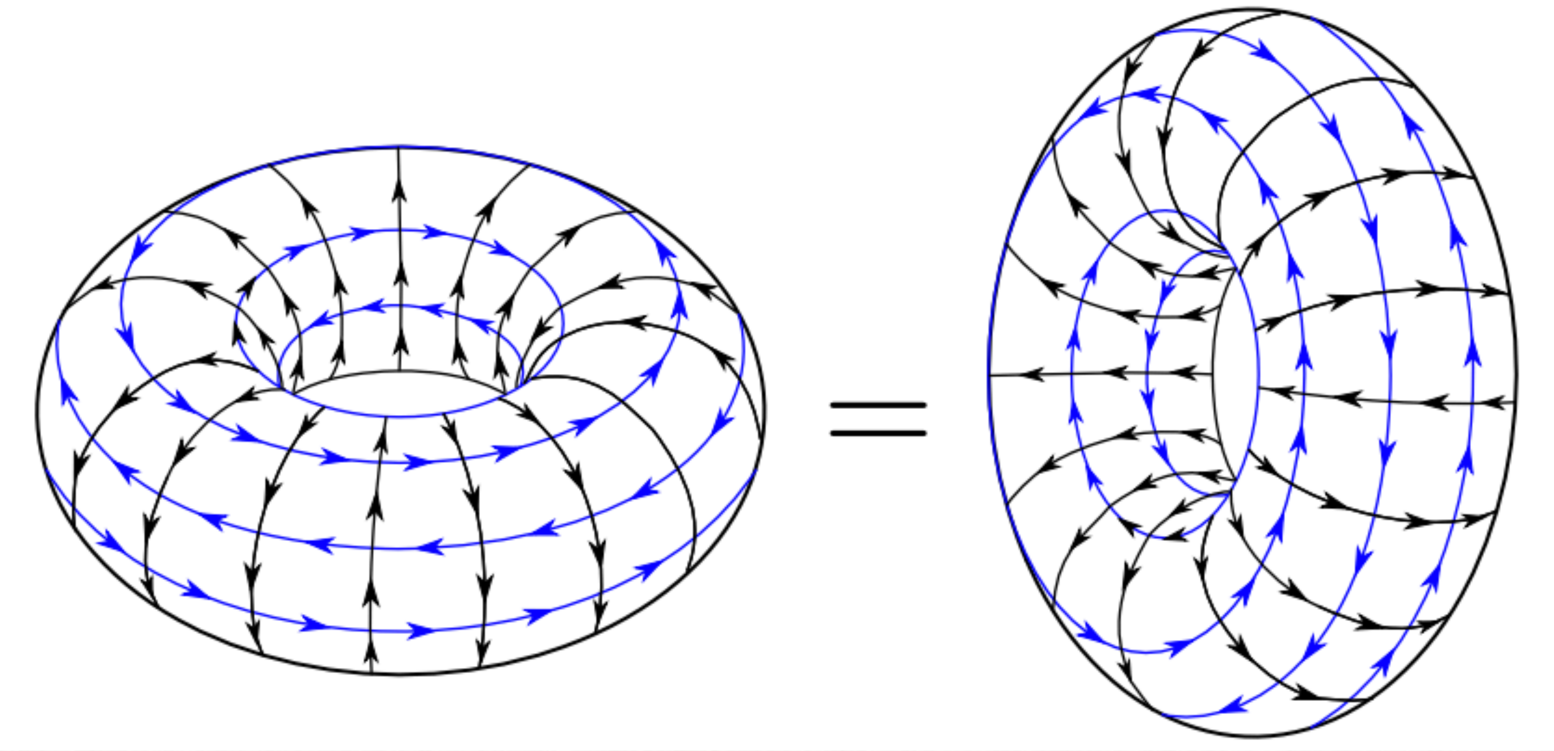
the convex hull implies

Half moment: $x, \tilde{x} \in \mathbb{R}^+$

$$\boxed{K(y), \quad K^{\text{shift},x}(y) \equiv K(y)|_{y^{(m,n)} \to y^{(m+1,n)}}, \quad K^{\text{shift},\tilde{x}}(y), \quad K^{\text{shift},x,\tilde{x}}(y) \geq 0}$$

These are the necessary (not sufficient) conditions of the hull

We will find that these two theory space is governed by the same product geometry.

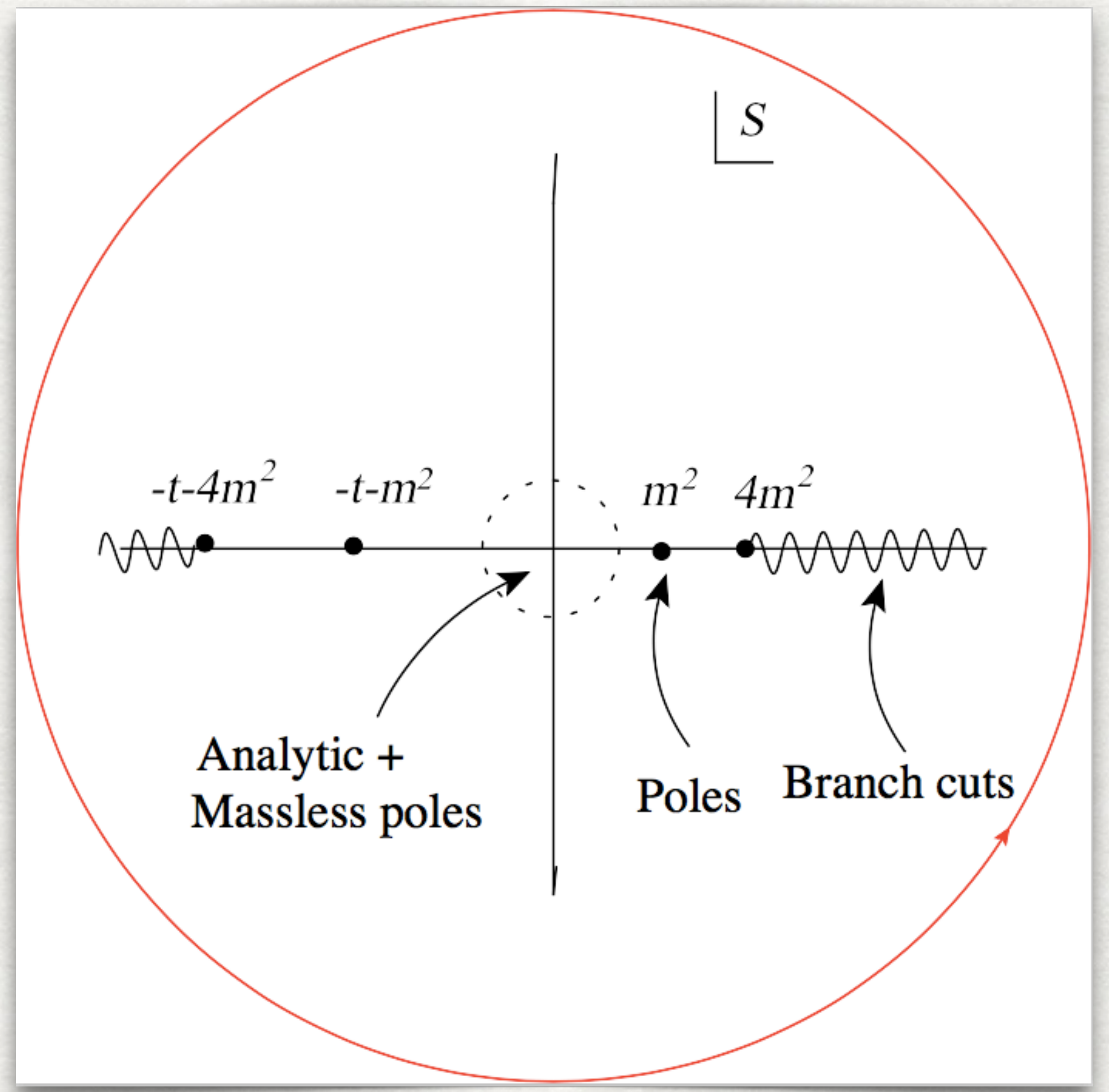


$$\mathcal{L} = \partial\phi\partial\phi + g_0\phi^4 + g_2(\partial\phi)^2\phi^2 + g_4(\partial\phi)^2(\partial\phi)^2 + g_6(\partial\phi)^2(\partial\phi)^2 \cdots$$

2D CFT

EFT

Product moment in the EFT



Unitarity and **Causality** tells us that M(s,t)< s^2 at large s for t>0

$$I = \frac{i}{2\pi}\int_{\infty}\frac{ds}{s^{n+1}}M(s,0)$$

=0 for n>1

The vanishing of the contour tell us that the **Taylor coefficients (g)** of the EFT is completely controlled by the discontinuity

$$\sum_{q=0}^{\infty} g_{n+q,q}t^q = \int \frac{ds}{s^{n+1}}Dis[M(s,t)]$$

The Taylor coefficients can be isolated by further expand in t

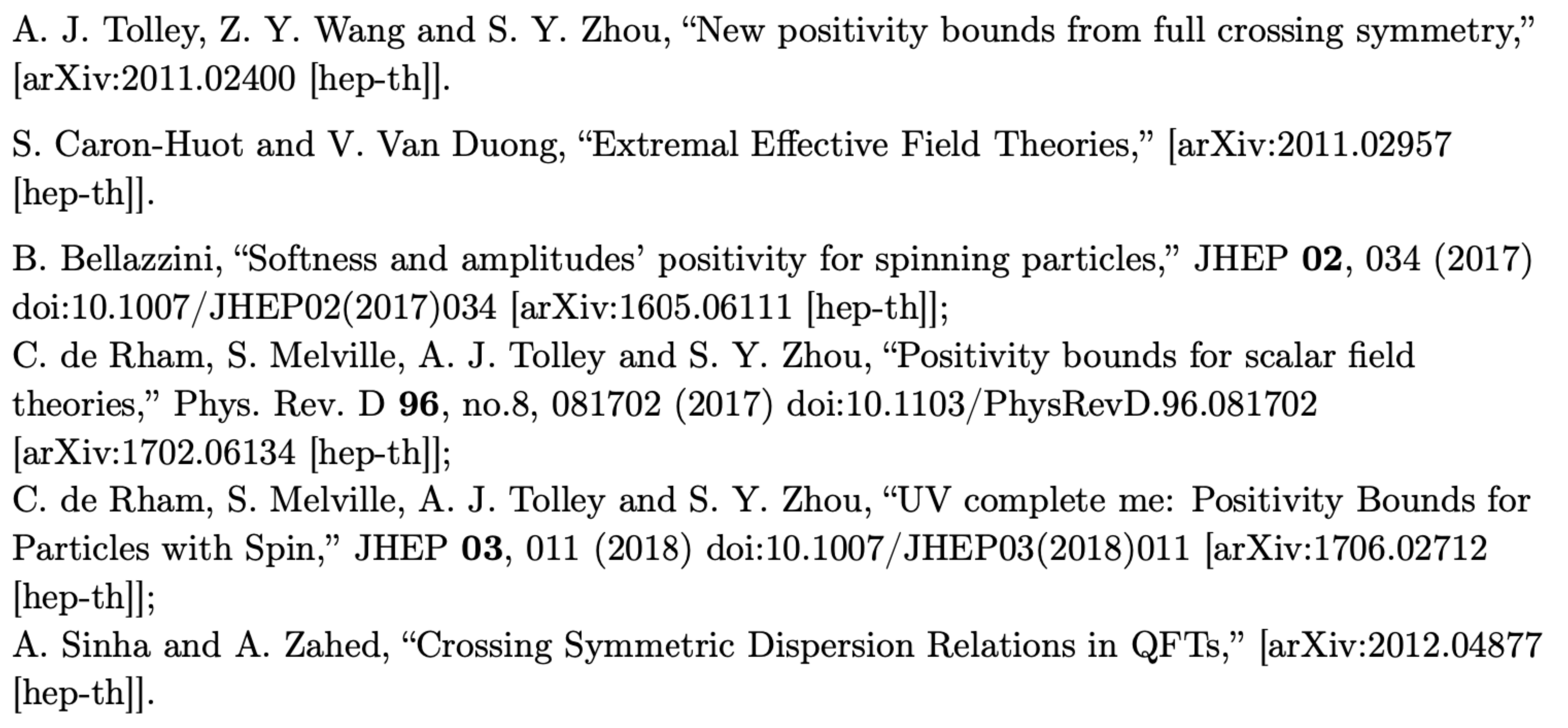
$$g_{k,q} = -\frac{1}{q!}\frac{\partial^q}{\partial t^q}\left(\sum_a \frac{Res_{s=m_a^2}M(s,t)}{(m_a^2)^{k-q+1}} + \int_{4m_a^2}\frac{ds'}{s'^{k-q+1}}DisM(s,t)\right)\Bigg|_{t=0} + \{u\}$$

$$\longrightarrow \quad g_{k,q} = \frac{1}{q!}\frac{d^q}{dt^q}\left(\sum_a \frac{\mathsf{p}_a G_{\ell_a}(1+2\frac{t}{m_a^2})}{(m_a^2)^{k-q+1}} + \sum_b \int ds'\mathsf{p}_{b,\ell}(s')\frac{G_\ell(1+2\frac{t}{s'})}{(s')^{k-q+1}} + \{u\}\right)\Bigg|_{t=0}$$

The Wilson coefficients are given by a convex hull

$$g_{k,q} = \frac{1}{q!}\frac{d^q}{dt^q}\left(\sum_a \frac{\mathsf{p}_a G_{\ell_a}(1+2\frac{t}{m_a^2})}{(m_a^2)^{k-q+1}} + \sum_b \int ds'\,\mathsf{p}_{b,\ell}(s')\frac{G_\ell(1+2\frac{t}{s'})}{(s')^{k-q+1}} + \{u\}\right)\Bigg|_{t=0}$$

The expansion in t is governed by the Taylor expansion of the Gegenbauer polynomial.



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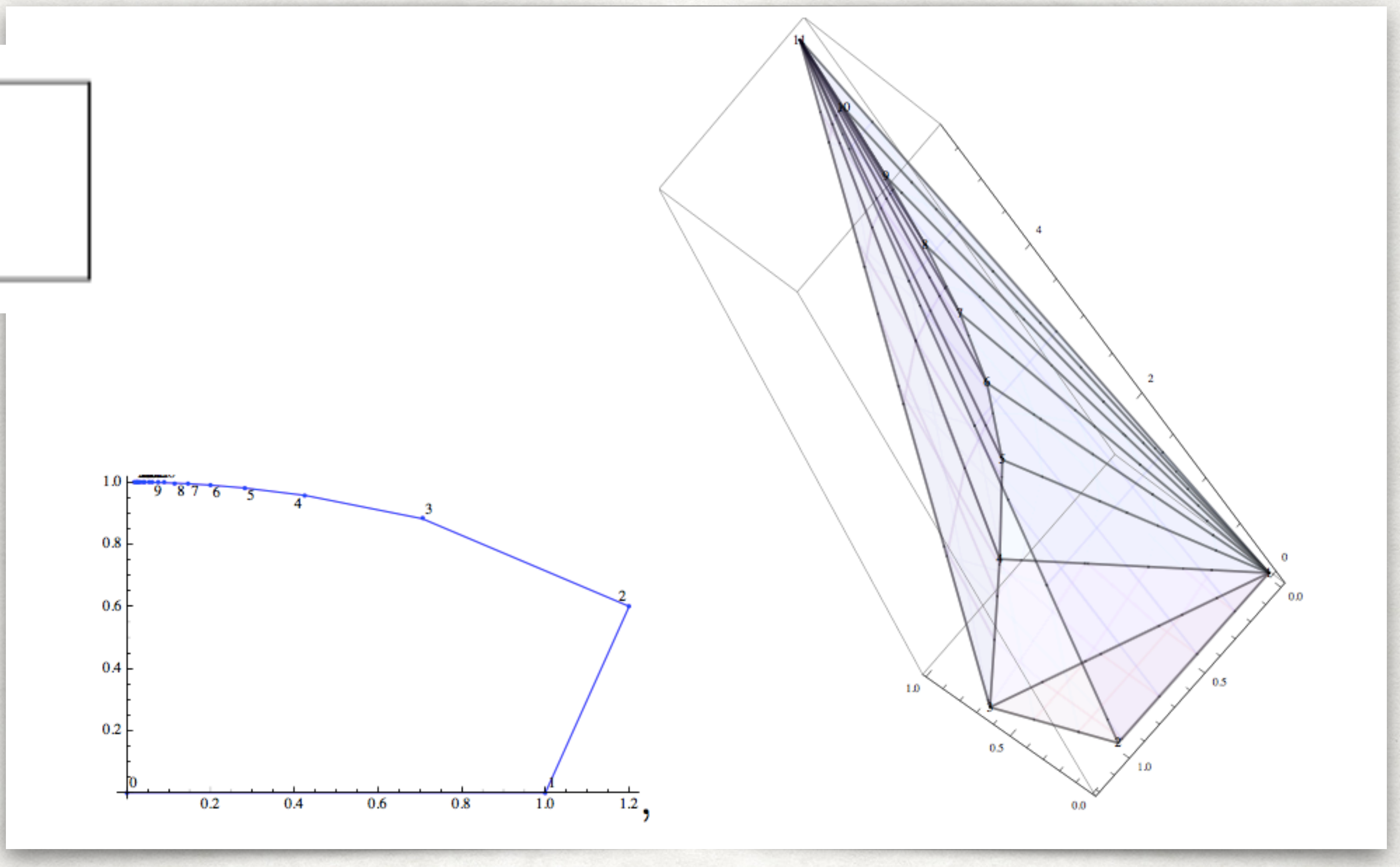
The **Wilson coefficients** are given by a **convex hull**

$$g_{k,q} = \frac{1}{q!}\frac{d^q}{dt^q}\left(\sum_a \frac{\mathsf{p}_a G_{\ell_a}(1+2\frac{t}{m_a^2})}{(m_a^2)^{k-q+1}} + \sum_b \int ds' \mathsf{p}_{b,\ell}(s')\frac{G_\ell(1+2\frac{t}{s'})}{(s')^{k-q+1}} + \{u\}\right)\Bigg|_{t=0}$$

The expansion in t is governed by the Taylor expansion of the Gegenbauer polynomial. We arrive at :

$$\boxed{g_{k,q} = \sum_a \mathsf{p}_a \frac{2^q u^{\alpha}_{\ell_a,k,q}}{(m_a^2)^{k+1}} \qquad \mathsf{p}_a > 0}$$

$$\begin{pmatrix} g_{k,0} \\ g_{k,1} \\ g_{k,2} \\ \vdots \end{pmatrix} = \sum_a \mathsf{p}_a \begin{pmatrix} u^{\alpha}_{\ell_a,k,0} \\ u^{\alpha}_{\ell_a,k,1} \\ u^{\alpha}_{\ell_a,k,2} \\ \vdots \end{pmatrix}$$



Let's look closer at the s-channel contribution in detail

$$g^s_{k,q} = \sum_a p_a \frac{v_{\ell_a,q}}{m_a^{2(k+1)}}$$

Note that we have a product geometry

$$\begin{pmatrix} g_{0,0} & & & \\ g_{1,0} & g_{1,1} & & \\ g_{2,0} & g_{2,1} & g_{2,2} & \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \sum_a p_a \begin{pmatrix} \frac{1}{m_a^2} \\ \frac{1}{m_a^4} \\ \frac{1}{m_a^6} \\ \frac{1}{m_a^8} \\ \vdots \end{pmatrix} \otimes (v_{\ell_a,0}, v_{\ell_a,1}, v_{\ell_a,2}, \cdots)$$

The analytic formula for the expansion coefficients (for D=4)

$$v_{\ell,q} = \frac{2^q}{q!(2-q)!}\frac{(\alpha)_{\ell+q}}{\prod_{a=1}^{q}(\alpha+2a-1)} = \frac{\prod_{a=0}^{q}(J-a(a-1))}{(q!)^2}$$

$$J = \ell(\ell+1)$$

After a linear transformation $\vec{a}_k = \mathbf{G}\cdot\vec{g}^s_k$ we have

$$\begin{pmatrix} a_{0,0} & & & \\ a_{1,0} & a_{1,1} & & \\ a_{2,0} & a_{2,1} & a_{2,2} & \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \sum_a p_a \begin{pmatrix} \frac{1}{m_a^2} \\ \frac{1}{m_a^4} \\ \frac{1}{m_a^6} \\ \frac{1}{m_a^8} \\ \vdots \end{pmatrix} \otimes (1, J, J^2, J^3, \cdots)$$

at the core, the couplings →
are governed by the hull of product moments

The linear transformed EFT couplings $\vec{a}_k = \mathbf{G} \cdot \vec{g}^s_k$ live inside our favorite product geometry !

$$\begin{pmatrix} a_{0,0} & & & \\ a_{1,0} & a_{1,1} & & \\ a_{2,0} & a_{2,1} & a_{2,2} & \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \sum_a p_a \begin{pmatrix} \frac{1}{m_a^2} \\ \frac{1}{m_a^4} \\ \frac{1}{m_a^6} \\ \frac{1}{m_a^8} \\ \vdots \end{pmatrix} \otimes (1, J, J^2, J^3, \cdots)$$

An inverse transform rotates us back to the physical coupling

“a” geometry

$$a_{k,q} = \sum p \frac{J^q}{m^k}$$

Rotation

“s” geometry

$$g^s_{k,q} = L(a_{k,q})$$

Intersection

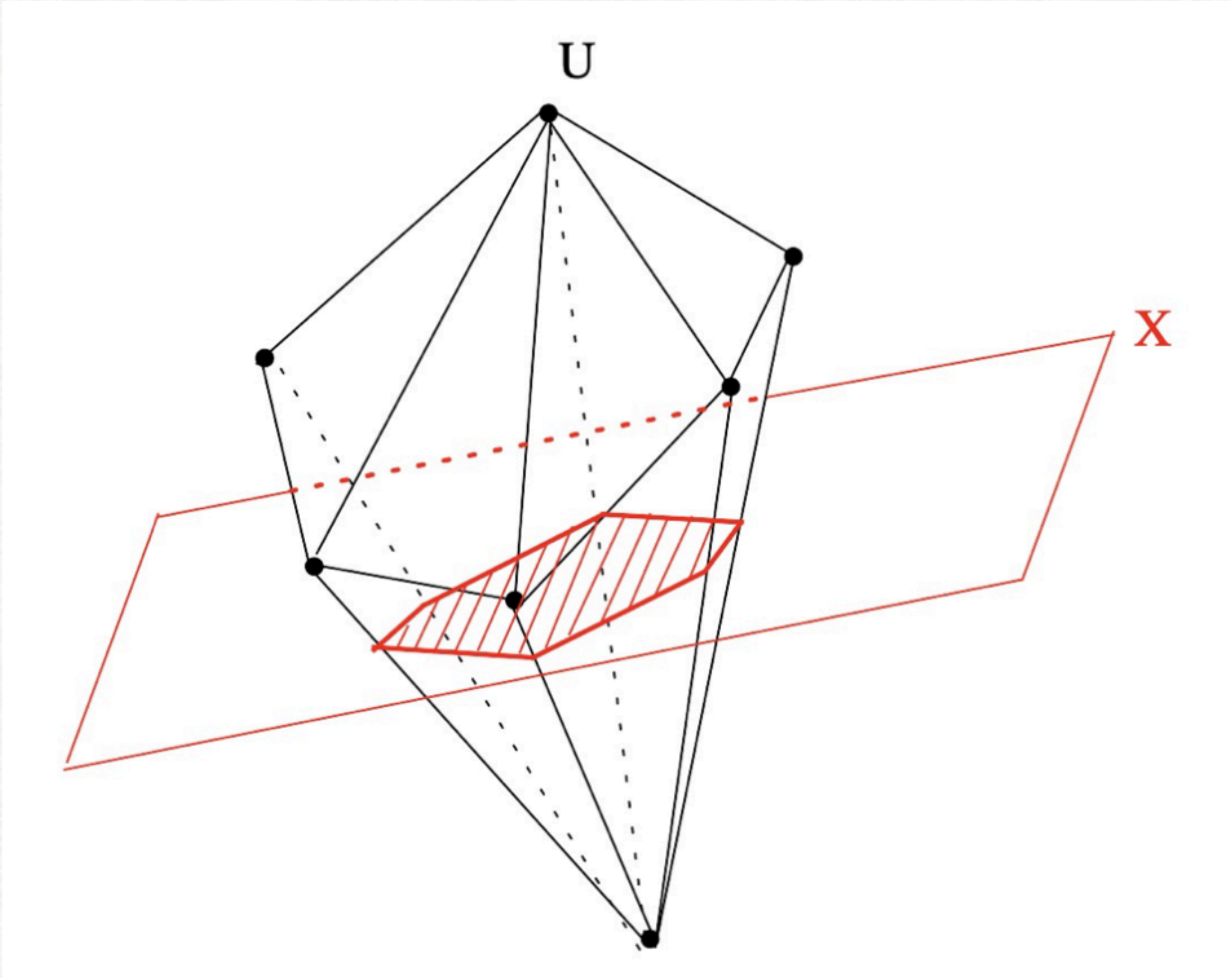
cyclic plane

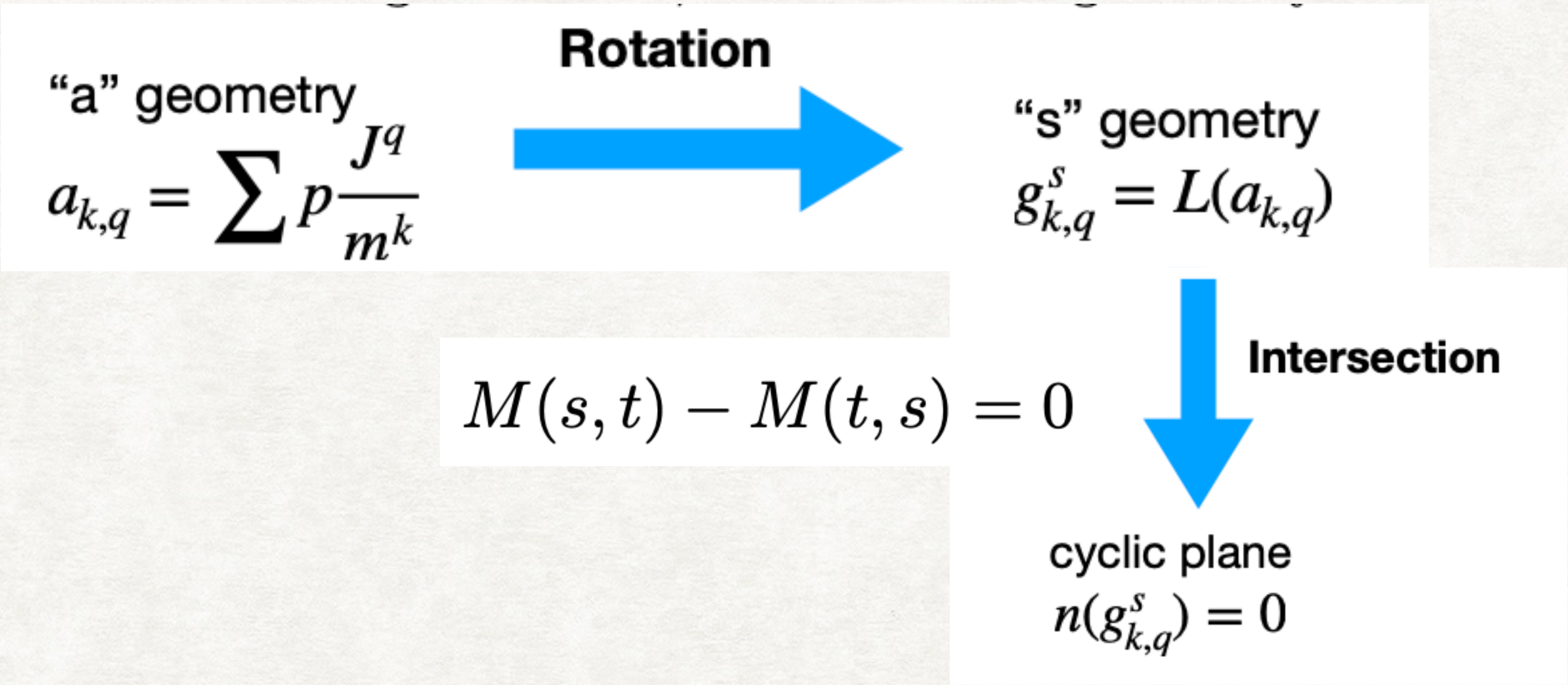
$$n(g^s_{k,q}) = 0$$

We can further impose cyclic symmetry

$$M(s,t) - M(t,s) = 0$$

$$g_{k,q} = g_{k,k-q}$$





Rotation

“a” geometry

$a_{k,q} = \sum p \frac{J^q}{m^k}$

“s” geometry

$g^s_{k,q} = L(a_{k,q})$

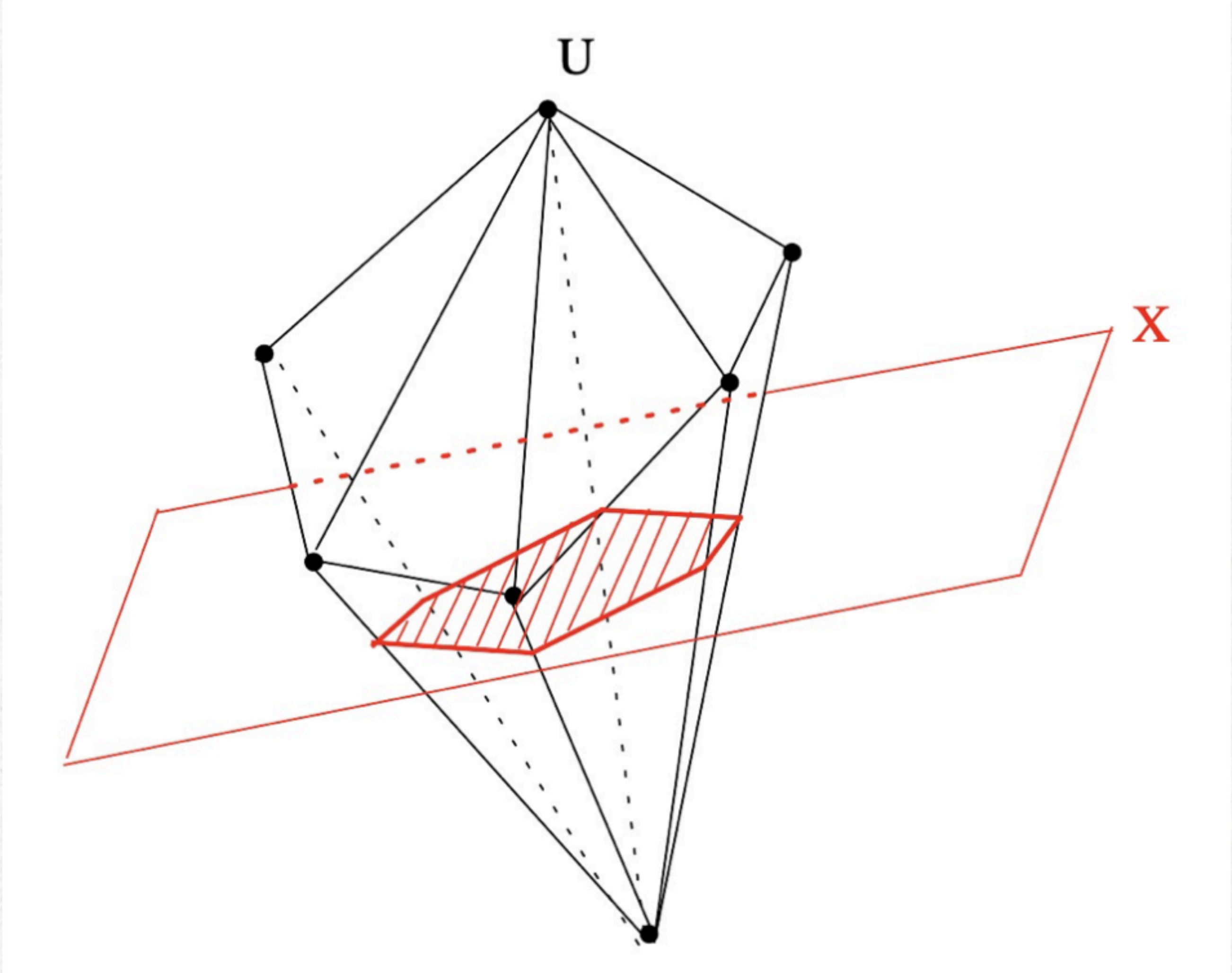
Intersection

cyclic plane

$n(g^s_{k,q}) = 0$

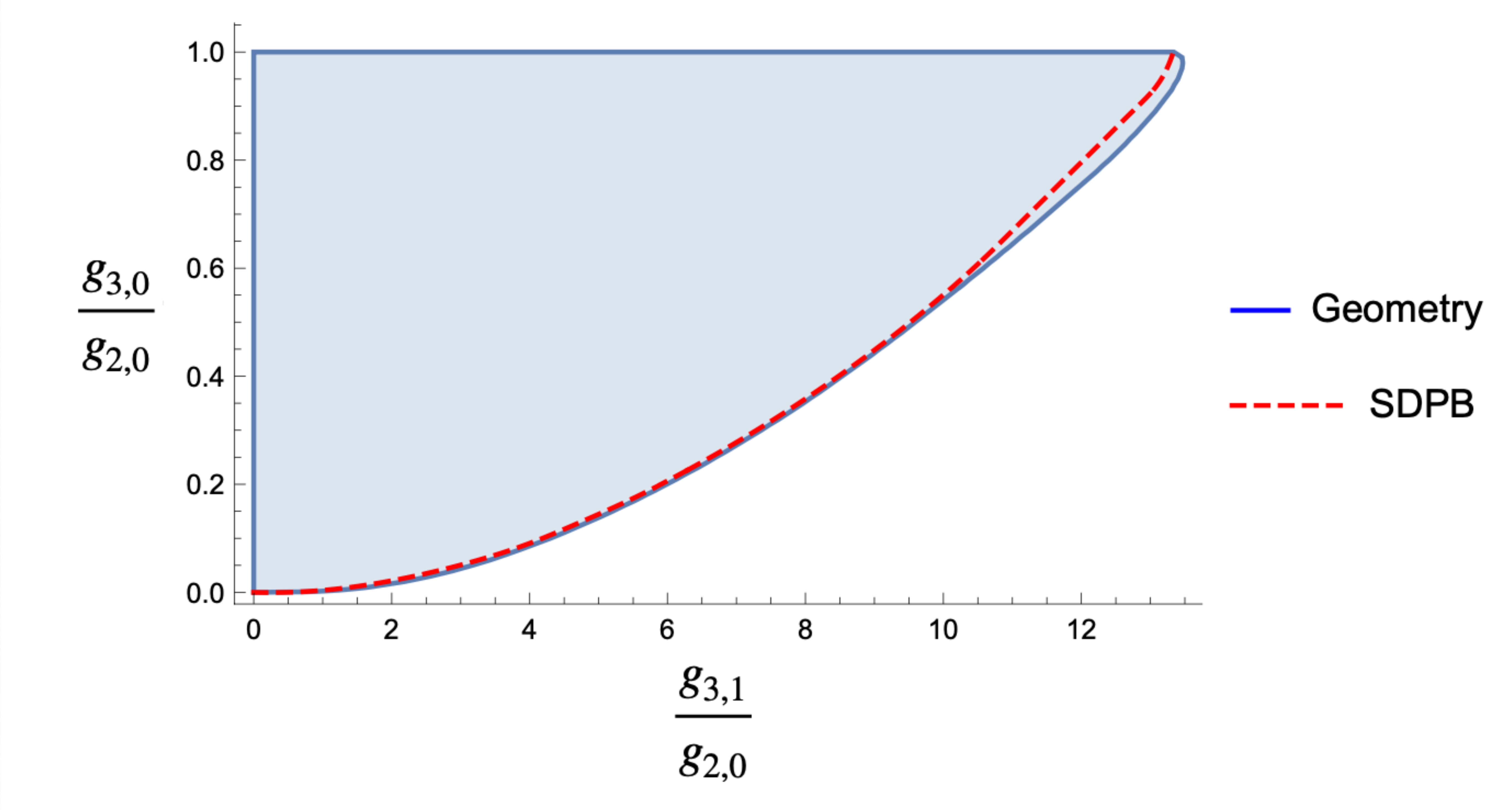
$$M(s,t) - M(t,s) = 0$$

The s-EFThedron



The s-EFThedron

$$M(s,t) = \cdots + g_{3,0}s^3 + g_{3,1}s^2t + g_{4,0}s^4 + g_{4,1}s^3t + g_{4,2}s^2t^2 \cdots \qquad (D^6\phi^4,\ D^8\phi^4)$$



Normalized with $M_{gap} = 1$ $g_{20} = 1$

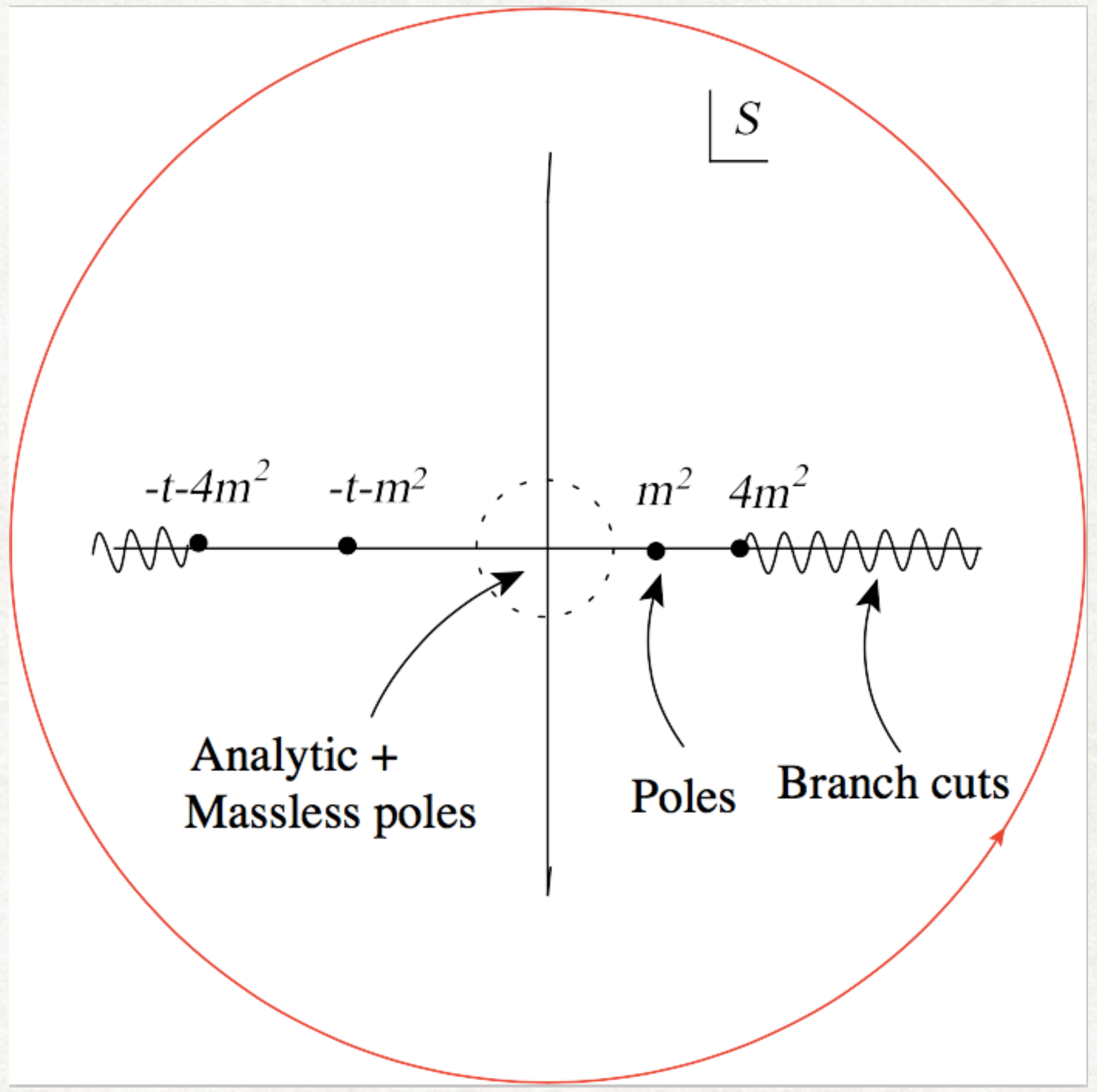
The Full-EFThedron

So far we've only considered s-channel singularities, in general at fixed t, we will have both s,u thresholds

Consider a,b -> a,b

$$M^{\rm IR}(a_1, b_2, b_2, a_3) = \sum_{k,q} g_{k,q} z^{k-q} t^q$$

$$s = -t/2+z,\ u = -t/2-z$$



once again the dispersion relations leads to

$$\sum_{k,q} g_{k,q} z^{k-q} t^q = -\sum_i p_i\, P_{\ell_i}(1+\frac{2t}{m_i^2}) \left(\frac{1}{-\frac{t}{2}-z-m_i^2} + \frac{1}{-\frac{t}{2}+z-m_i^2} \right)$$

expanding in z, t

$$g_{k,q} = \sum_i p_i \frac{u_{\ell_i,k,q}}{m_i^{2(k+1)}}$$

$$\vec{u}_{\ell,k} = \begin{pmatrix} u_{\ell,k,0} \\ 0 \\ u_{\ell,k,2} \\ \vdots \\ 0 \\ u_{\ell,k,k} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{(k-1)_2}{2}\frac{1}{2^2} & (k-1)_1 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{(1)_k}{k!}2^{-k} & -\frac{(1)_{k-1}}{k-1!}2^{2-k} & \frac{(1)_{k-2}}{k-2!}2^{4-k} & \cdots & -2^{k-2} & 1 \end{pmatrix} \begin{pmatrix} v_{\ell,0} \\ v_{\ell,1} \\ v_{\ell,2} \\ \vdots \\ v_{\ell,k-1} \\ v_{\ell,k} \end{pmatrix}$$

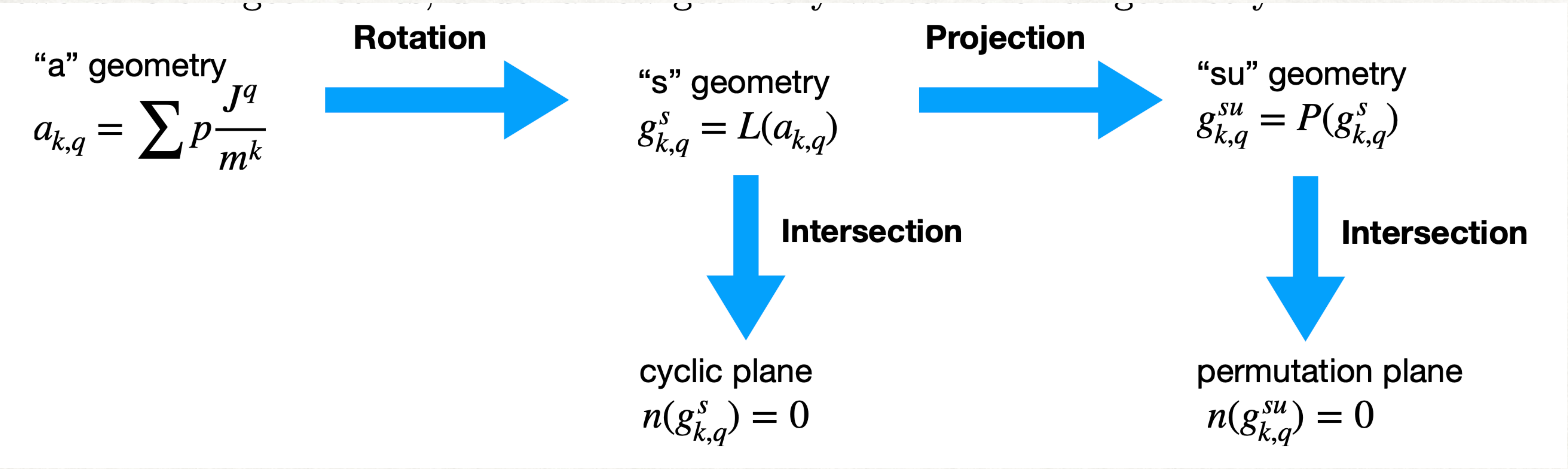
The Full-EFThedron

So far we've only considered s-channel singularities, in general at fixed t, we will have both s,u thresholds

Consider a,b -> a,b

$$M^{\rm IR}(a_1,b_2,b_2,a_3)=\sum_{k,q} g_{k,q}z^{k-q}t^q$$

$$s=-t/2+z,\ u=-t/2-z$$



"a" geometry

$a_{k,q}=\sum p\frac{J^q}{m^k}$

Rotation

"s" geometry

$g^s_{k,q}=L(a_{k,q})$

Projection

"su" geometry

$g^{su}_{k,q}=P(g^s_{k,q})$

Intersection

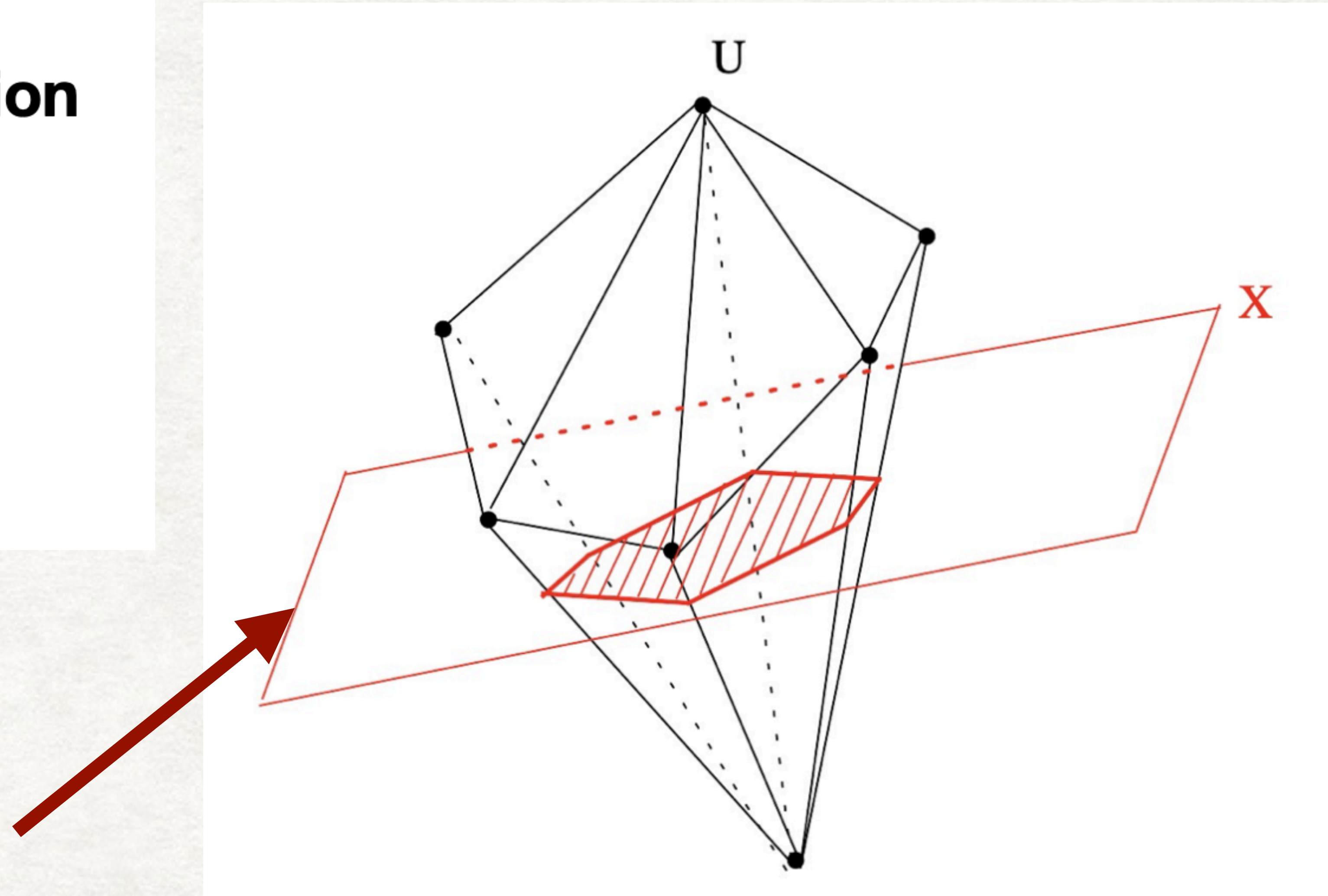
cyclic plane

$n(g^s_{k,q})=0$

Intersection

permutation plane

$n(g^{su}_{k,q})=0$

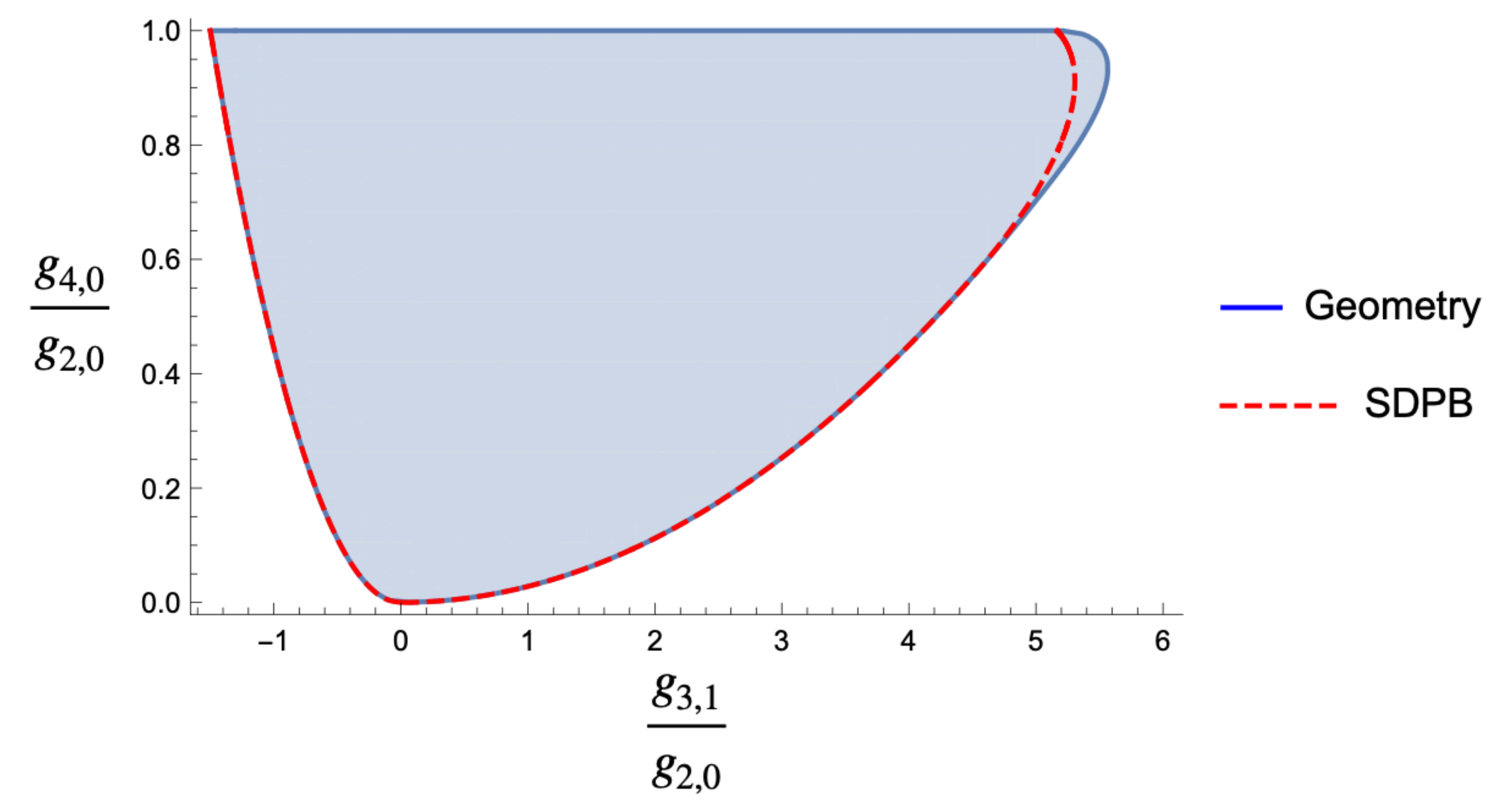


For identical scalars we further impose permutation invariance

$$\mathbf{X}_{\rm perm}\to M(s,t)=M(s,-t-s)=M(-t-s,t)$$

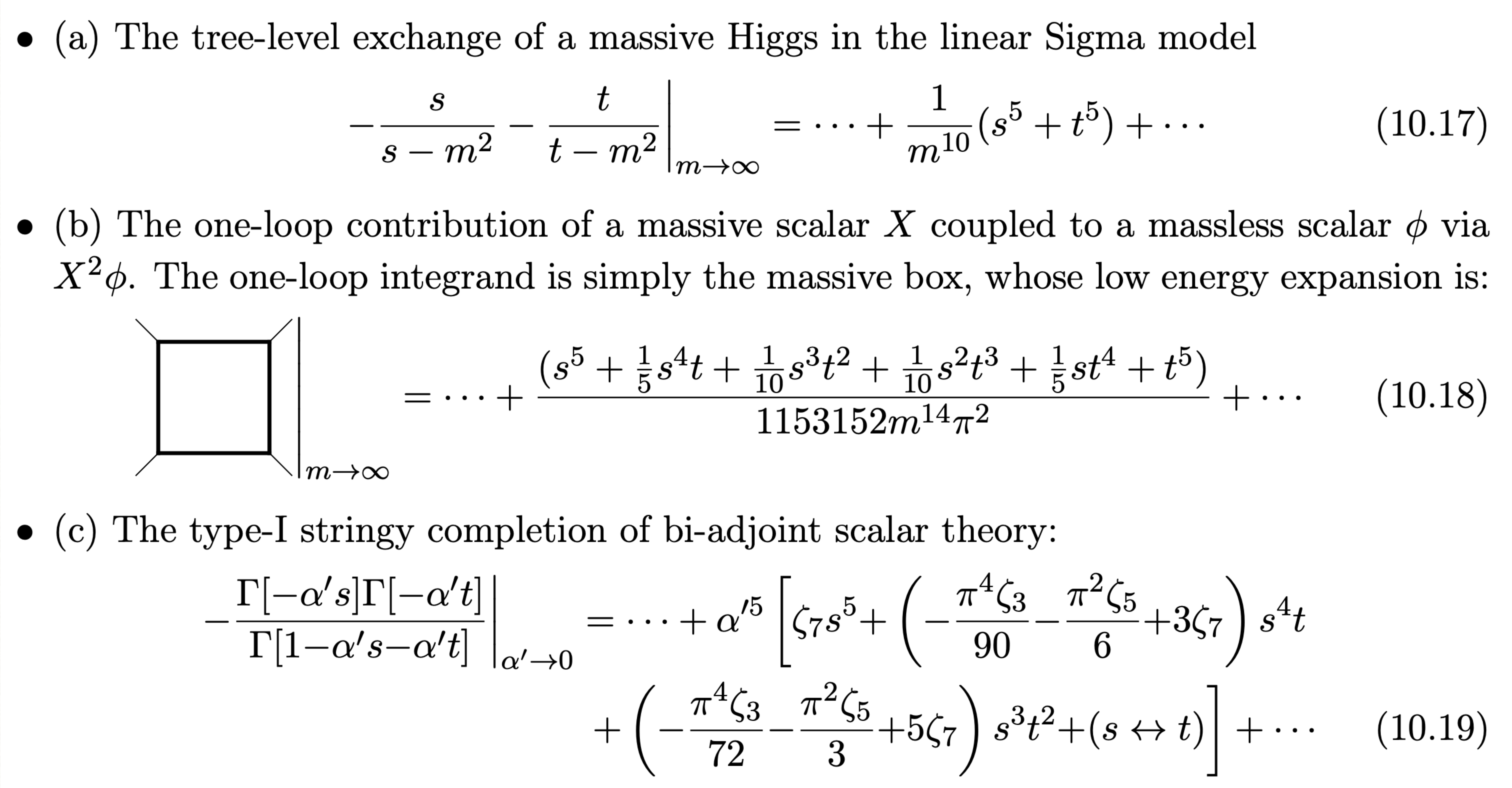
The Full-EFThedron

So far we've only considered s-channel singularities, in general at fixed t, we will have both s,u thresholds



Explicit EFTs

$$g_{5,0}s^5+g_{5,1}s^4t+g_{5,2}s^3t^2+g_{5,3}s^2t^3 = s^5+xs^4t+ys^3t^2+ys^2t^3$$



- (a) The tree-level exchange of a massive Higgs in the linear Sigma model

$$-\frac{s}{s-m^2}-\frac{t}{t-m^2}\bigg|_{m\to\infty} = \cdots + \frac{1}{m^{10}}(s^5+t^5)+\cdots \tag{10.17}$$

- (b) The one-loop contribution of a massive scalar X coupled to a massless scalar ϕ via $X^2\phi$. The one-loop integrand is simply the massive box, whose low energy expansion is:

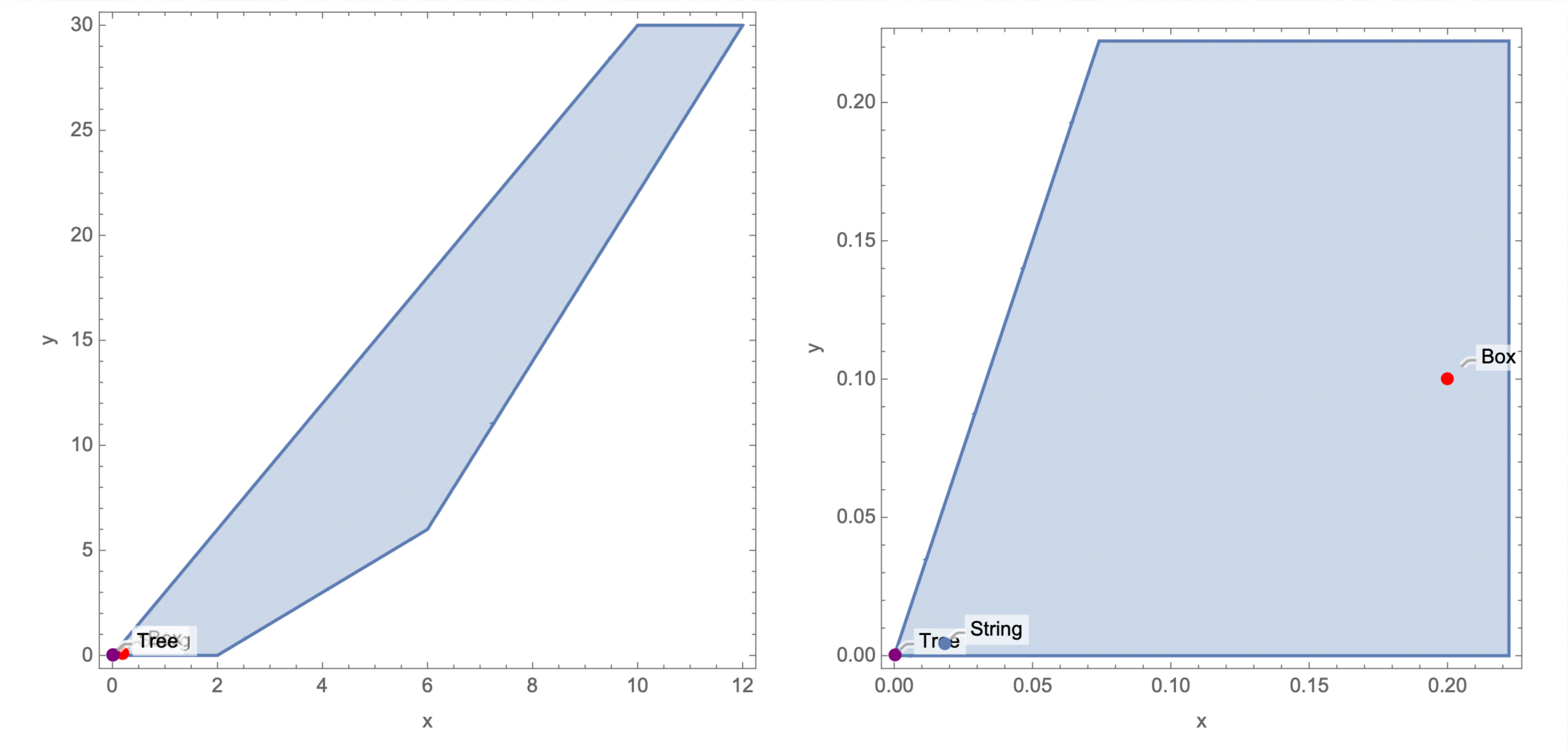
$$\bigg|_{m\to\infty} = \cdots + \frac{(s^5+\frac{1}{5}s^4t+\frac{1}{10}s^3t^2+\frac{1}{10}s^2t^3+\frac{1}{5}st^4+t^5)}{1153152m^{14}\pi^2}+\cdots \tag{10.18}$$

- (c) The type-I stringy completion of bi-adjoint scalar theory:

$$-\frac{\Gamma[-\alpha's]\Gamma[-\alpha't]}{\Gamma[1-\alpha's-\alpha't]}\bigg|_{\alpha'\to 0} = \cdots + \alpha'^5\left[\zeta_7 s^5+\left(-\frac{\pi^4\zeta_3}{90}-\frac{\pi^2\zeta_5}{6}+3\zeta_7\right)s^4t \right.$$
$$\left. +\left(-\frac{\pi^4\zeta_3}{72}-\frac{\pi^2\zeta_5}{3}+5\zeta_7\right)s^3t^2+(s\leftrightarrow t)\right]+\cdots \tag{10.19}$$

For example

$$g_{5,0}s^5+g_{5,1}s^4t+g_{5,2}s^3t^2+g_{5,3}s^2t^3 = s^5+xs^4t+ys^3t^2+ys^2t^3$$

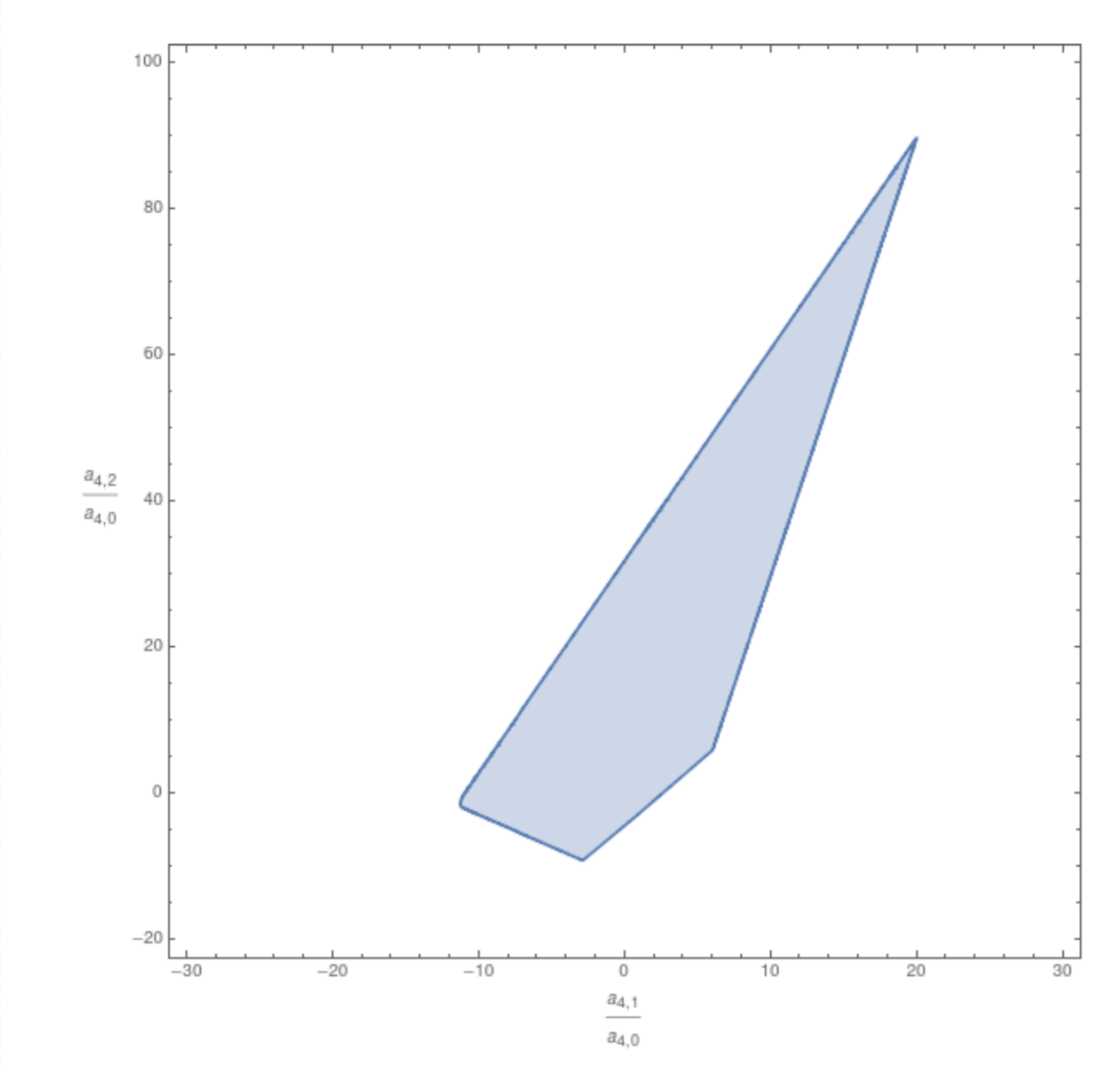


We can generalize to spinning external states $M(+h,+h,-h,-h)$

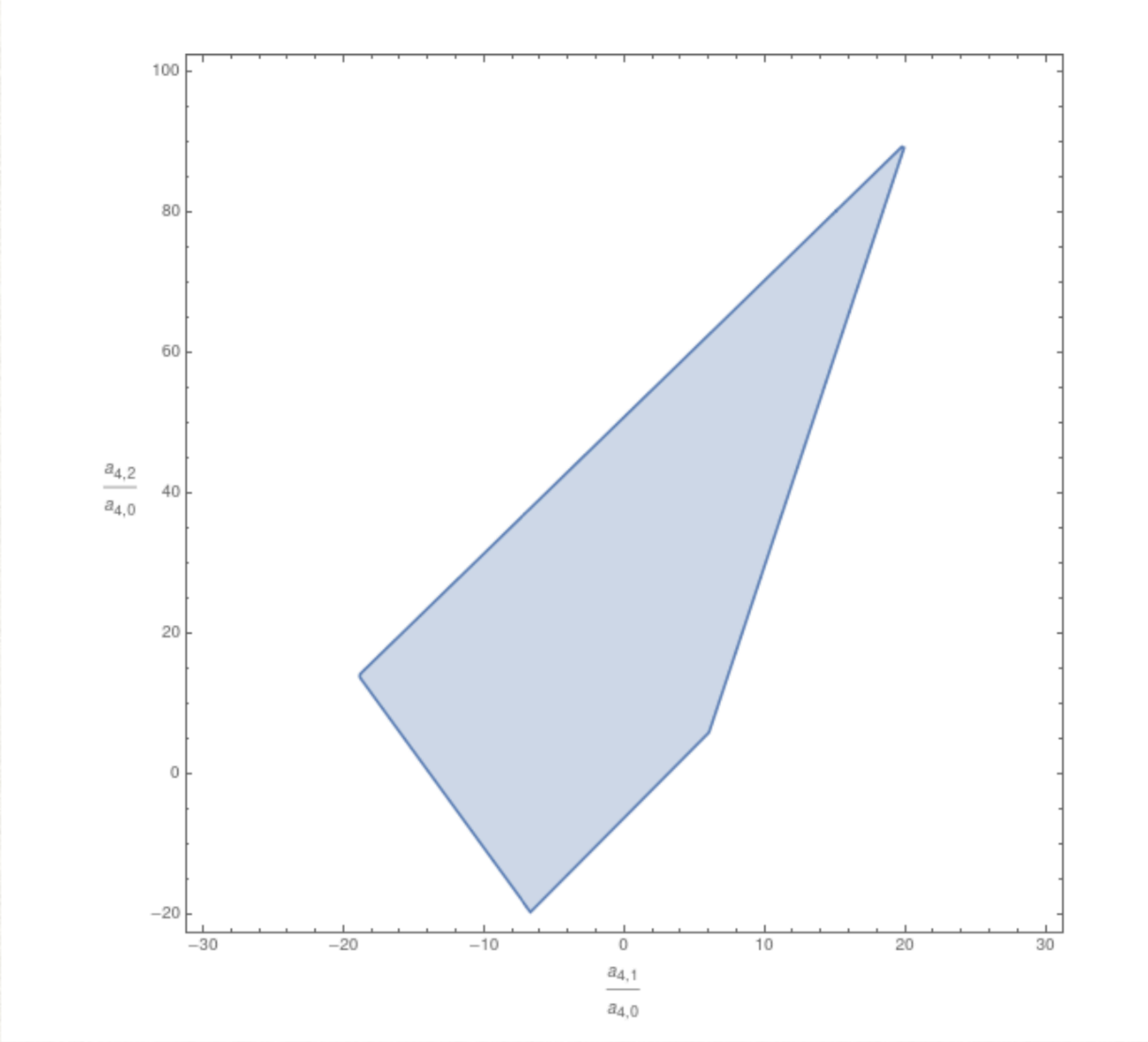
leads to EFTs of photons and gravitons

$$[12]^{2h}\langle 34\rangle^{2h}(\sum_{k,q} a_{k,q}s^{k-q}t^{q}) = -[12]^{2h}\langle 34\rangle^{2h}\left(\sum_{\ell_a\geq 0} p_i\,\frac{d^{\ell_i=even}_{0,0}(\theta)}{s-m_i^2} + \sum_{\ell_j\geq 2h} p_j\,\frac{\tilde{d}^{\ell_j}_{2h,2h}(\theta)}{-t-s-m_j^2}\right)$$

D^8F^4

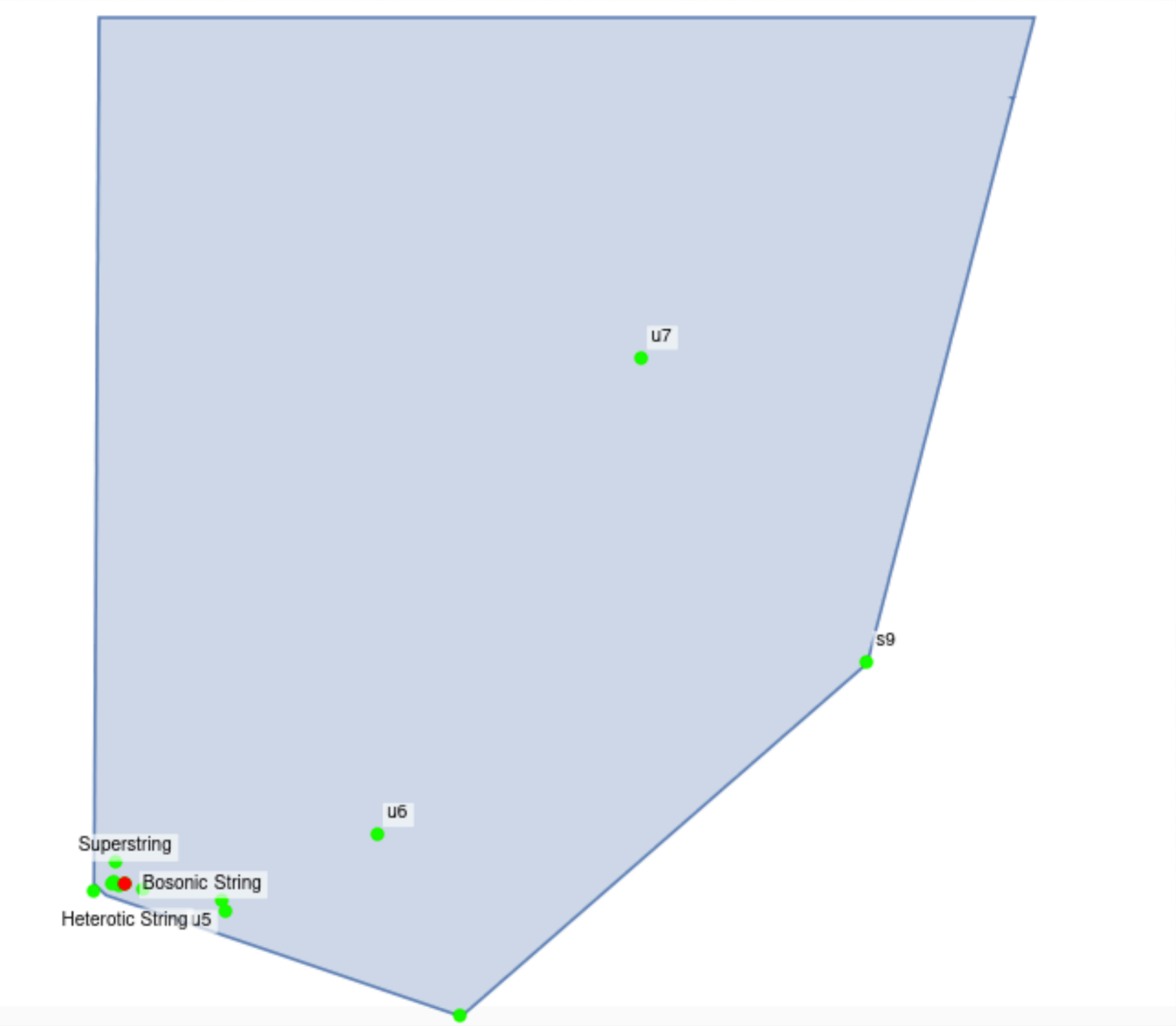


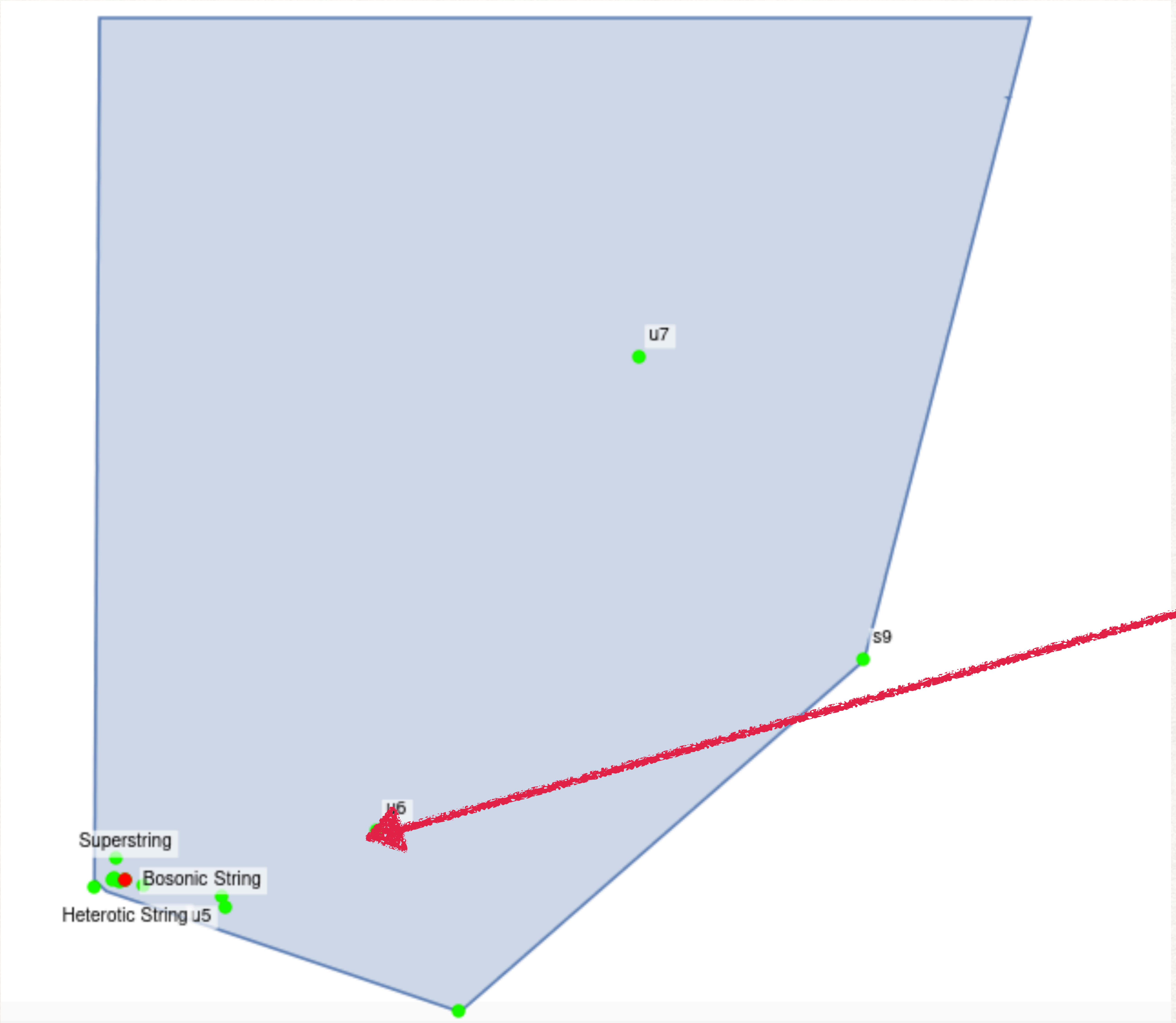
D^8R^4

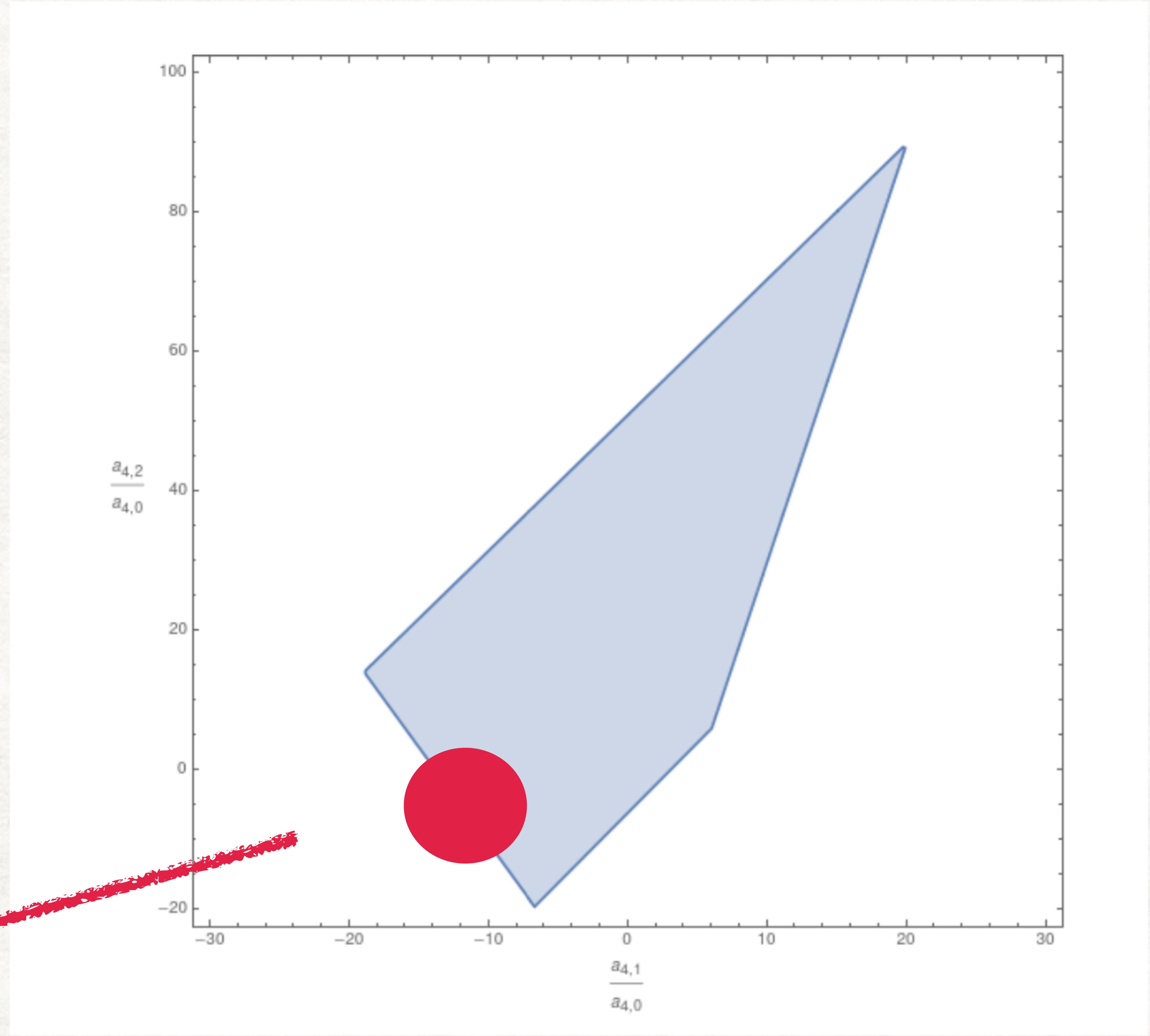


We can generalize to spinning external states $M(+h, +h, -h, -h)$

Compare to explicit closed string EFTs







Why are known EFTs isolated in a corner tiny corner of the EFT-hedron ?

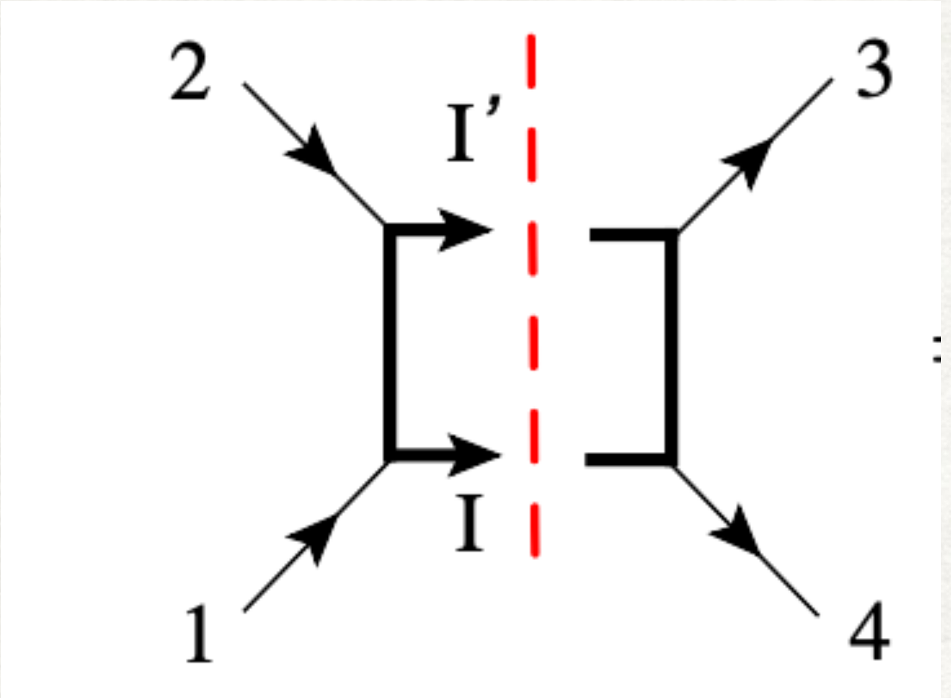
Low Spin-Dominance

Recall that in the defining the EFThedron, we used

$$g_{k,q} = \sum_a \mathsf{p}_a \frac{2^q u^{\alpha}_{\ell_a,k,q}}{(m_a^2)^{k+1}} \qquad \mathsf{p}_a > 0$$

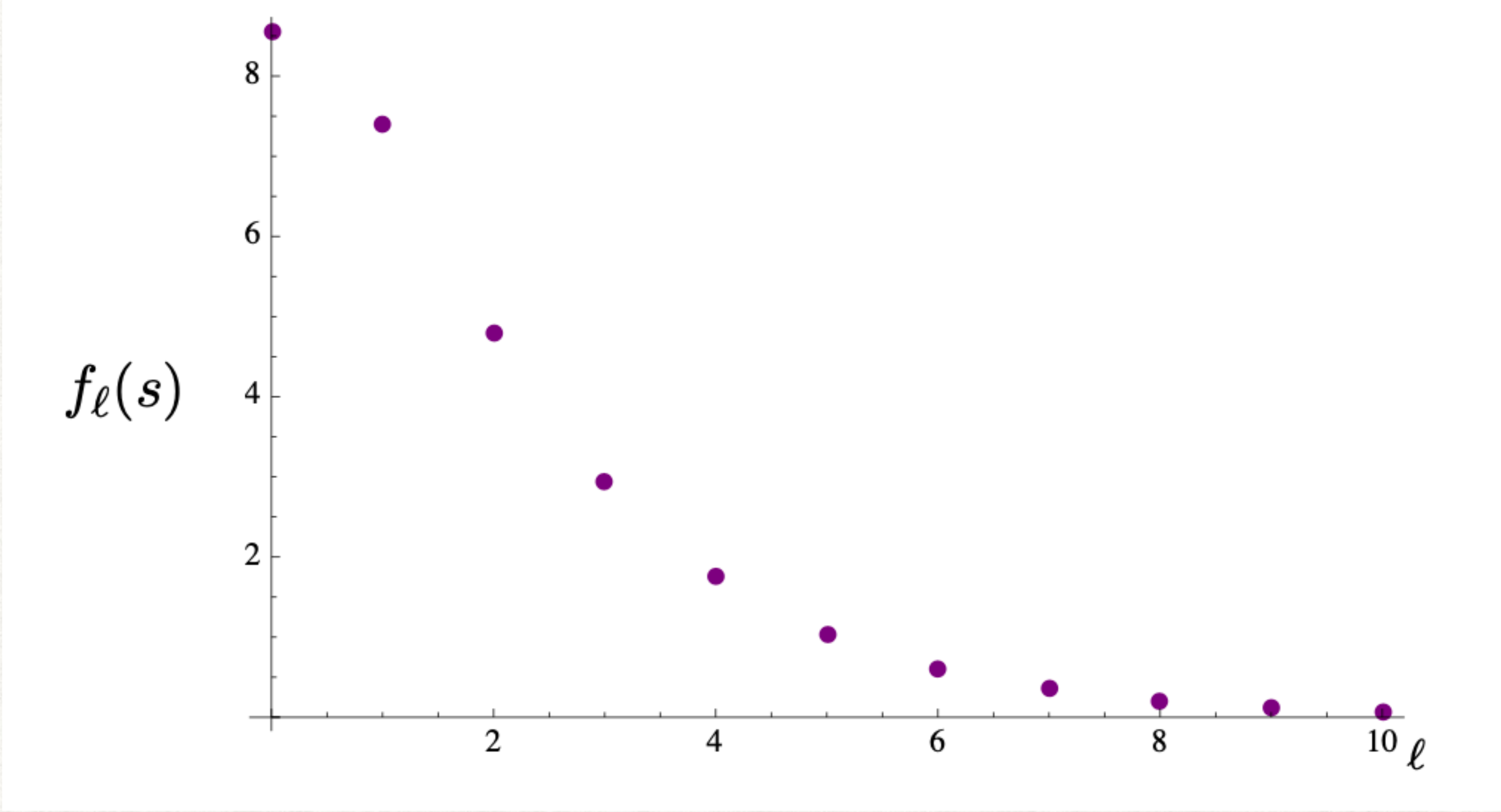
We only require this to be positive

But for any generic consistent UV completion



$$\langle \hat{p}_{in}|T^\dagger T|\hat{p}_{out}\rangle = \int_{4m^2}^{\infty} ds \frac{4J_s}{s^2} \sum_\ell \mathsf{p}_\ell(s) \frac{2}{2\ell+1} G_\ell^{\frac{1}{2}}(\cos\theta),$$

$$\mathsf{p}_\ell(s) \equiv |f_\ell(s)|^2$$



Suppressed at large spins !

Low Spin-Dominance

Recall that in the defining the EFThedron, we used

$$\boxed{g_{k,q} = \sum_a \mathsf{p}_a \frac{2^q u^{\alpha}_{\ell_a,k,q}}{(m_a^2)^{k+1}} \qquad \mathsf{p}_a > 0}$$

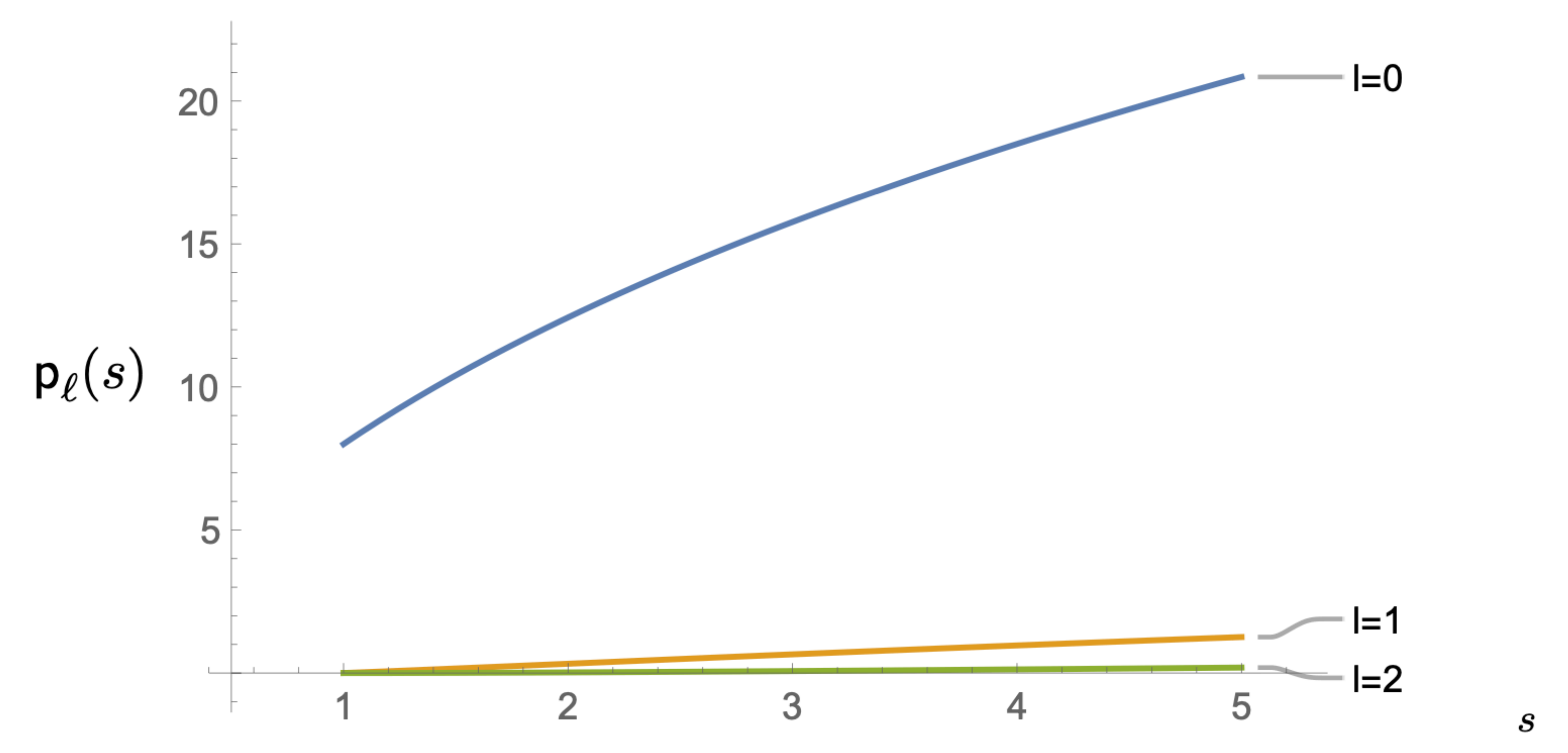
We only require this to be positive

But for any generic consistent UV completion

Open string

$\ell \backslash n$	1	2	3	4	5
0	1				$\frac{1}{11880}$
1		$\frac{1}{14}$		$\frac{1}{924}$	
2			$\frac{1}{84}$		$\frac{25}{39312}$
3				$\frac{2}{693}$	
4					$\frac{125}{144144}$

,



Suppressed at large spins !

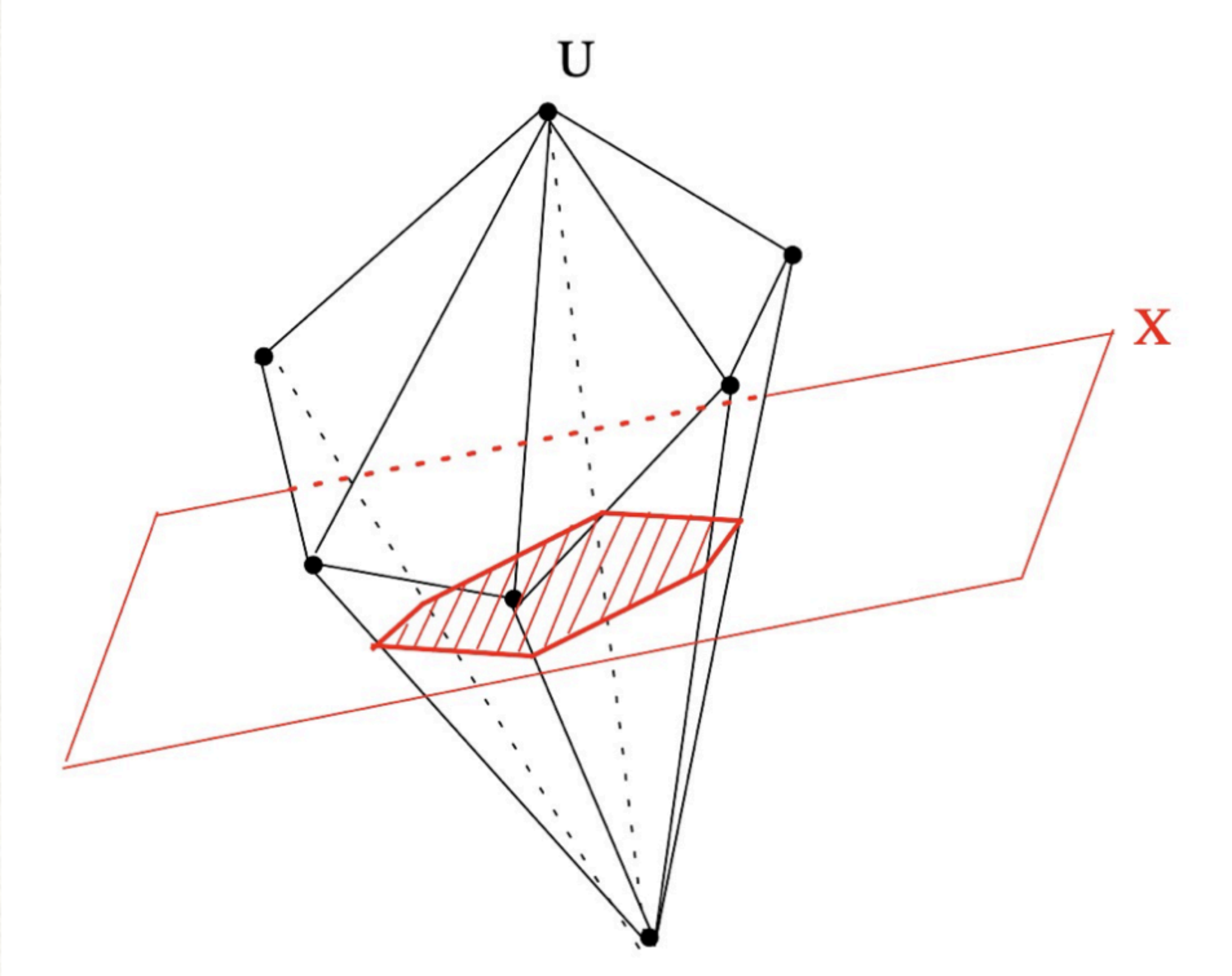
See also Z. Bern D. Kosmopoulos, A Zhiboedov 2103.12729

Does large spin suppression emerge from the geometry of the EFThedron ? If so to which extent ?

What constraints on the UV spectrum does the geometry imposes ?

The geometry by nature is an intersection geometry:

A physical spectrum must live on the symmetry plane. A convenient way of formulating this condition is the statement that the hull must have zero image under the projection of the symmetry plane, i.e. the hull must have zero components perpendicular to the symmetry plane.



This leads to "null constraints"

$$n_8 : \quad g_{8,4}-\frac{21}{8}g_{8,0}+\frac{1}{4}g_{8,2}=0, \quad g_{8,6}-\frac{21}{8}g_{8,0}+\frac{5}{16}g_{8,2}=0 .$$

When using the dispersive representation

$$n_k = \sum_i \frac{p_{\ell_i}}{(m_i^2)^{k+1}}\omega_k(\ell_i) = 0 ,$$

where w(l) is a polynomial. The roots impose non-trivial constraint on the spectrum

$$n_k = \sum_i \frac{p_{\ell_i}}{(m_i^2)^{k+1}} \omega_k(\ell_i) = 0 ,$$

For example for k=4

$$n_4 = \sum_i \frac{p_i}{m^4} \ell_i(\ell_i+1)(\ell_i^2+\ell_i-8) = 0$$

for most of the spins w(l) is positive

ℓ	0	2	4	6	$\ldots$
$\omega_4(\ell)$	0	$-$	$+$	$+$	$+$

Spin-2 must be part of the spectrum

For k=7 we have two null constraints

$$n_7 \equiv g_{7,3} - \frac{4}{5} g_{7,1} = 0, \quad n_7' \equiv g_{7,5} - \frac{16}{3} g_{7,1} = 0$$

→

$$n_7 + n_7' = \sum_i \frac{p_i}{m_i^7} (\ell_i - 2)\ell_i(\ell_i+1)(\ell_i+3)\left(\ell_i^2 + \ell_i - \frac{49}{2}\right) = 0$$

ℓ	0	2	4	6	$\ldots$
$\omega_7(\ell)$	0	0	-	$+$	$+$

Spin-4 must be part of the spectrum

$$n_k = \sum_i \frac{p_{\ell_i}}{(m_i^2)^{k+1}} \omega_k(\ell_i) = 0\,,$$

For k=3a+1 we can always arrange for

$$\omega_{3a+1}(\ell) = (\ell^2 + \ell - f_p(a)) \prod_{i=0}^{a-1} (\ell - 2i) \prod_{j=0}^{a-1} (\ell + 2j + 1)$$

Up to a<15 the sign pattern looks

ℓ	0	2	$\ldots$	$2a-2$	$2a$	$2a+2$	$\ldots$
$\omega_{3a+1}(\ell)$	0	0	0	0	-	+	+

All spins below 28 must be present !

For higher spins more work is needed

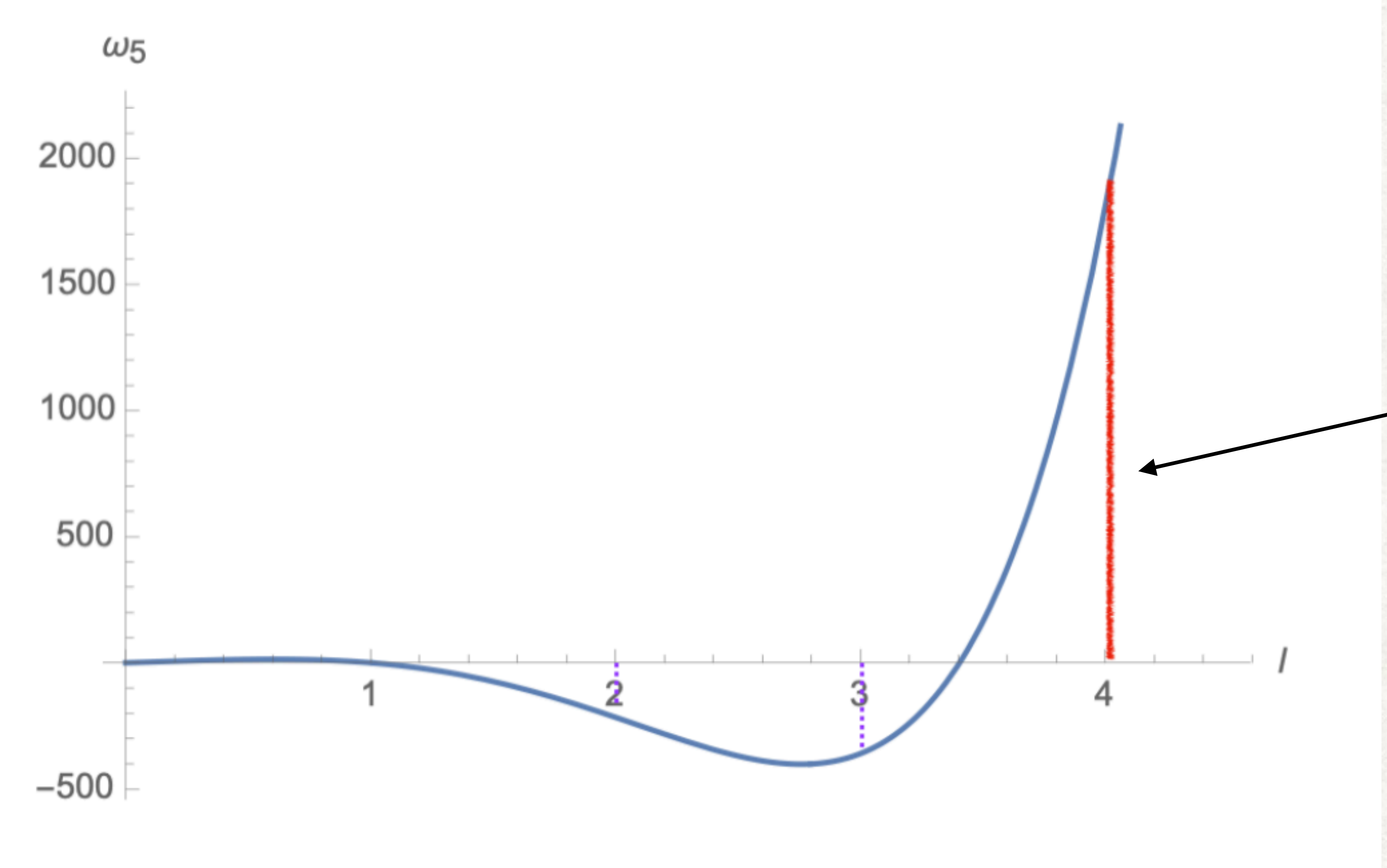
The geometry further imposes constraint on the magnitude for contributions of each spin!

Define the average spinning spectral function

$$\langle p_{k,\ell} \rangle \equiv \sum_{\{i, \ell_i = \ell\}} \frac{p_{\ell_i}}{m_i^{2(k+1)}}$$

The they are reflected in the null constraints as

$$n_k = \sum_i \frac{p_{\ell_i}}{(m_i^2)^{k+1}} \omega_k(\ell_i) = \sum_\ell \langle p_{k,\ell} \rangle \omega_k(\ell) = 0$$



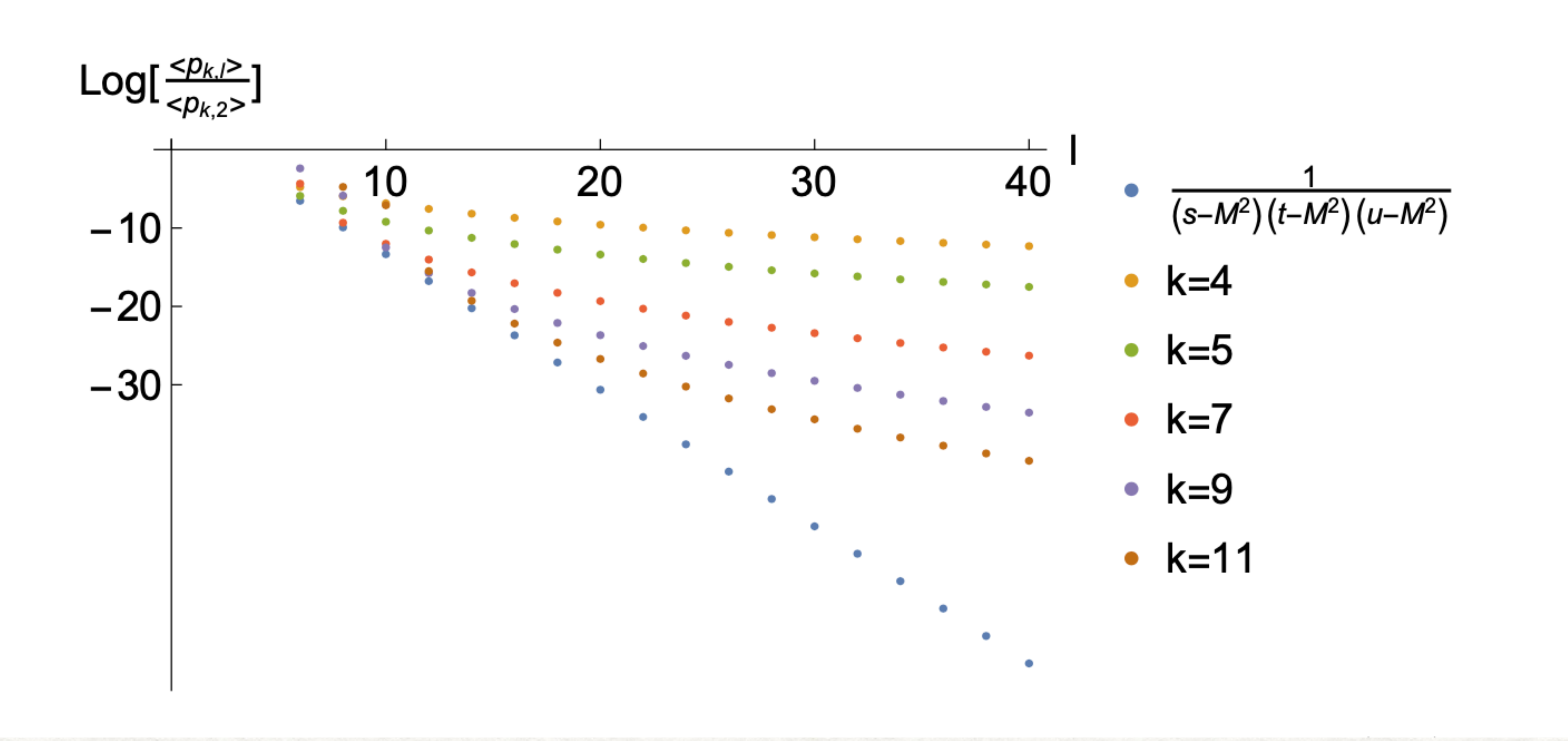
The maximal allowed value for any spin is bounded by the spins with negative contribution, i.e. there’s an Upper bound on the ratio

The geometry further imposes constraint on the magnitude for contributions of each spin!

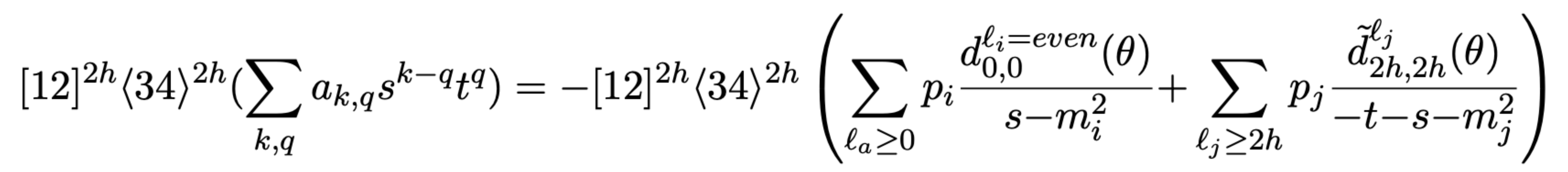
Since we know that spin-2 states must exists, we can bound the ratio of spin-l/spin-2

$$\langle p_{k,\ell}\rangle \equiv \sum_{\{i,\ell_i=\ell\}} \frac{p_{\ell_i}}{m_i^{2(k+1)}}$$

$$\frac{\langle p_{4,\ell}\rangle}{\langle p_{4,2}\rangle} \leq \frac{12}{\ell(\ell+1)\left(\ell^2+\ell-8\right)}, \quad (\ell \geq 4)$$
$$\frac{\langle p_{5,\ell}\rangle}{\langle p_{5,2}\rangle} \leq \frac{216}{\ell(\ell+1)(\ell(\ell+1)(2\ell(\ell+1)-43)+150)}, \quad (\ell \geq 4)$$

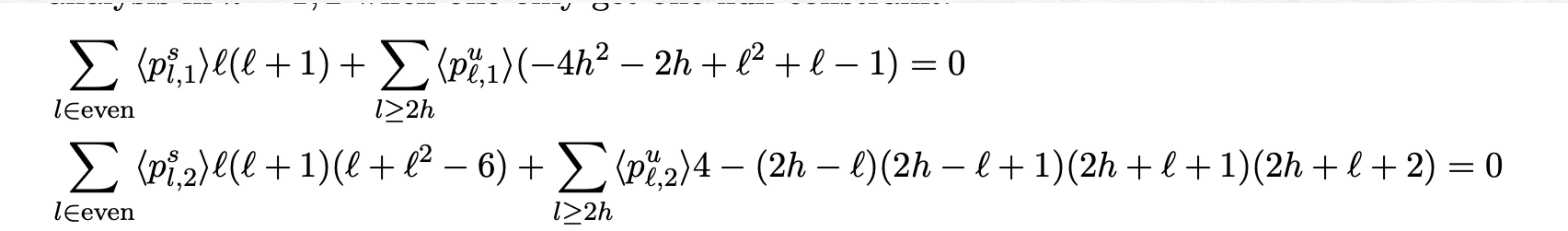


We can apply similar analysis to spinning external states



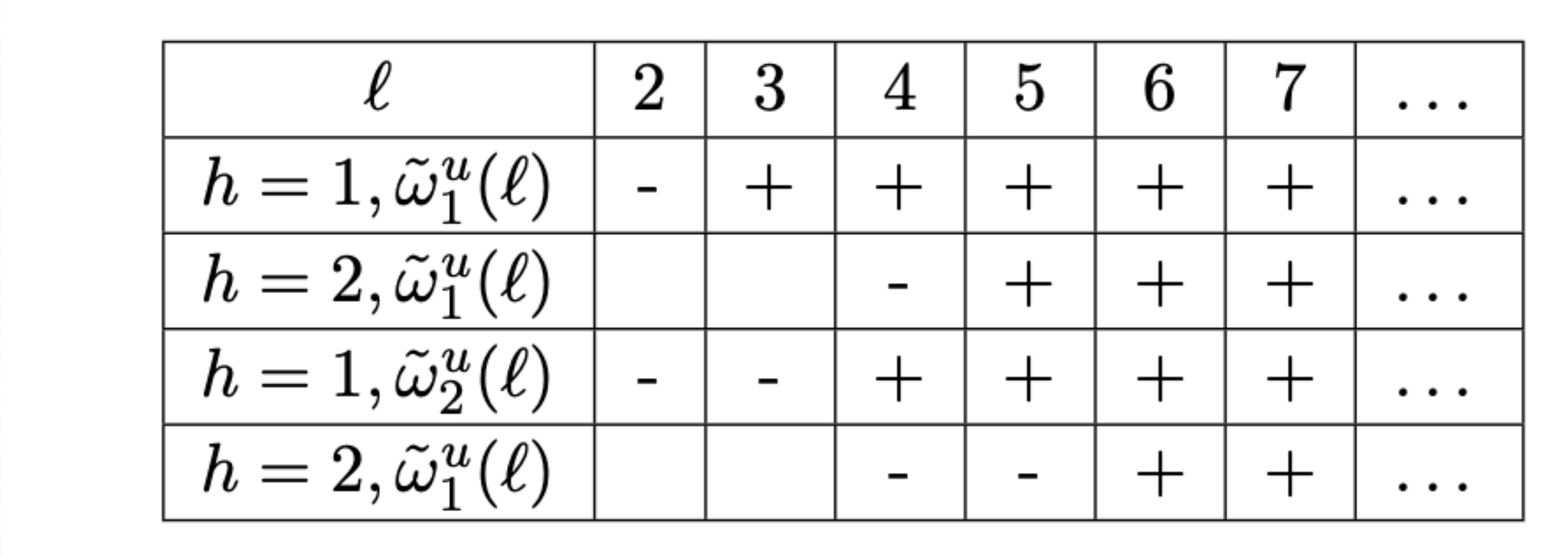
$$[12]^{2h}\langle 34\rangle^{2h}(\sum_{k,q} a_{k,q}s^{k-q}t^{q}) = -[12]^{2h}\langle 34\rangle^{2h}\left(\sum_{\ell_a\geq 0} p_i\frac{d^{\ell_i=even}_{0,0}(\theta)}{s-m_i^2}+\sum_{\ell_j\geq 2h} p_j\frac{\tilde{d}^{\ell_j}_{2h,2h}(\theta)}{-t-s-m_j^2}\right)$$

1<->2 symmetry also leads to null constraints



$$\sum_{l\in\text{even}}\langle p^s_{l,1}\rangle\ell(\ell+1)+\sum_{l\geq 2h}\langle p^u_{\ell,1}\rangle(-4h^2-2h+\ell^2+\ell-1)=0$$

$$\sum_{l\in\text{even}}\langle p^s_{l,2}\rangle\ell(\ell+1)(\ell+\ell^2-6)+\sum_{l\geq 2h}\langle p^u_{\ell,2}\rangle 4-(2h-\ell)(2h-\ell+1)(2h+\ell+1)(2h+\ell+2)=0$$



ℓ	2	3	4	5	6	7	$\dots$
$h=1,\tilde{\omega}^u_1(\ell)$	-	+	+	+	+	+	$\dots$
$h=2,\tilde{\omega}^u_1(\ell)$			-	+	+	+	$\dots$
$h=1,\tilde{\omega}^u_2(\ell)$	-	-	+	+	+	+	$\dots$
$h=2,\tilde{\omega}^u_1(\ell)$			-	-	+	+	$\dots$

We must have l=2 for E&M and l=4 for gravity

We can apply similar analysis to spinning external states

$$[12]^{2h}\langle 34\rangle^{2h}(\sum_{k,q} a_{k,q} s^{k-q} t^{q}) = -[12]^{2h}\langle 34\rangle^{2h}\left(\sum_{\ell_a\geq 0} p_i \frac{d^{\ell_i=even}_{0,0}(\theta)}{s-m_i^2} + \sum_{\ell_j\geq 2h} p_j \frac{\tilde{d}^{\ell_j}_{2h,2h}(\theta)}{-t-s-m_j^2}\right)$$

1<->2 symmetry also leads to null constraints

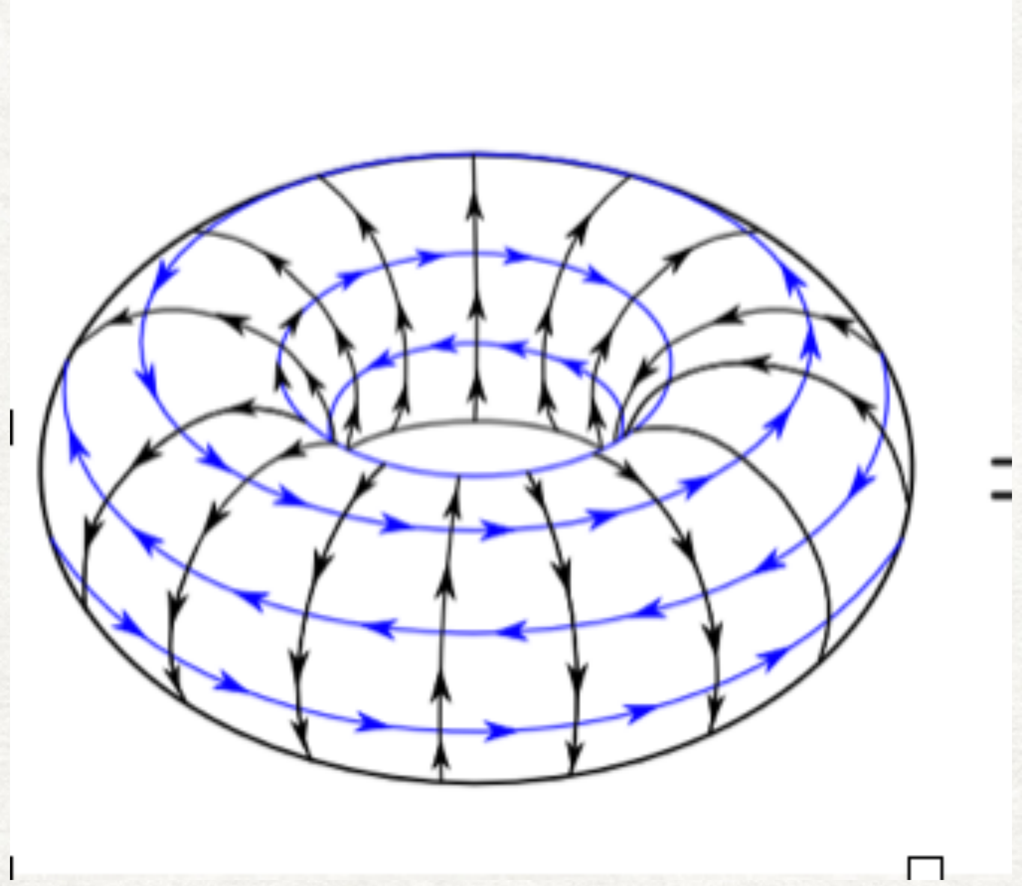
We must have l=2 for E&M and l=4 for gravity

$$\begin{array}{lll}
h=1: & \frac{\langle p^s_{\ell,1}\rangle}{\langle p^u_{2,1}\rangle} \leq \frac{1}{\ell(\ell+1)} \quad (\ell \geq 2), & \frac{\langle p^u_{\ell,1}\rangle}{\langle p^u_{2,1}\rangle} \leq \frac{1}{\ell^2+\ell-7} \quad (\ell \geq 3) \\
h=2: & \frac{\langle p^s_{\ell,1}\rangle}{\langle p^u_{4,1}\rangle} \leq \frac{1}{\ell(\ell+1)} \quad (\ell \geq 2), & \frac{\langle p^u_{\ell,1}\rangle}{\langle p^u_{4,1}\rangle} \leq \frac{1}{\ell^2+\ell-21} \quad (\ell \geq 5)
\end{array}$$

Product moment in the Modular bootstrap

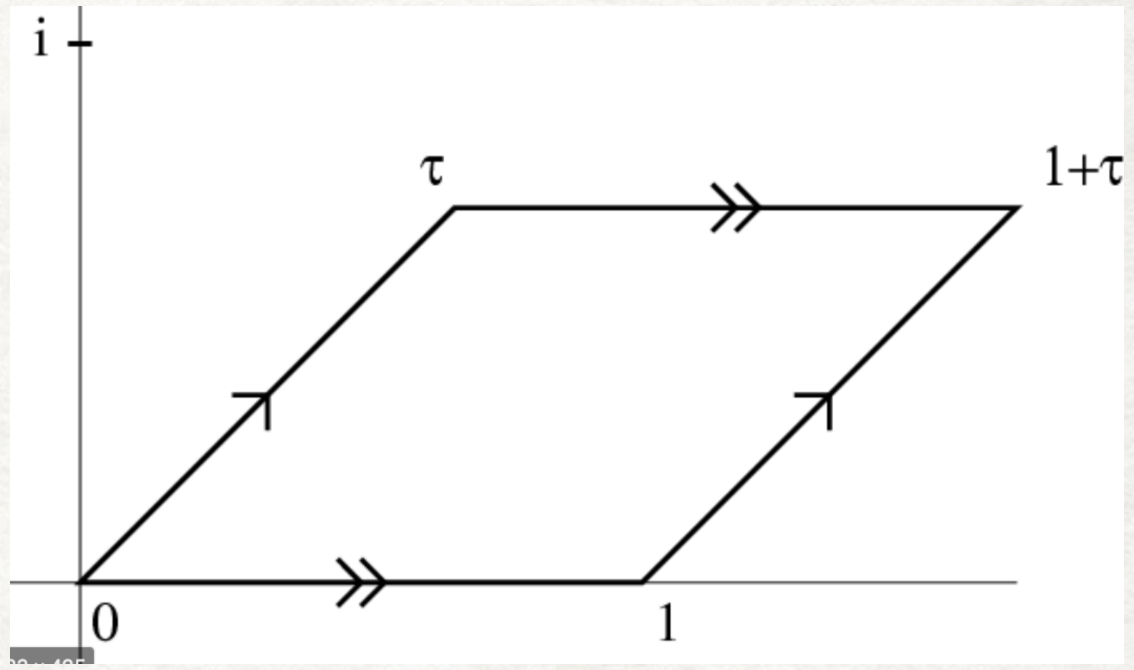
To torus partition function

The degeneracy number a positive integer



$$Z(\tau,\bar\tau)=\sum_{h,\bar h} n_{h,\bar h}\chi_h(q)\chi_{\bar h}(\bar q),\quad q=e^{i2\pi\tau}$$

$$\chi_0(q)=q^{-\frac{(c-1)}{24}}\frac{1-q}{\eta(q)},\quad \chi_h(q)=q^{h-\frac{(c-1)}{24}}\frac{1}{\eta(q)}\quad \forall h>0\,,$$



Let us consider the expansion of the torus moduli around some fixed point

$$z^{(a,b)}\equiv(\tau\partial_\tau)^a(\bar\tau\partial_{\bar\tau})^b Z|_{\tau=i,\bar\tau=-i},\ \chi_h^{(a)}\equiv(\tau\partial_\tau)^a\chi_h|_{\tau=i}$$

The partition function is given by a product geometry

$$[\mathbf{Z}]\equiv\begin{pmatrix} z^{(0,0)} & z^{(0,1)} & z^{(0,2)} & \dots\\ z^{(1,0)} & z^{(1,1)} & z^{(1,2)} & \dots\\ z^{(2,0)} & z^{(2,1)} & z^{(2,2)} & \dots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}$$

$$[\mathbf{Z}]=\sum_{h,\bar h} n_{h,\bar h}\quad \vec\chi_h\,(\vec\chi_{\bar h})^T,\qquad \vec\chi_h\equiv\begin{pmatrix}\chi_h^{(0)}\\ \chi_h^{(1)}\\ \chi_h^{(2)}\\ \vdots\\ \chi_h^{(d)}\end{pmatrix},\ n_{h,\bar h}>0$$

Product moment in the Modular bootstrap

Using the reduced partition function and characters $\hat{Z}(\tau,\bar{\tau}) = |\tau|^{\frac{1}{2}}|\eta(\tau)|^2 Z(\tau,\bar{\tau})$

$$\hat{\chi}_h^{(n)} \equiv (\tau\partial_\tau)^n \hat{\chi}_h(\tau)|_{\tau=i} = ((-1)^{\frac{1}{8}} e^{\frac{(c-1-24h)\pi}{12}})(a_{n,n}+X(\cdots(a_{n,3}+X(a_{n,2}+X(a_{n,1}+X))$$
$$\hat{\chi}_{\bar{h}}^{(n)} \equiv (\bar{\tau}\partial_{\bar{\tau}})^n \hat{\chi}_{\bar{h}}(\bar{\tau})|_{\bar{\tau}=-i} = ((-1)^{\frac{15}{8}} e^{\frac{(c-1-24\bar{h})\pi}{12}})(a_{n,n}+\bar{X}(\cdots(a_{n,3}+\bar{X}(a_{n,2}+\bar{X}(a_{n,1}+\bar{X}))$$

$$X = (c-1-24h)\pi,\ \bar{X} = (c-1-24\bar{h})\pi$$

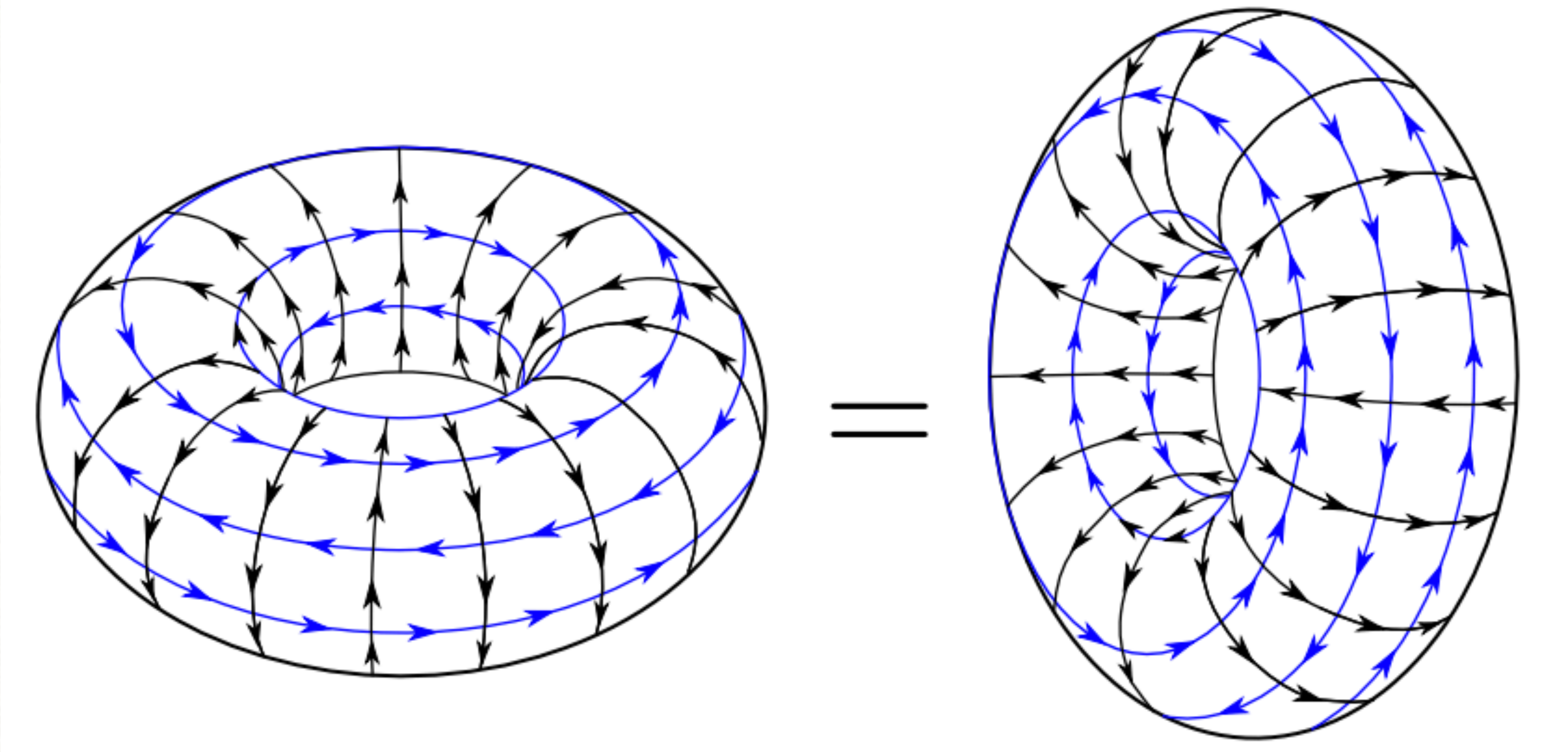
we see that the coefficients are simply polynomials, its just an GL transform away from moments!

$$M(c) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\pi(c-1)+3}{24\pi} & -\frac{1}{2\pi} & 0 \\ \frac{\pi(c-1)(\pi(c-1)+6)+45}{576\pi^2} & \frac{-\pi c+\pi-9}{24\pi^2} & \frac{1}{4\pi^2} \end{pmatrix}$$
$$\to \quad M(c)\begin{pmatrix} \hat{\chi}_h^{(0)} \\ \hat{\chi}_h^{(1)} \\ \hat{\chi}_h^{(2)} \end{pmatrix} = \alpha_{c,h} M(c) \begin{pmatrix} 1 \\ \frac{1}{12}(3+X) \\ \frac{1}{(12)^2}(9+X(18+X)) \end{pmatrix} = \alpha_{c,h} \begin{pmatrix} 1 \\ h \\ h^2 \end{pmatrix}$$

The subtracted partition function lives in the convex hull of the product moments

$$[\hat{\mathbf{Z}}] = M(c)\left([\mathbf{Z}] - \vec{\hat{\chi}}_0 \vec{\hat{\chi}}_0^T\right) M^T(c) = \sum_{h,\bar{h}} \tilde{n}_{h,\bar{h}} \begin{pmatrix} 1 \\ h \\ h^2 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 \\ \bar{h} \\ \bar{h}^2 \\ \vdots \end{pmatrix}^T$$

Modular invariance



$$Z\left[\tau,\bar{\tau}\right] = Z\left[-\frac{1}{\tau},-\frac{1}{\bar{\tau}}\right]$$

Modular invariance then requires

$$\left(\tau\frac{\partial}{\partial\tau}\right)^m\left(\bar{\tau}\frac{\partial}{\partial\bar{\tau}}\right)^n Z(\tau,\bar{\tau})|_{\tau=i} = 0\,, \quad \forall\, m{+}n \in odd$$

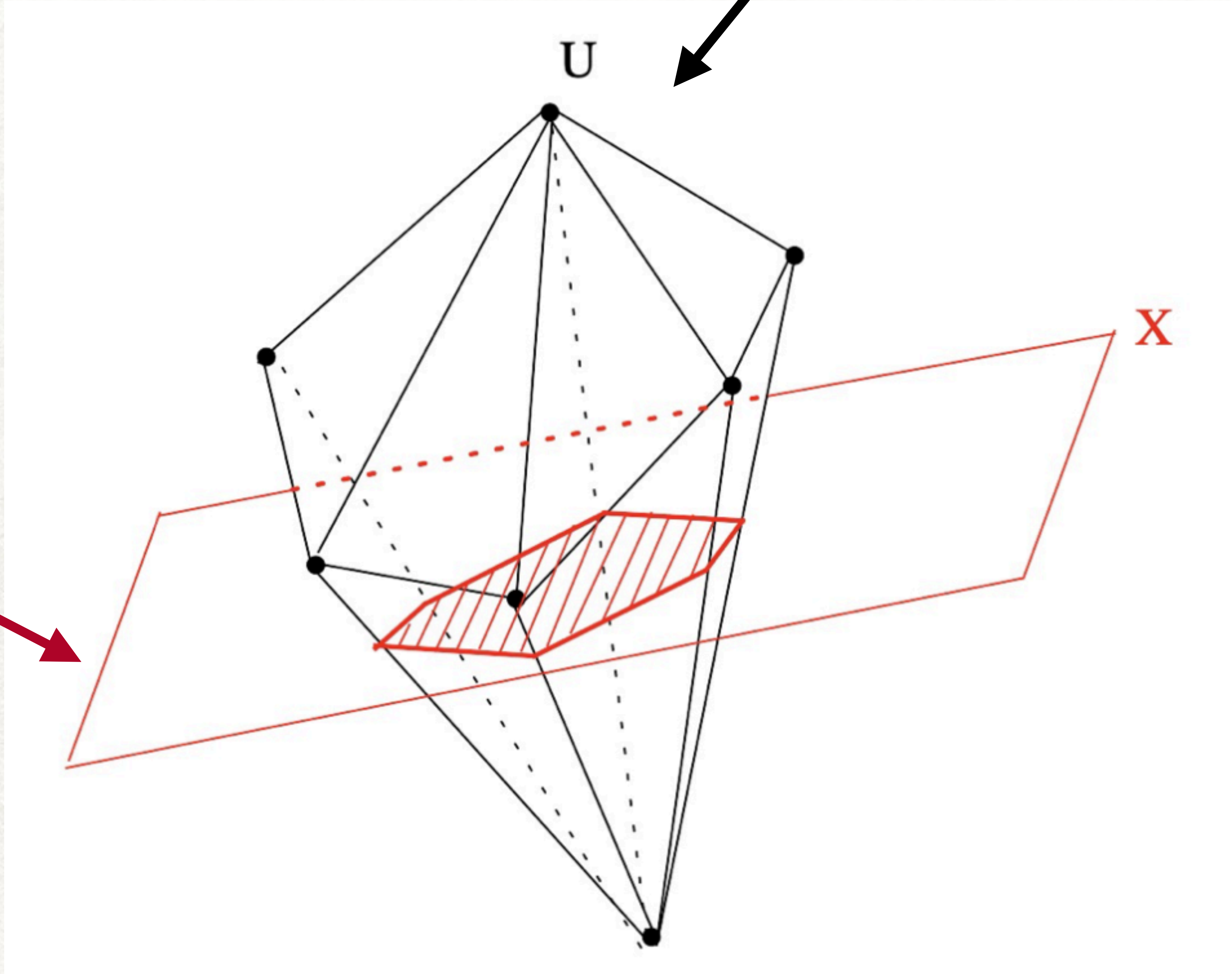
thus the partition function lives on the modular subplane $\mathbf{X}_{\mathrm{mod}}$

$$\mathbf{X}_{\mathrm{mod}} = \begin{pmatrix} x_1 & 0 & x_2 & 0 & \cdots \\ 0 & x_3 & 0 & x_5 & \cdots \\ x_4 & 0 & x_6 & 0 & \cdots \\ 0 & x_7 & 0 & x_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

The geometry behind modular bootstrap

$$M_1(h_g)M(c)([\mathbf{Z}]-\mathbf{G}_0)M^T(c)M_1^T(\bar{h}_g) = \sum_{h,\bar{h}>0} \tilde{n}_{h,\bar{h}} \begin{pmatrix} 1 \\ h-h_g \\ \vdots \\ (h-h_g)^d \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \bar{h}-\bar{h}_g \\ \vdots \\ (\bar{h}-\bar{h}_g)^d \end{pmatrix}^T$$

$$\mathbf{X}_{\text{mod}} = \begin{pmatrix} x_1 & 0 & x_2 & 0 & \cdots \\ 0 & x_3 & 0 & x_5 & \cdots \\ x_4 & 0 & x_6 & 0 & \cdots \\ 0 & x_7 & 0 & x_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$



Single moment limits

1. taking fixed columns

$$\vec{Z}_m = \begin{pmatrix}\hat{Z}_{0,m}\\ \hat{Z}_{1,m}\\ \hat{Z}_{2,m}\\ \vdots\end{pmatrix} = \sum_{h,\bar{h}} \tilde{n}_{h,\bar{h}} \bar{h}^m \begin{pmatrix}1\\ h\\ h^2\\ \vdots\end{pmatrix} \longrightarrow M_2(h_g)\vec{Z} = \sum_{h\geq h_g} \tilde{n}_h \begin{pmatrix}1\\ h-h_g\\ \vdots\\ (h-h_g)^d\end{pmatrix} = \sum_{h\geq 0} \tilde{n}_h \begin{pmatrix}1\\ h\\ \vdots\\ h^d\end{pmatrix}$$

The twist-gap bootstrap $\vec{y} = M_2(h_g)\vec{Z}(\mathbf{X}_{\text{mod}})\,, \quad \rightarrow \quad \det\left(K[\vec{y}]\right) \geq 0\,, \quad \det\left(K^{\text{shift}}[\vec{y}]\right) \geq 0$

2. linear combinations

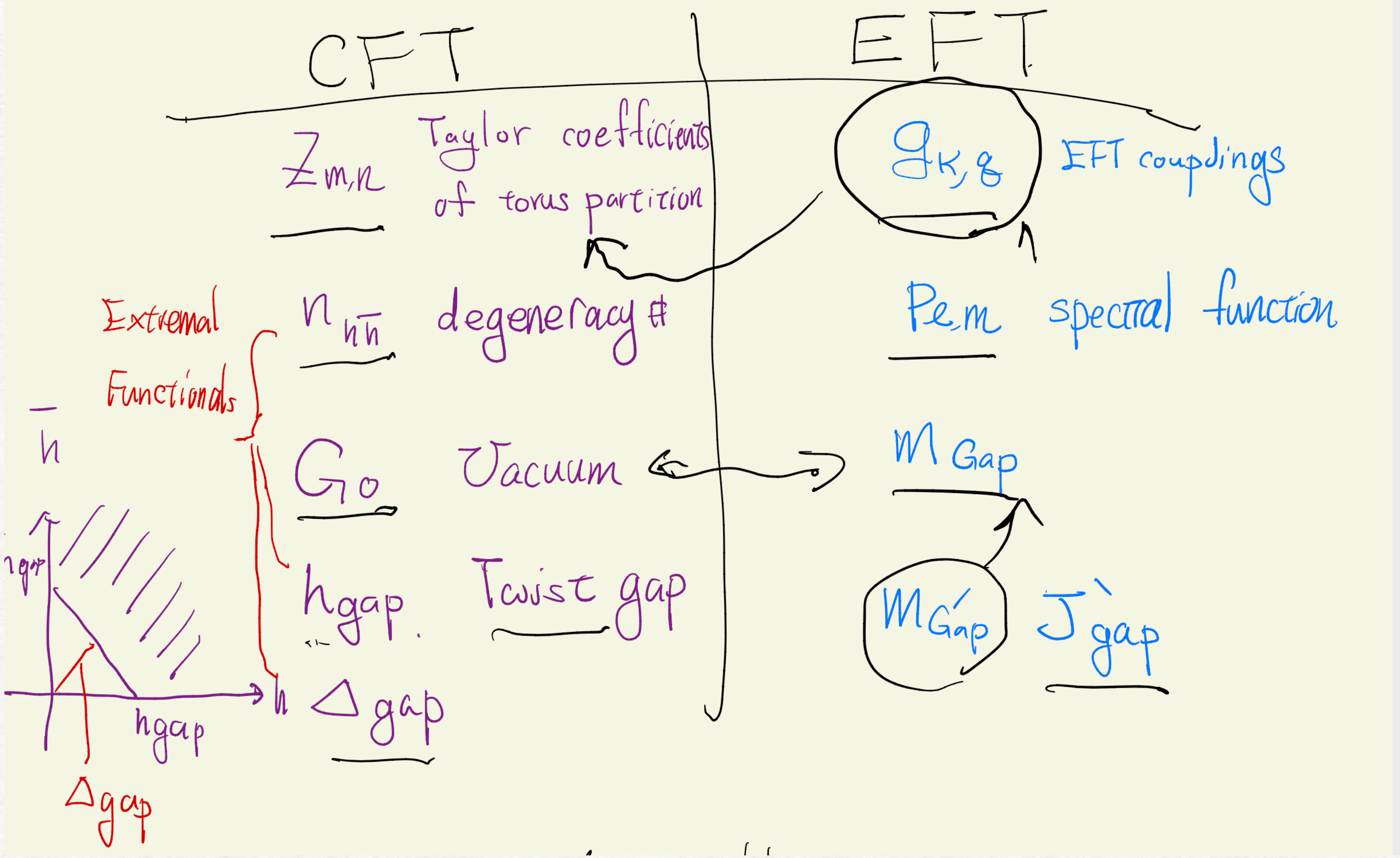
$$\hat{Z}'_0 = \hat{Z}_{0,0} \quad \hat{Z}'_1 = \hat{Z}_{1,0} + \hat{Z}_{0,1}\,, \quad \hat{Z}'_2 = \hat{Z}_{2,0} + 2\hat{Z}_{1,1} + \hat{Z}_{0,2}\,, \quad e.t.c. \longrightarrow \vec{Z}' = \begin{pmatrix}\hat{Z}'_0\\ \hat{Z}'_1\\ \hat{Z}'_2\\ \vdots\end{pmatrix} = \sum_{\Delta} \tilde{n}_\Delta \begin{pmatrix}1\\ \Delta\\ \Delta^2\\ \vdots\end{pmatrix}$$

$$\longrightarrow M_2(\Delta_g)\vec{Z}' = \sum_{\Delta\geq\Delta_g} \tilde{n}_\Delta \begin{pmatrix}1\\ \Delta-\Delta_g\\ \vdots\\ (\Delta-\Delta_g)^d\end{pmatrix} = \sum_{\Delta\geq 0} \tilde{n}_\Delta \begin{pmatrix}1\\ \Delta\\ \vdots\\ \Delta^d\end{pmatrix}$$

The spinless modular bootstrap $\vec{y} = M_2(\Delta_g)\vec{Z}'(\mathbf{X}_{\text{mod}})\,, \quad \rightarrow \quad \det\left(K[\vec{y}]\right) \geq 0\,, \quad \det\left(K^{\text{shift}}[\vec{y}]\right) \geq 0$

We have seen that the theory space of

This tells us that there is a map between questions on the two sides



- The convex hull of product moments is the linchpin of the geometry behind modular and EFT bootstrap (see Francesco’s talk)
- The intersection of the symmetry plane imposes constraint on the spectrum
- The missing corners of the truncated moment problem?
- The geometry of non-identical states ?
- The spectral parameter have a natural upper bound from unitarity alone (p<2)
- Into the EFThedron

$$M(s,t) = \{massless/massive\ poles\} + \sum_{k,q} a^{\Lambda}_{k,q} s^{k-q} t^{q}$$

$$\sum_{k,q} a^{(n)}_{k,q} z^{k-q} t^{q} = \frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]}{\Gamma[1+s]\Gamma[1+u]\Gamma[+t]} - \left[\sum_{a=1}^{n} R_a(t)\left(\frac{1}{s-a}+\frac{1}{u-a}\right)\right]$$