Resummed lattice QCD equation of state at finite baryon density: strangeness neutrality and beyond

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June 15, 2022

Strangeness in Quark Matter 2022

PRD 105 (2022) 114504

with:
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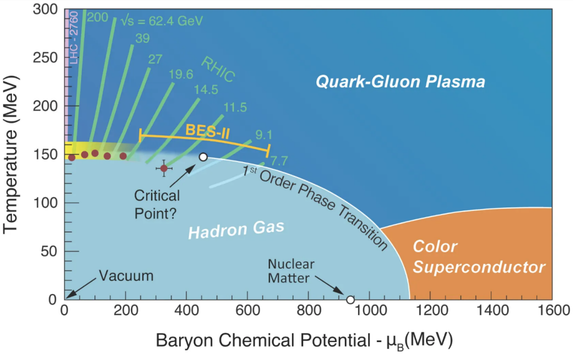
The equation of state of QCD

What do we know about QCD thermodynamics at finite T, μ_B?

From a combination of approaches (experiment, models, first principle calculations, ...), we have *some knowledge* of the phase diagram.

- Ordinary nuclear matter at $T \simeq 0$ and $\mu_B \simeq 922\,\mathrm{MeV}$
- Deconfinement transition at $\mu_B = 0$ is a smooth crossover at $T \simeq 155 - 160\,\mathrm{MeV}$
- Transition line at finite μ_B is known to some precision (+ freeze-out extraction)
- EoS of QCD: expansion up to $\mu_B \simeq 2 - 2.5T$
- Critical point? Exotic phases?



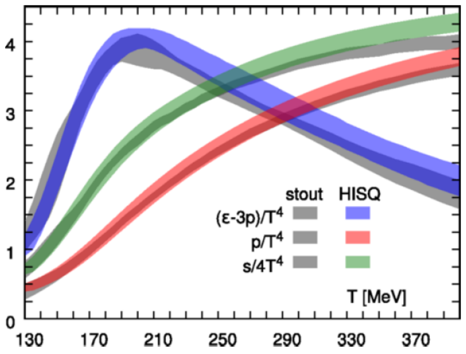
The equation of state (EoS) of QCD is invaluable. Knowing it would mean we can *really* draw the phase diagram of QCD.

The EoS of QCD at $\mu_B = 0$

- A crucial input to hydrodynamic simulations of e.g., heavy-ion collisions
- Known at $\mu_B = 0$ to high precision for a few years now (continuum limit, physical quark masses) $\longrightarrow$ Agreement between different calculations

From grancanonical partition function $\mathcal{Z}$

* **Pressure**: $p = -k_B T \frac{\partial \ln \mathcal{Z}}{\partial V}$
* **Entropy density**: $s = \left(\frac{\partial p}{\partial T}\right)_{\mu_i}$
* **Charge densities**: $n_i = \left(\frac{\partial p}{\partial \mu_i}\right)_{T,\mu_{j\neq i}}$
* **Energy density**: $\epsilon = Ts - p + \sum_i \mu_i n_i$
* More (**Fluctuations**, etc...)



WB: Borsányi *et al.*, PLB 370 (2014) 99-104, HotQCD: Bazavov *et al.* PRD 90 (2014) 094503

Finite density: the sign/complex action problem

Euclidean path integrals on the lattice are calculated with MC methods using importance sampling, interpreting the factor $\det M[U]\, e^{-S_G[U]}$ as the Boltzmann weight for the configuration U

$$Z(V,T,\mu) = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{-S_F(U,\psi,\bar{\psi})-S_G(U)}$$
$$= \int \mathcal{D}U\; \det M(U) e^{-S_G(U)}$$

- If there is particle-antiparticle-symmetry ($\mu = 0$) $\det M(U)$ is real
- For real chemical potential ($\mu^2 > 0$) $\rightarrow$ $\det M(U)$ is complex (**complex action problem**) and has wildly oscillating phase (**sign problem**)
  $\Rightarrow$ It cannot serve as a statistical weight
- For *purely imaginary* chemical potential ($\mu^2 < 0$) $\rightarrow$ $\det M(U)$ is real again, simulations can be made!

Finite density: alternatives

In lattice QCD one tries to work around the **sign problem** directly (still exploratory)

- Reweighting techniques → exciting new results
- Complex Langevin
- Lefschetz thimbles
- ...

or indirectly:

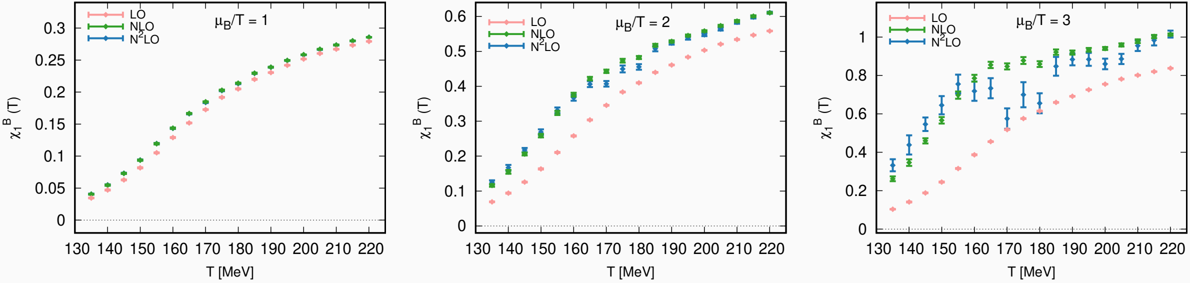
- **Taylor expansion around** $\mu_B = 0$

$$\frac{p(T,\mu_B)}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) \left(\frac{\mu_B}{T}\right)^{2n} \quad , \qquad c_n(T) = \frac{1}{n!}\chi_n^B(T,\mu_B = 0)$$

- **Analytical continuation from imaginary** μ_B

Lattice QCD at finite μ_B - Taylor expansion

- Thermodynamic quantities at large chemical potential become problematic
- Higher orders do not help with the convergence of the series



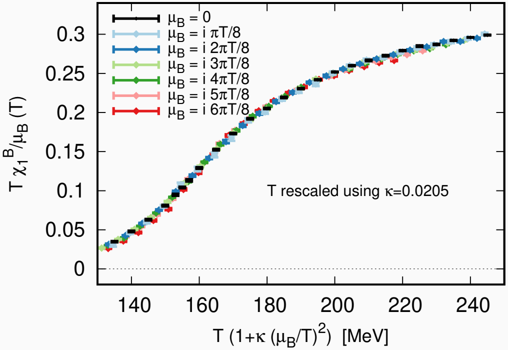
- Inherent problem with Taylor expansion: carried out at $T = \text{const}$. This doesn't cope well with $\hat{\mu}_B-$dependent transition temperature
- Can we find an alternative expansion to improve finite-$\hat{\mu}_B$ behavior?

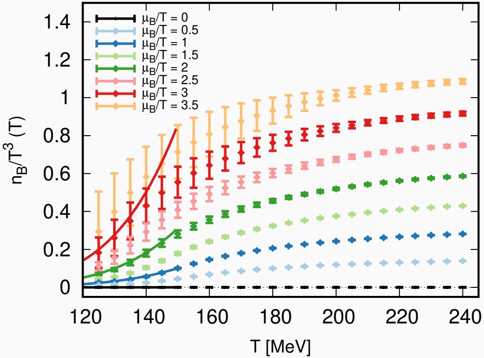
Borsányi, PP *et al.* PRL 126 (2021) 232001

The alternative approach at $\mu_Q = \mu_S = 0$

From an *observation* at imaginary μ_B we constructed (**Borsányi *et al.*, PRL 126 (2021) 232001**) an ansatz to determine thermodynamics at finite (real) μ_B:

$$\frac{\chi_1^B(T, \hat{\mu}_B)}{\hat{\mu}_B} = \chi_2^B(T', 0) \ , \quad T' = T\left(1 + \kappa_2(T)\,\hat{\mu}_B^2 + \kappa_4(T)\,\hat{\mu}_B^4 + \mathcal{O}(\hat{\mu}_B^6)\right)$$

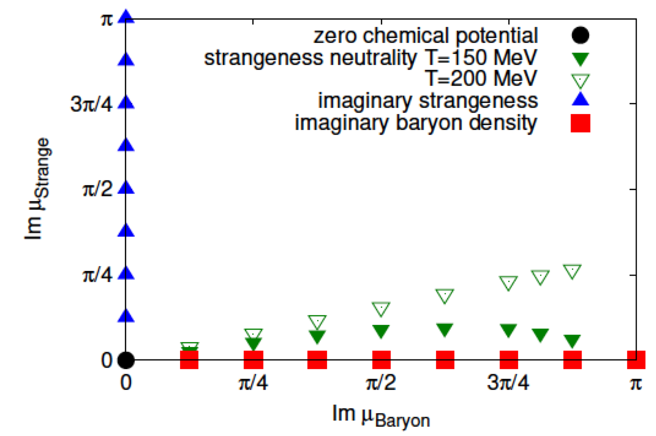




Borsányi, PP *et al.* PRL 126 (2021) 232001

Imaginary μ_B: strangeness neutrality

With the alternative scheme previously introduced at $\mu_Q = \mu_S = 0$, we now move to strangeness neutrality $\langle n_S \rangle = 0$, with $\mu_Q = 0$.



The idea is to follow lines of constant "observable", instead of constant T.

Rigorous formulation

- The $\hat{\mu}_B$-dependence of certain observables amounts to a simple rescaling of the temperature T
- For a certain observable F, we can write:

$$F(T, \hat{\mu}_B) = F(T', 0), \, , \quad \mathrm{T}' = \mathrm{T}\left(1 + \kappa_2^{\mathrm{F}}(\mathrm{T})\, \hat{\mu}_{\mathrm{B}}^2 + \kappa_4^{\mathrm{F}}(\mathrm{T})\, \hat{\mu}_{\mathrm{B}}^4 + \mathcal{O}(\hat{\mu}_{\mathrm{B}}^6)\right)$$

- **Important:** this is a re-organization (resummation) of the Taylor expansion via an expansion in the shift

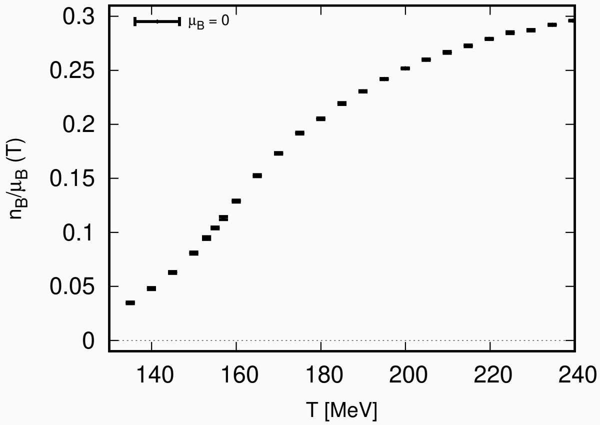
$$\Delta T = T - T' = \left(\kappa_2^F(T)\, \hat{\mu}_B^2 + \kappa_4^F(T)\, \hat{\mu}_B^4 + \mathcal{O}(\hat{\mu}_B^6)\right)$$

- In fact, the coefficients of the (Taylor) expansion in $\hat{\mu}_B$ and those of our expansion in ΔT are related directly , e.g. at $\mu_Q = \mu_S = 0$ for $\chi_1^B / \hat{\mu}_B$:

$$\kappa_2(T) = \frac{1}{6T} \frac{\chi_4^B(T)}{\chi_2^{B'}(T)} \qquad \kappa_4(T) = \frac{1}{360 \chi_2^{B'}(T)^3} \left(3\chi_2^{B'}(T)^2 \chi_6^B(T) - 5\chi_2^{B''}(T)\chi_4^B(T)^2\right)$$

Determine κ_n

The procedure, visualized:



Spline fit both at $\hat{\mu}_B = 0$ and $\hat{\mu}_B \neq 0$, then determine $T - T'$ (horizontal segments)

Determine κ_n

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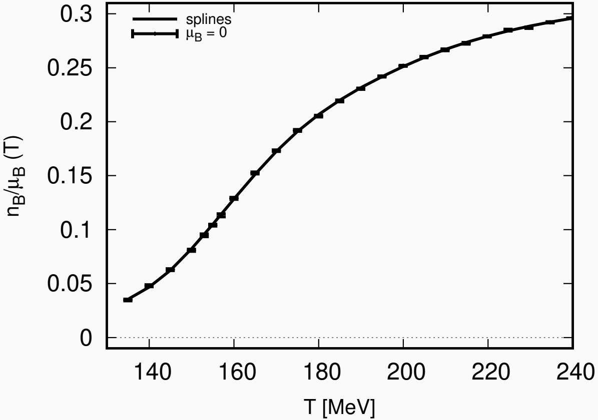

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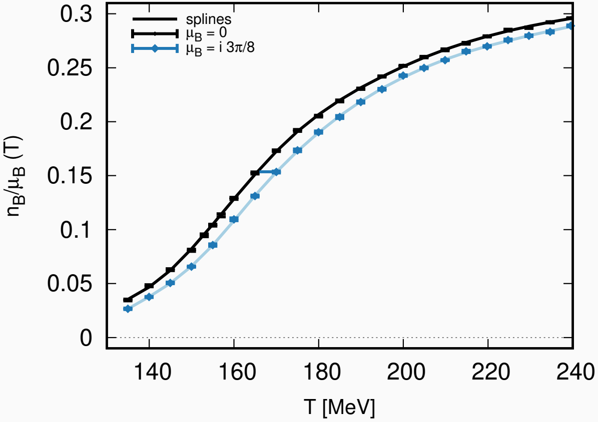

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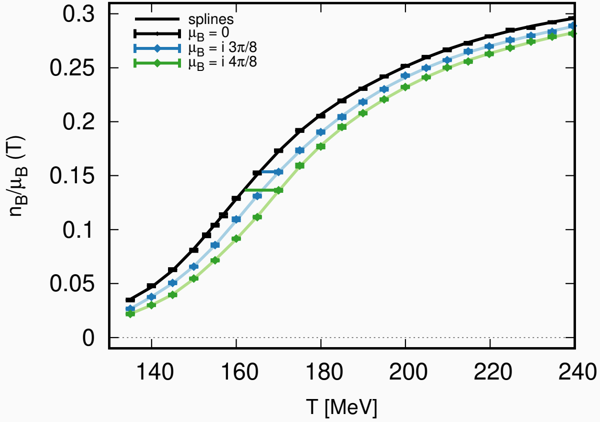

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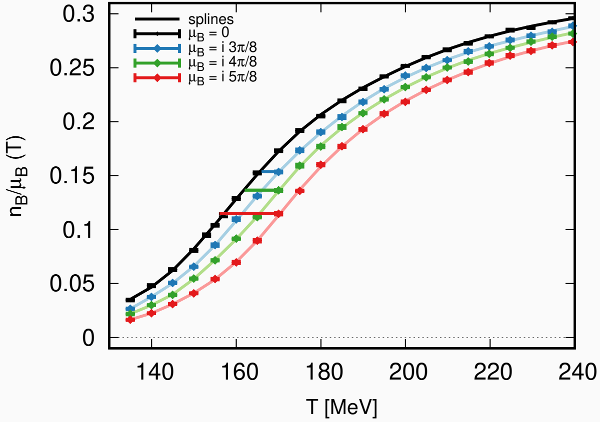

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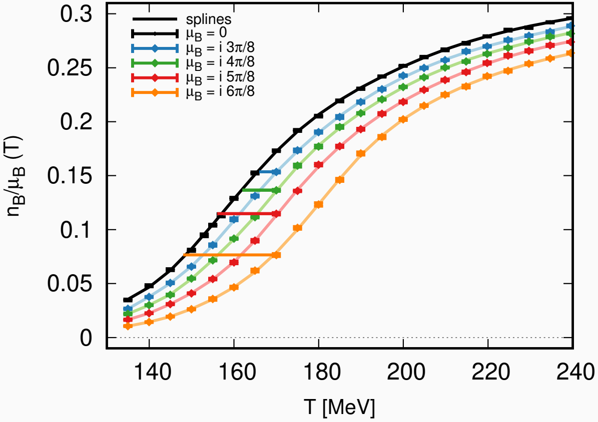


Spline fit both at $\hat{\mu}_B = 0$ and $\hat{\mu}_B \neq 0$, then determine $T - T'$ (horizontal segments)

Strangeness neutrality

- In this work, we look at three observables:

$$c_1^B(\hat{\mu}_B, T)\,, \qquad \mathrm{M}(\hat{\mu}_B, T) = \frac{\mu_S}{\mu_B}(\hat{\mu}_B, T)\,, \qquad \chi_2^S(\hat{\mu}_B, T)\,,$$

  where

$$c_n^B = \frac{d^n}{d\hat{\mu}_B^n}\frac{p}{T^4} = \left(\frac{\partial}{\partial\,\hat{\mu}_B} + \frac{d\hat{\mu}_S}{d\hat{\mu}_B}\frac{\partial}{\partial\,\hat{\mu}_S}\right)^n \frac{p}{T^4} = \left(\frac{\partial}{\partial\,\hat{\mu}_B} - \frac{\chi_{11}^{BS}}{\chi_2^S}\frac{\partial}{\partial\,\hat{\mu}_S}\right)^n \frac{p}{T^4} \quad \Rightarrow c_1^B \equiv \chi_1^B$$

  are the Taylor coefficients of the pressure along the strangeness neutral line, and μ_S realizes strangeness neutrality.

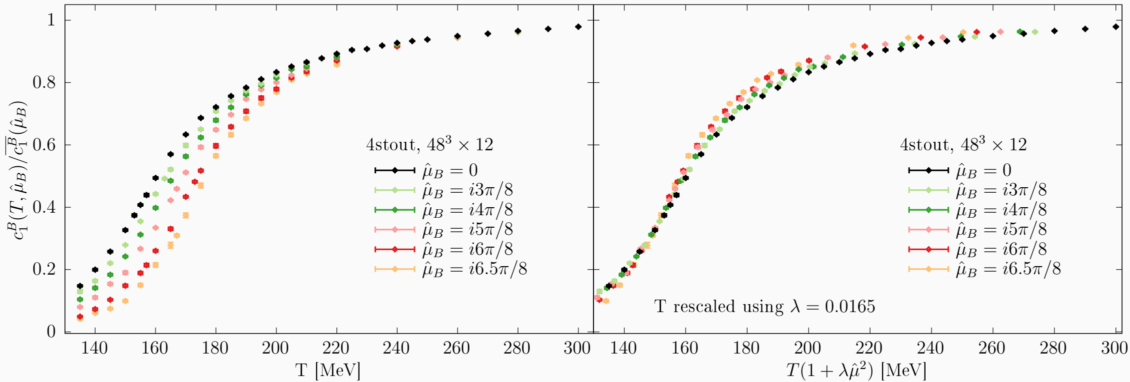
- We introduce a “Stefan-Boltzmann” (SB) correction, in that we normalize every quantity wrt its ($\hat{\mu}_B$-dependent) SB limit. This ensures the method is applicable (and improves results) at large T.

  Note: this can be done in the non-strangeness neutral case too.

The alternative approach at strangeness neutrality

With SB correction:

$$\frac{c_1^B(T,\,\hat{\mu}_B)}{\bar{c}_1^B(\,\hat{\mu}_B)} = \frac{c_2^B(T',0)}{\bar{c}_2^B(0)}\ , \qquad T' = T\left(1+\lambda\,\hat{\mu}_B^2\right)$$



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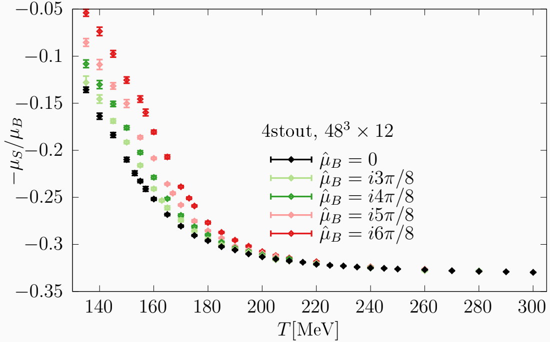
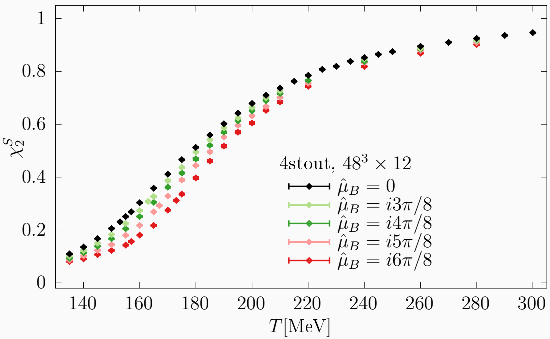
The alternative approach at strangeness neutrality

Similarly, for μ_S/μ_B and χ_2^S:

$$\frac{\mathrm{M}(\mathrm{T},\hat{\mu}_\mathrm{B})}{\overline{\mathrm{M}}(\hat{\mu}_B)} = \frac{\mathrm{M}(\mathrm{T}'_{\mathrm{BS}},0)}{\overline{\mathrm{M}}(0)} \,, \qquad \frac{\chi_2^\mathrm{S}(\mathrm{T},\hat{\mu}_\mathrm{B})}{\overline{\chi_2^\mathrm{S}}(\hat{\mu}_\mathrm{B})} = \frac{\chi_2^\mathrm{S}(\mathrm{T}'_{\mathrm{SS}},0)}{\overline{\chi_2^\mathrm{S}}(0)},$$





The SB correction has no effect here, because both $\overline{\mathrm{M}}(\hat{\mu}_B) = \overline{\mathrm{M}}(0)$ and $\overline{\chi_2^S}(\hat{\mu}_B) = \overline{\chi_2^S}(0)$

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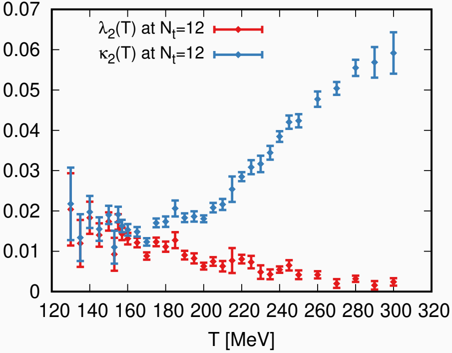
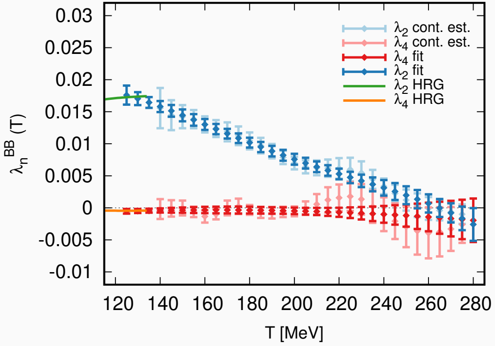
The alternative approach at strangeness neutrality

We give the new coefficients the name λ, because they define a different (although closely related) expansion





As expected, λ_2 goes to zero, making the expansion applicable at larger T and $\hat{\mu}_B$

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Thermodynamics at finite (real) μ_B

Thermodynamic quantities at finite (real) μ_B can be reconstruted from the same ansazt:

$$\frac{n_B(T,\hat{\mu}_B)}{T^3} = c_1^B(T,\hat{\mu}_B) = c_2^B(T',0)\,\frac{\overline{c_1^B}(\hat{\mu}_B)}{\overline{c_2^B}(0)},$$

with $T' = T(1 + \lambda_2^{BB}(T)\,\hat{\mu}_B^2 + \lambda_4^{BB}(T)\,\hat{\mu}_B^4)$.

From the baryon density n_B one finds the pressure:

$$\frac{p(T,\,\hat{\mu}_B)}{T^4} = \frac{p(T,0)}{T^4} + \int_0^{\hat{\mu}_B} \mathrm{d}\hat{\mu}'_B\,\frac{n_B(T,\,\hat{\mu}'_B)}{T^3}$$

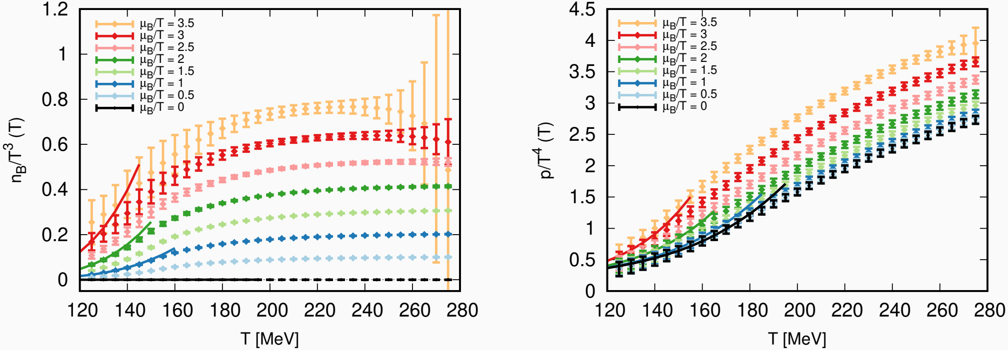
then the entropy, energy density:

$$\frac{s(T,\,\hat{\mu}_B)}{T^4} = 4\frac{p(T,\,\hat{\mu}_B)}{T^4} + T\,\left.\frac{\partial p(T,\,\hat{\mu}_B)}{\partial T}\right|_{\hat{\mu}_B} - \hat{\mu}_B\,\frac{n_B(T,\,\hat{\mu}_B)}{T^3}$$

$$\frac{\epsilon(T,\,\hat{\mu}_B)}{T^4} = \frac{s(T,\,\hat{\mu}_B)}{T^3} - \frac{p(T,\,\hat{\mu}_B)}{T^4} + \hat{\mu}_B\,\frac{n_B(T,\,\hat{\mu}_B)}{T^3}$$

Thermodynamics at finite (real) μ_B - strangenesss neutrality

- We can reach out to $\hat{\mu}_B \simeq 3.5$ with reasonable uncertainties
- Good agreement with HRG
- No pathological (non-monotonic) behavior is present



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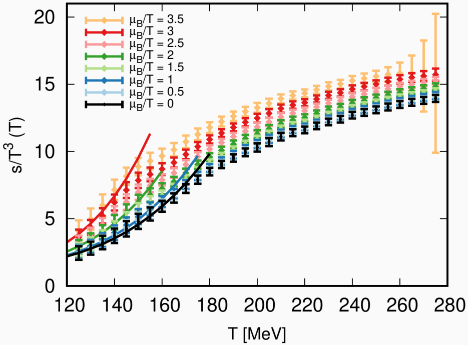

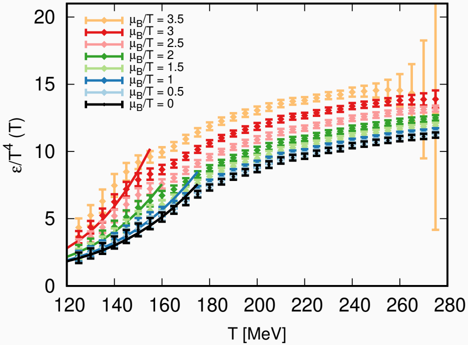

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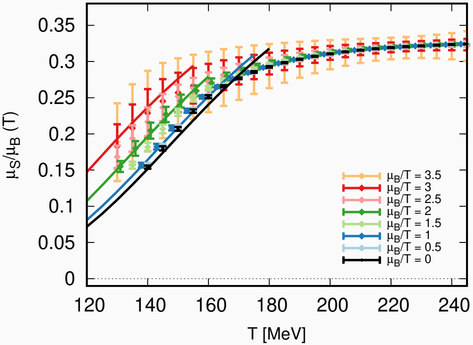

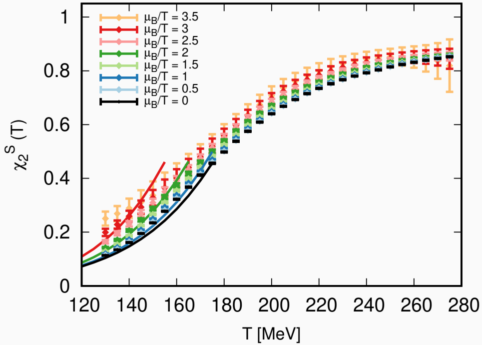

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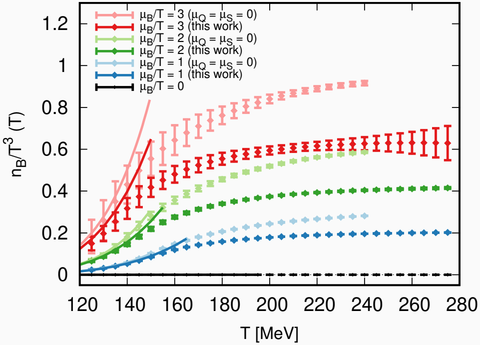
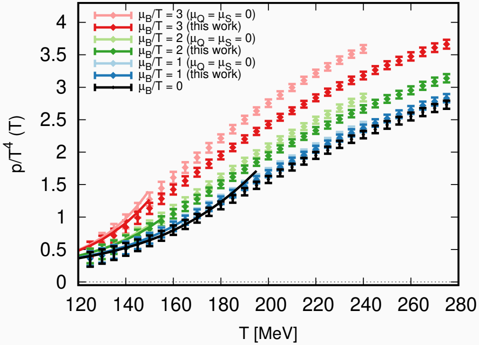
What is different with strangeness neutrality?

- The difference between the two cases is simply driven by different chemical potentials
- The quality of the results is comparable





Difference in the pressure is less visible, because dominated by $\mu_B = 0$ contribution.

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Beyond strangeness neutrality

Move away from the strangeness neutrality $\langle n_S \rangle = 0$, where $\hat{\mu}_S = \hat{\mu}_S^\star$, by an amount $\Delta\hat{\mu}_S \equiv \hat{\mu}_S - \hat{\mu}_S^\star$:

$$\chi_1^S(\hat{\mu}_S) \approx \chi_2^S(\hat{\mu}_S^\star)\Delta\hat{\mu}_S \qquad \chi_1^B(\hat{\mu}_S) \approx \chi_1^B(\hat{\mu}_S^\star) + \chi_{11}^{BS}(\hat{\mu}_S^\star)\Delta\hat{\mu}_S$$

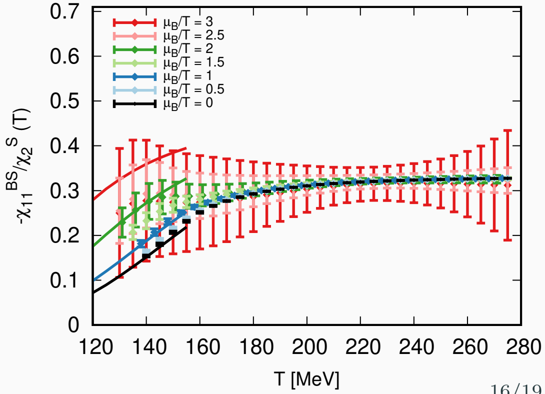
Expand in strangeness-to-baryon ratio R:

$$R = \frac{\chi_1^S}{\chi_1^B} = \frac{\chi_2^S(\hat{\mu}_S^\star)\Delta\hat{\mu}_S}{\chi_1^B(\hat{\mu}_S^\star)\Delta\hat{\mu}_S + \chi_{11}^{BS}(\hat{\mu}_S^\star)}$$

which gives:

$$\Delta\hat{\mu}_S = \frac{R\hat{\chi}_1^B(\hat{\mu}_S^\star)}{\chi_2^S(\hat{\mu}_S^\star) - R\chi_{11}^{BS}(\hat{\mu}_S^\star)}$$

The other quantity we need is $\chi_{11}^{BS}(\hat{\mu}_S^\star)$ (or $\chi_{11}^{BS}(\hat{\mu}_S^\star)/\chi_2^S(\hat{\mu}_S^\star)$).



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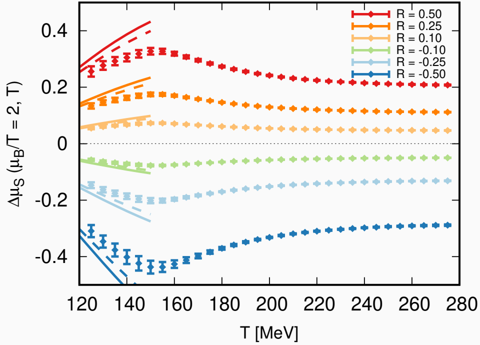
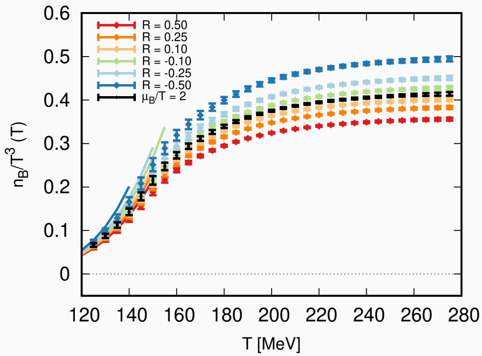
Beyond strangeness neutrality

We then get the chemical potential shift $\Delta\,\hat{\mu}_S$, and from it the baryon density follows trivially





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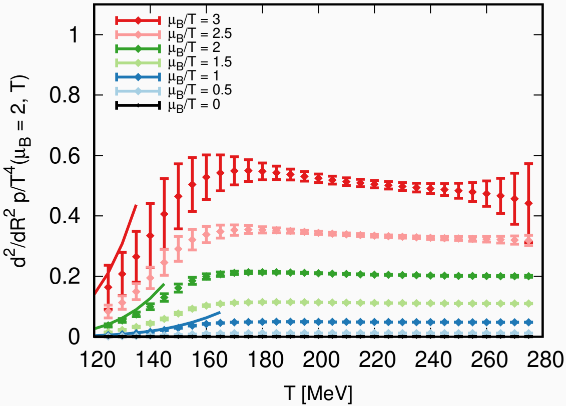
Beyond strangeness neutrality

The pressure receives no correction at $\mathcal{O}(R)$ (it would be $\sim \chi_1^S$):

$$\hat{p}(T,\,\hat{\mu}_B,R) \approx \hat{p}(T,\,\hat{\mu}_B,0)+\frac{1}{2}\frac{\mathrm{d}^2\hat{p}}{\mathrm{d}R^2}\,(T,\,\hat{\mu}_B)\,R^2$$

with:

$$\frac{\mathrm{d}^2\hat{p}}{\mathrm{d}R^2}\,(T,\,\hat{\mu}_B) = \frac{\left(\chi_1^B\,(T,\,\hat{\mu}_B)\right)^2}{\chi_2^S\,(T,\,\hat{\mu}_B)}$$



This is the beginning of the extrapolation beyond $n_S = 0$, better precision will be required

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Summary

- The EoS for QCD at large chemical potential is highly demanded in heavy-ion collisions community, especially for hydrodynamic simulations
- Historical approach of Taylor expansion for EoS has shortcomings
  - Because of technical/numerical challenges
  - Because of phase structure of the theory
- An alternative expansion scheme tailored to the specific behavior of relevant observables seems a better approach (better convergence). Thermodynamic quantities up to $\hat{\mu}_B \simeq 3.5$ have very reasonable uncertainties
- Successfully applied our procedure to strangeness neutrality, and moved beyond

Outlook

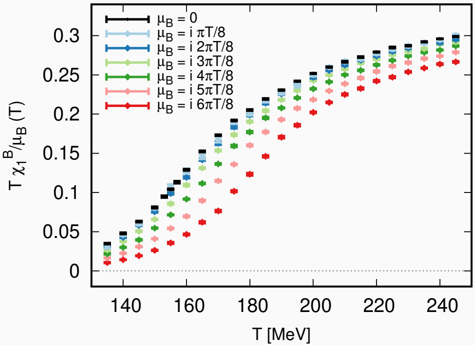
- Signal can be improved with better statistics
- Improved EoS at $\mu_B = 0$ would have big impact on errors

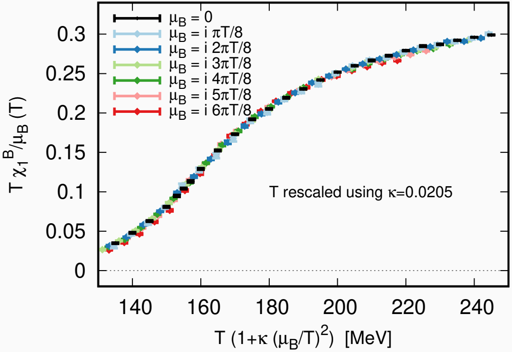
BACKUP

An alternative approach

From simulations at imaginary μ_B we *observe* that $\chi_1^B(T,\,\hat{\mu}_B)$ at (imaginary) $\hat{\mu}_B$ appears to be differing from $\chi_2^B(T,0)$ mostly by a rescaling of T:

$$\frac{\chi_1^B(T,\,\hat{\mu}_B)}{\hat{\mu}_B} = \chi_2^B(T',0)\;, \quad T' = T\left(1+\kappa\,\hat{\mu}_B^2\right)$$

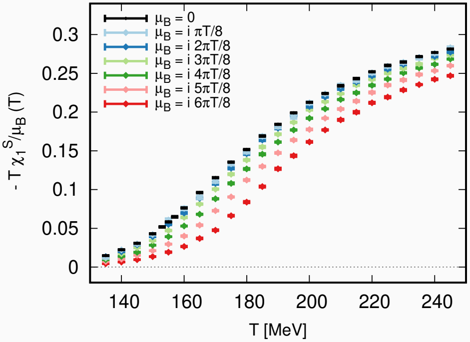


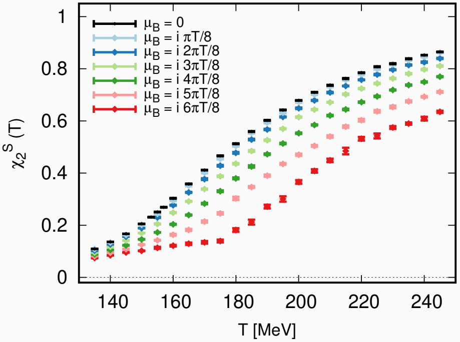


An alternative approach

The other (BS) second order susceptibilities display a similar scenario:

$$\frac{\chi_1^S}{\hat{\mu}_B}(T,\,\hat{\mu}_B) = \chi_{11}^{BS}(T',0)\ , \qquad \chi_2^S(T,\,\hat{\mu}_B) = \chi_2^S(T',0)$$





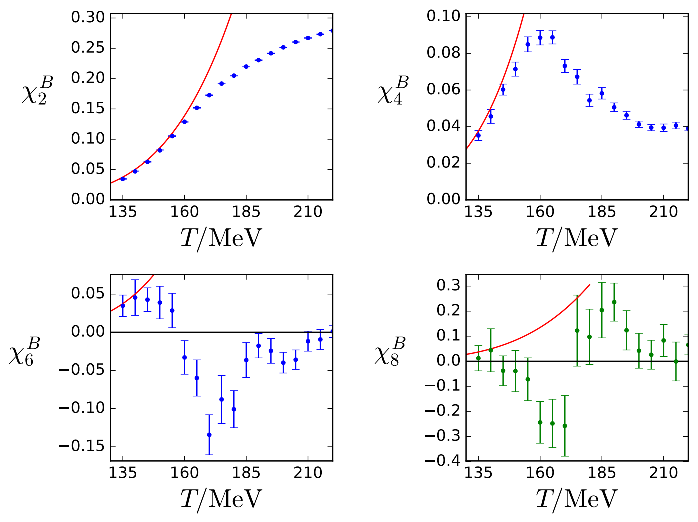
Lattice QCD at finite μ_B - Taylor coefficients

- Fluctuations of baryon number are the Taylor expansion coefficients of the pressure

$$\chi_{ijk}^{BQS}(T) = \left.\frac{\partial^{i+j+k} p/T^4}{\partial\hat{\mu}_B^i \partial\hat{\mu}_Q^j \partial\hat{\mu}_S^k}\right|_{\vec{\mu}=0}$$

- Signal extraction is increasingly difficult with higher orders, especially in the transition region

- Higher order coefficients present a more complicated structure



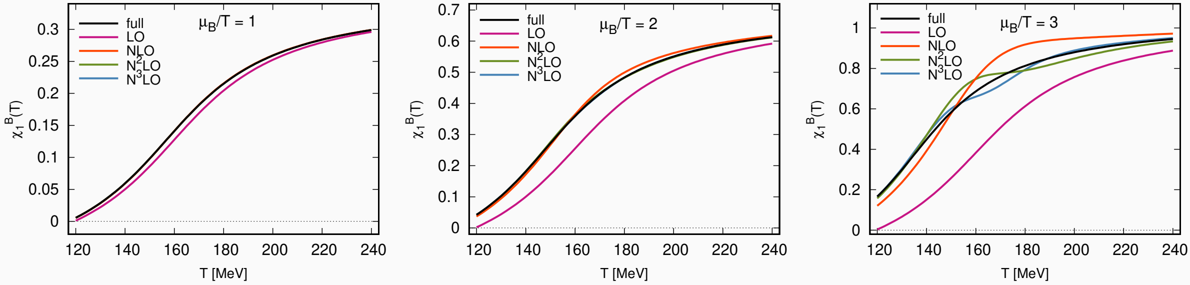
WB: Borsányi *et al.* JHEP 10 (2018) 205;
(also e.g., HotQCD: Bazavov *et al.* PRD101 (2020), 074502)

Taylor expanding a (shifting) sigmoid

Assume we have a sigmoid function $f(T)$ which shifts with $\hat{\mu}$, with a simple T-independent shifting parameter κ. How does Taylor cope with it?

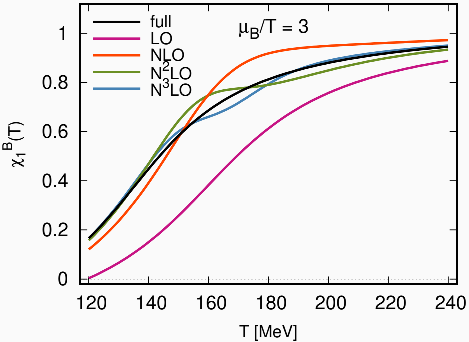
$$f(T,\,\hat{\mu}) = f(T',0)\ , \qquad\qquad T' = T(1+\kappa\,\hat{\mu}^2)\ ,$$

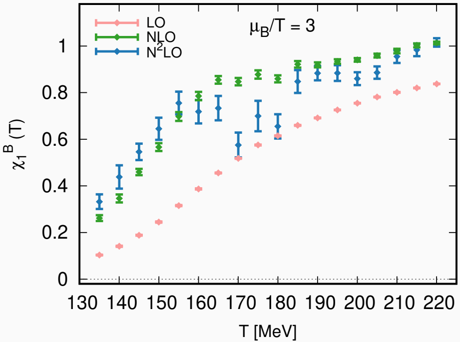
We fitted $f(T,0) = a + b\arctan(c(T-d))$ to $\chi_2^B(T,0)$ data for a 48×12 lattice



Taylor expanding a (shifting) sigmoid

- The Taylor expansion seems to have problems reproducing the original function (left)
- Quite suggestive comparison with actual Taylor-expanded lattice data (right)





- Problems at T slightly larger than $T_{pc} \Rightarrow$ influence from structure in χ_6^B and χ_8^B

Determine κ_n

I. Directly determine $\kappa_2(T)$ at $\hat{\mu}_B = 0$ from the previous relation

II. From our imaginary-$\hat{\mu}_B$ simulations ($\hat{\mu}_Q = \hat{\mu}_S = 0$) we calculate:

$$\frac{T' - T}{T\,\hat{\mu}_B^2} = \kappa_2(T) + \kappa_4(T)\,\hat{\mu}_B^2 + \mathcal{O}(\hat{\mu}_B^4) = \Pi(T)$$

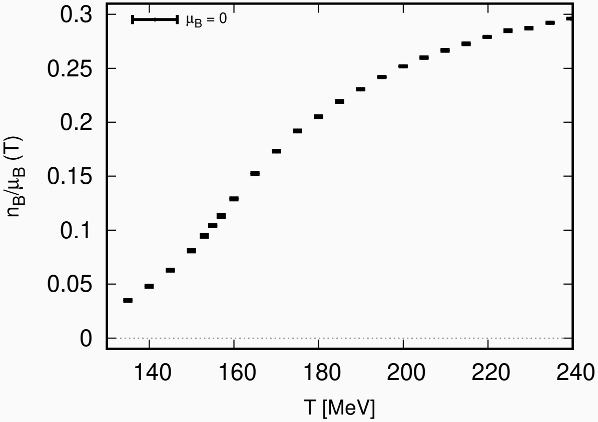
III. Calculate $\Pi(T, N_\tau, \hat{\mu}_B^2)$ for $\hat{\mu}_B = in\pi/8$ and $N_\tau = 10, 12, 16$

IV. Perform a combined fit of the $\hat{\mu}_B^2$ and $1/N_\tau^2$ dependence of $\Pi(T)$ at each temperature, yielding a continuum estimate for the coefficients

$\Rightarrow$ The $\mathcal{O}(1)$ and $\mathcal{O}(\hat{\mu}_B^2)$ coefficients of the fit are $\kappa_2(T)$ and $\kappa_4(T)$

Determine κ_n

The procedure, visualized:



Spline fit both at $\mu_B = 0$ and $\mu_B \neq 0$, then determine $T - T'$ (horizontal segments)

Determine κ_n

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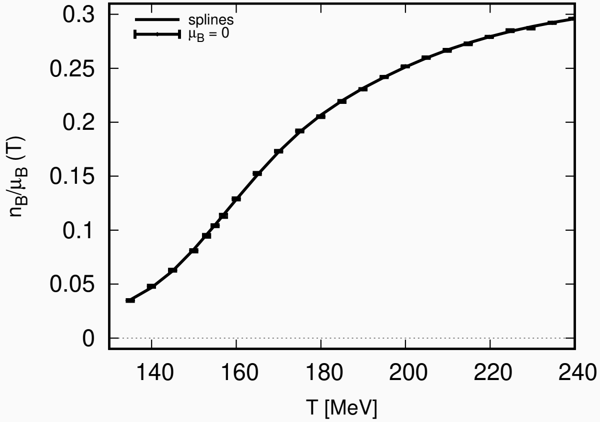

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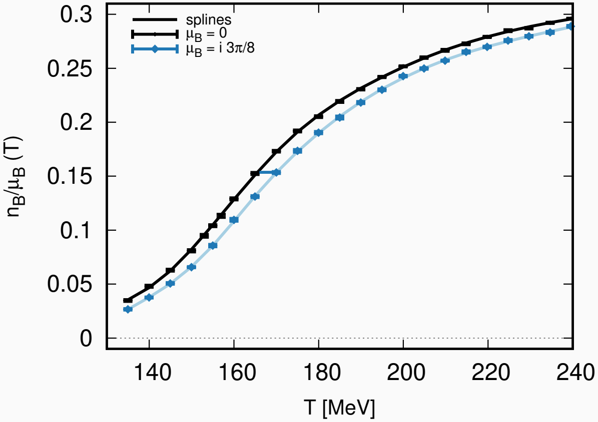

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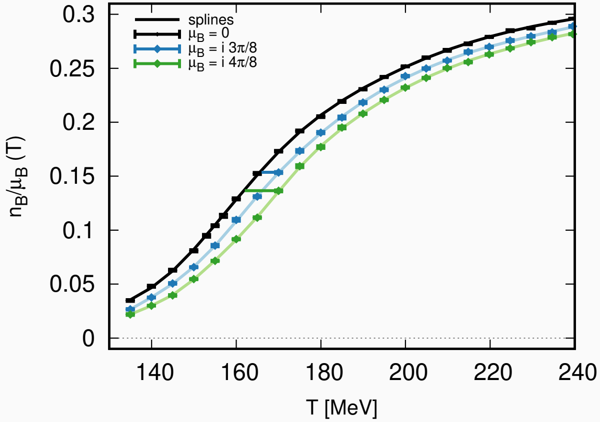

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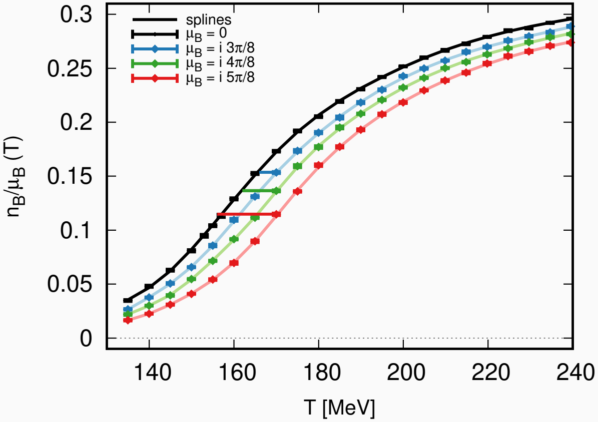

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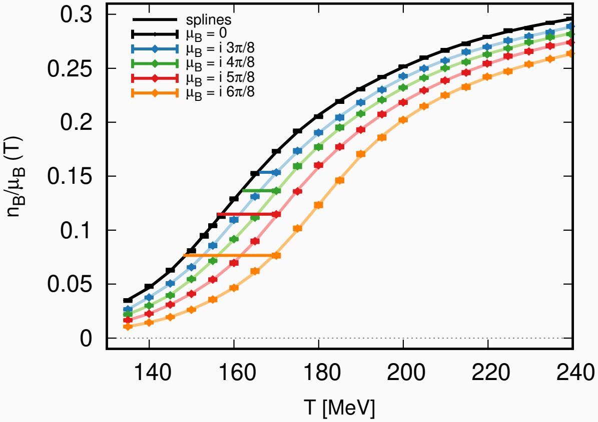


Spline fit both at $\hat{\mu}_B = 0$ and $\hat{\mu}_B \neq 0$, then determine $T - T'$ (horizontal segments)

Rigorous formulation: $\mu_Q = \mu_S = 0$

Similar relations can be derived analogously from:

$$\frac{\chi_1^S}{\hat{\mu}_B}(T,\,\hat{\mu}_B) = \chi_{11}^{BS}(T',0)\ , \qquad \chi_2^S(T,\,\hat{\mu}_B) = \chi_2^S(T',0)$$

yielding:

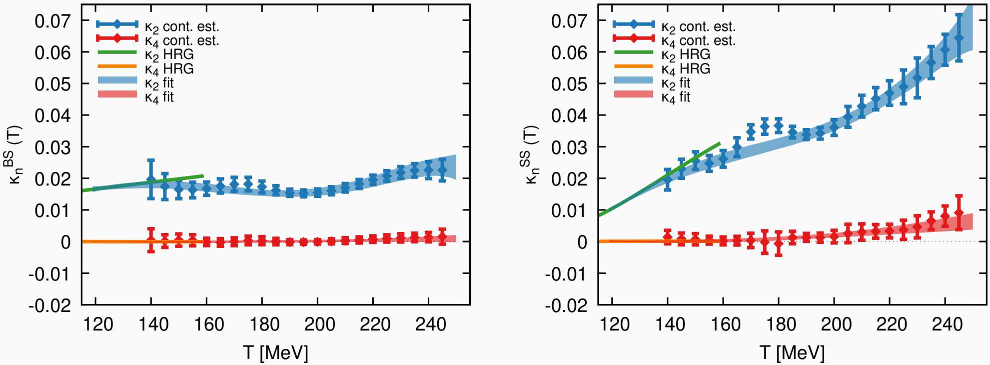
$$\kappa_2^{BS}(T) = \frac{1}{6T}\frac{\chi_{31}^{BS}(T)}{\chi_{11}^{BS\prime}(T)} \qquad \kappa_2^{S}(T) = \frac{1}{2T}\frac{\chi_{22}^{BS}(T)}{\chi_{2}^{S\prime}(T)}$$

$$\kappa_4^{BS}(T) = \frac{1}{360\chi_{11}^{BS\prime}(T)^3}\Big(3\chi_{11}^{BS\prime}(T)^2\chi_{51}^{BS}(T) - 5\chi_{11}^{BS\prime\prime}(T)\chi_{31}^{BS}(T)^2\Big)$$

$$\kappa_4^{S}(T) = \frac{1}{24\chi_{2}^{S\prime}(T)^3}\Big(\chi_{2}^{S\prime}(T)^2\chi_{42}^{BS}(T) - 3\chi_{2}^{S\prime\prime}(T)\chi_{22}^{BS}(T)^2\Big)$$

The results for $\kappa_2(T)$, $\kappa_4(T)$

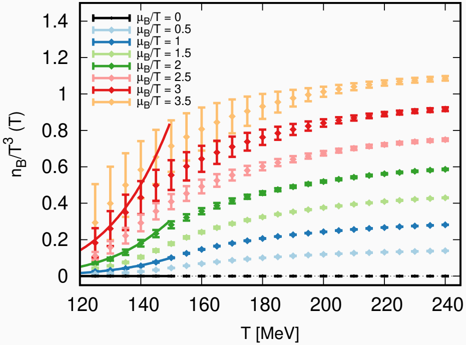
A similar picture appears for κ_n^{BS} and κ_n^{SS}

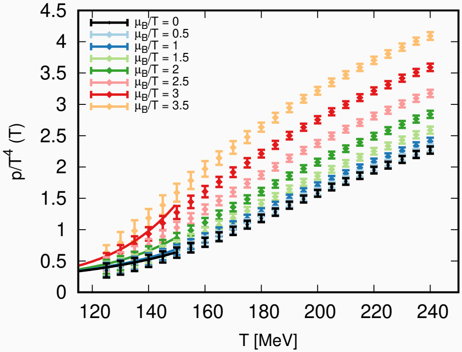


NOTE: polynomial fits take into account both statistical and systematic correlations.

Thermodynamics at finite (real) μ_B

- We reconstruct thermodynamic quantities up to $\hat{\mu}_B \simeq 3.5$ with uncertainties well under control
- Agreement with HRG model calculations at small temperatures
- No pathological (non-monotonic) behavior is present





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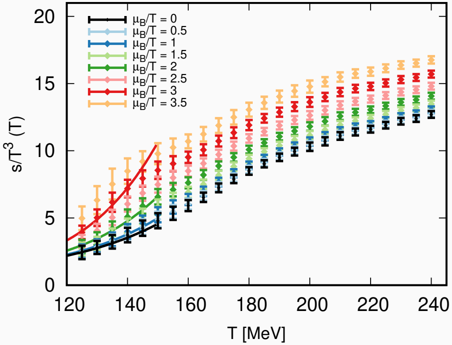

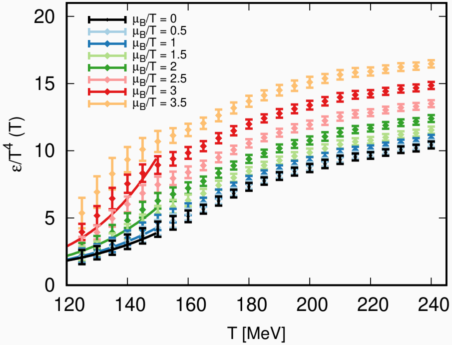

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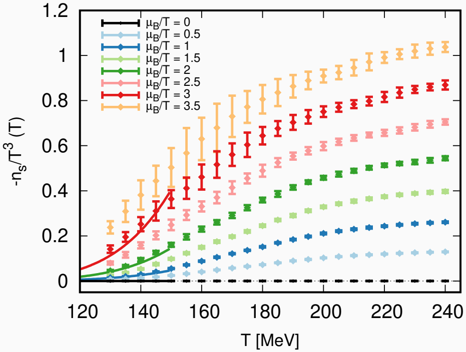
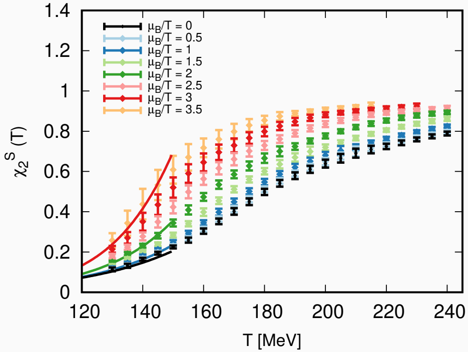
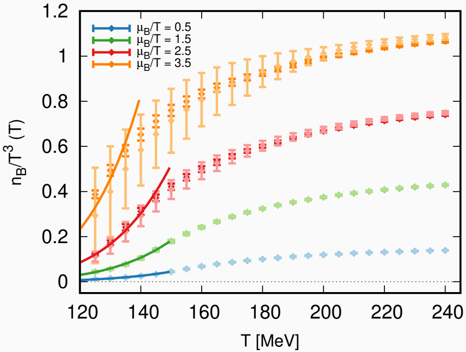


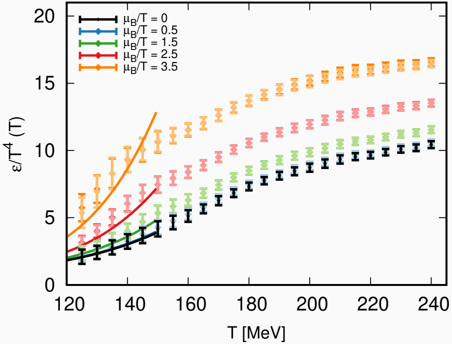


Thermodynamics at finite (real) μ_B

- We also check the results without the inclusion of $\kappa_4(T)$ (darker shades)
- Including $\kappa_4(T)$ only results in added error, but does not "move" the results

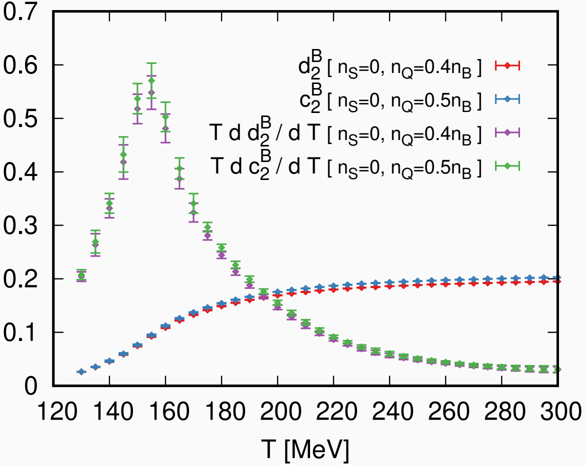
$\longrightarrow$ <u>Good convergence</u>





Strangeness neutrality vs strangeness neutrality

Comparing strangeness neutrality with $\mu_Q = 0$ (i.e. $n_Q = 0.5 n_B$) against strangeness neutrality with $n_Q = 0.4 n_B$ (heavy-ion)



Formulae with the SB correction

For the expansion coefficient of the baryon density, we get:

$$\lambda_2^{\mathrm{BB}} = \frac{1}{6Tf'(T)}\left(c_4^B(0,T) - \frac{\overline{c_4^B}(0)}{\overline{c_2^B}(0)} f(T)\right) ,$$

where $f(T) = \frac{d^2 \log Z}{d\mu_B^2}(\mu_B = 0, T)$. For the expansion coefficient of the strangeness chemical potential we get:

$$\lambda_2^{\mathrm{BS}} = \frac{1}{Tf'(T)} s_3(T) = \frac{1}{6Tf'(T)} \frac{d^3\,\hat{\mu}_S}{d\,\hat{\mu}_B^3}(T) ,$$

where $\frac{\hat{\mu}_S}{\hat{\mu}_B}(\,\hat{\mu}_B, T) = s_1(T) + s_3(T)\,\hat{\mu}_B^2 + s_5(T)\,\hat{\mu}_B^4 + \ldots$ and $f(T) = \lim_{\hat{\mu}_B \to 0} \frac{\hat{\mu}_S}{\hat{\mu}_B}(\mu_B, T) = -\frac{\chi_{11}^{BS}}{\chi_2^S}(0, T)$. For the expansion coefficient of the strangeness susceptibility we get:

$$\lambda_2^{\mathrm{SS}} = \frac{1}{2\,Tf'(T)} S_{2,\mathrm{sym}}^{\mathrm{NLO}}(0, T) ,$$

where $f(T) = \chi_2^S(\mu_B = 0, T)$.

Formulae with the SB correction

In principle, the λ_4 coefficients can also be expressed using the Taylor coefficients at $\mu \equiv 0$. For these one needs the Taylor coefficients up to sixth order and the second temperature derivative of the second order coefficients. For the quantities discussed in this paper we have:

$$\lambda_4^{\mathrm{BB}}(T) = \frac{1}{360T} \frac{1}{\overline{c}_2^B(0)^2 f'(T)^3} \cdot \left[3\,\overline{c}_2^B(0)^2 c_6^B(0,T) f'(T)^2 \right.$$
$$-10\,\overline{c}_4^B(0)\, f'(T)^2 \left(\overline{c}_2^B(0) c_4^B(0,T) - \overline{c}_4^B(0) f(T)\right)$$
$$\left. -5\, f''(T) \left(\overline{c}_2^B(0) c_4^B(0,T) - \overline{c}_4^B(0) f(T)\right)^2\right] ,$$

where $f(T) = \frac{d^2 \log Z}{d\mu_B^2}(\mu_B = 0, T)$.

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In principle, the λ_4 coefficients can also be expressed using the Taylor coefficients at $\mu \equiv 0$. For these one needs the Taylor coefficients up to sixth order and the second temperature derivative of the second order coefficients. For the quantities discussed in this paper we have:

$$\begin{aligned}\lambda_4^{\mathrm{BS}}(T) &= \frac{s_5(T)}{T f'(T)} - \frac{s_3(T)^2 f''(T)}{2T f'(T)^3} \\ &= \frac{1}{120 T f'(T)} \frac{d^5\,\hat{\mu}_S}{d\,\hat{\mu}_B^5}(T) - \frac{f''(T)}{72 T f'(T)^3} \left(\frac{d^3\,\hat{\mu}_S}{d\,\hat{\mu}_B^3}(T) \right)^2 ,\end{aligned}$$

where $\frac{\hat{\mu}_S}{\hat{\mu}_B}(\,\hat{\mu}_B, T) = s_1(T) + s_3(T)\,\hat{\mu}_B^2 + s_5(T)\,\hat{\mu}_B^4 + \ldots$ and $f(T) = \lim_{\,\hat{\mu}_B \to 0} \frac{\hat{\mu}_S}{\hat{\mu}_B}(\mu_B, T) = -\frac{\chi_{11}^{BS}}{\chi_2^S}(0, T)$.

Formulae with the SB correction

In principle, the λ_4 coefficients can also be expressed using the Taylor coefficients at $\mu \equiv 0$. For these one needs the Taylor coefficients up to sixth order and the second temperature derivative of the second order coefficients. For the quantities discussed in this paper we have:

$$\lambda_4^{\rm SS}(T) = \frac{1}{24 T f'(T)^3} \Big(S_{2,\rm sym}^{\rm NNLO}(0,T) f'(T)^2 - 3 f''(T) S_{2,\rm sym}^{\rm NLO}(0,T)^2 \Big) \, ,$$

where $f(T) = \chi_2^S(\mu_B = 0, T)$, and

$$\begin{aligned} S_{2,\rm sym}^{\rm NLO}(0,T) &= \chi_{22}^{BS}(0,T) \\ &+ 2 s_1(T) \chi_{13}^{BS}(0,T) + s_1(T)^2 \chi_4^S(0,T) \\ S_{2,\rm sym}^{\rm NNLO}(0,T) &= \chi_{42}^{BS}(0,T) + 4 s_1(T) \chi_{33}^{BS}(0,T) \\ &+ 6 s_1(T)^2 \chi_{24}^{BS}(0,T) + 4 s_1(T)^3 \chi_{15}^{BS}(0,T) \\ &+ s_1(T)^4 \chi_6^S(0,T) + 24 s_3(T) \chi_{13}^{BS}(0,T) \\ &+ 24 \chi_4^S(0,T) s_1(T) s_3(T) \end{aligned}$$

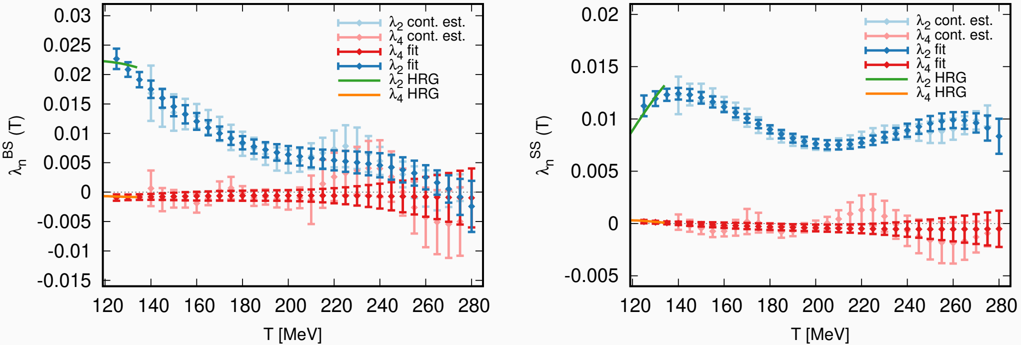
Formulae with the SB correction

In addition, we used the expansion coefficients of $\hat{\mu}_S(\hat{\mu}_B)$:

$$
\begin{aligned}
s_1 &= -\frac{\chi_{11}^{BS}}{\chi_2^S} \\
s_3 &= -\frac{1}{6\chi_2^S}\left[\chi_4^S s_1^3 + 3\chi_{13}^{BS} s_1^2 + 3\chi_{22}^{BS} s_1 + \chi_{31}^{BS}\right] \\
s_5 &= -\frac{1}{120\chi_2^S}\left[+\chi_6^S s_1^5 + 5\chi_{15}^{BS} s_1^4 + 10\chi_{24}^{BS} s_1^3\right. \\
&\quad +60\chi_4^S s_1^2 s_3 + 120\chi_{13}^{BS} s_1 s_3 + 60\chi_{22}^{BS} s_3 \\
&\quad \left.+10\chi_{33}^{BS} s_1^2 + 5\chi_{42}^{BS} s_1 + \chi_{51}^{BS}\right] .
\end{aligned}
$$

The alternative approach at strangeness neutrality

The coefficients for μ_S/μ_B and χ_2^S:



Here SB has no effect, though λ_2^{BS} still goes to zero

Borsányi, PP *et al.* PRD 105 (2022) 114504

Systematics

For an analysis of the systematic uncertainties, we consider:

- 2x scale settings (w_0 and f_π)
- 2x choices of $\hat{\mu}_B$ fitting range ($\hat{\mu}_B = in\pi/8$ with $n \in \{0, 3 - 5.5\}$ or $n \in \{0, 3 - 6.5\}$)
- 2x fit functions. Always linear in $1/N_\tau^2$, and linear or parabolic in $\hat{\mu}_B^2$
- 3x splines at $\hat{\mu}_B = 0$
- 2x splines at $\hat{\mu}_B \neq 0$
- Included (or not) $N_\tau = 8$

for a total of 96x analyses for each T.

At each temperature, the 96x analyses are combined with uniform weights, if $Q > 0.01$.