Constraints on hadron resonance gas interactions via first-principles Lattice QCD susceptibilities

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Strongly-interacting matter proceeds through several different phases during a collision event → HIC modeling/phenomenology
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Freeze-out Stages in HICs

- After a heavy-ion collision, the hot, dense system cools and expands, eventually producing color-neutral hadrons that are measured by the detectors.

- **Chemical freeze-out**: inelastic collisions cease; the chemical composition is fixed (particle yields and fluctuations)

- **Kinetic freeze-out**: elastic collisions cease; spectra and correlations are fixed
The Hadron Resonance Gas Model

- The ideal HRG model agrees well with Lattice QCD results on the Equation of State below the cross-over transition temperature:

\[
\frac{P}{T^4} = \frac{1}{VT^3} \sum_i \ln Z_i(T, V, \bar{\mu})
\]

\[
\ln Z_i^{M/B} = \mp \frac{V d_i}{(2\pi)^3} \int d^3p \ln(1 \mp \exp[-(\epsilon_i - \mu_a X^a_i)/T])
\]

energy \(\epsilon_i = \sqrt{p^2 + m_i^2}\)

conserved charges \(\bar{X}_i = (B_i, S_i, Q_i)\)

degeneracy \(d_i\), mass \(m_i\), volume \(V\)

S. Borsanyi et al (WB collaboration), PLB (2014), A. Bazavov et al (HotQCD collaboration), PRD (2014)
Susceptibilities in the HRG Model

Susceptibilities are fluctuations of conserved charges from a theoretical perspective:

\[ \chi_{l,m,n}^{BSQ} = \frac{\partial^{l+m+n} p / T^4}{\partial (\mu_B / T)^l \partial (\mu_S / T)^m \partial (\mu_Q / T)^n} \]

- mean: \( M = \chi_1 \)
- variance: \( \sigma^2 = \chi_2 \)
- skewness: \( S = \chi_3 / \chi_2^{3/2} \)
- kurtosis: \( \kappa = \chi_4 / \chi_2^2 \)

\[ \sigma^2 / M = \chi_2 / \chi_1 \quad S \sigma^3 / M = \chi_3 / \chi_1 \]

\[ S \sigma = \chi_3 / \chi_2 \quad \kappa \sigma^2 = \chi_4 / \chi_2 \]

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\[
\frac{\sigma^2}{M} = \frac{\chi_2}{\chi_1} \\
\frac{S\sigma^3}{M} = \frac{\chi_3}{\chi_1} \\
S\sigma = \frac{\chi_3}{\chi_2} \\
\kappa\sigma^2 = \frac{\chi_4}{\chi_2}
\]

\[
\kappa_B\sigma_B^2 = \frac{\chi_4^B(T, \mu_B)}{\chi_2^B(T, \mu_B)} = r_{42}^B + \ldots
\]

Even-order ratios predicted to be unity in ideal HRG

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S. Ejiri, F. Karsch, K. Redlich, PLB (2006),
B. Friman et al, EPJ (2011),
S. Borsanyi et al (WB collaboration), JHEP (2018),
P. Alba et al (WB collaboration), PRD (2017)
Susceptibilities in the HRG Model

- Susceptibilities are fluctuations of conserved charges from a theoretical perspective:

\[ \chi_{lmn}^{BSQ} = \frac{\partial^{l+m+n} p/T^4}{\partial (\mu_B/T)^l \partial (\mu_S/T)^m \partial (\mu_Q/T)^n} \]

**mean**: \(M = \chi_1\)

**variance**: \(\sigma^2 = \chi_2\)

**skewness**: \(S = \chi_3/\chi_2^{3/2}\)

**kurtosis**: \(\kappa = \chi_4/\chi_2^2\)

\[ \sigma^2/M = \chi_2/\chi_1 \]

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Some differential quantities are not well-described within ideal HRG

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Susceptibilities in the HRG Model

➤ Susceptibilities are fluctuations of conserved charges from a theoretical perspective:

$$\chi_{B^{SM}}^{n\mu} = \frac{\partial \lambda^{m+n} p / T^4}{\partial (\mu_B / T)^m \partial (\mu_\Sigma / T)^n \partial (\mu_Q / T)^n}$$

- mean: $$M = \chi_1$$
- variance: $$\sigma^2 = \chi_2$$
- skewness: $$S = \chi_3 / \chi_2^{3/2}$$
- kurtosis: $$\kappa = \chi_4 / \chi_2^2$$

$$\frac{\sigma^2}{M} = \chi_2 / \chi_1$$  $$S\sigma^3 / M = \chi_3 / \chi_1$$  $$S\sigma = \chi_3 / \chi_2$$  $$\kappa\sigma^2 = \chi_4 / \chi_2$$

Some differential quantities are not well-described within ideal HRG

Susceptibilities beyond the Ideal HRG Model

Outline

I. Extensions of the ideal HRG model
   - Excluded volume repulsive interactions
   - Additional states in the hadronic spectrum than those that are well-established by the Particle Data Group

II. Confronting EV-HRG Susceptibilities with Lattice QCD
   - Excluded volume interaction sensitive susceptibility ratios
   - Hadronic spectrum specific susceptibility ratios
   - Strangeness susceptibilities
I. Extensions of the ideal HRG model
Modified HRG Model

- Improved HRG description of higher order cumulants with repulsive interactions
  - Minimalistic extension

- Complementary yet distinct effects can be constrained separately with specific susceptibility ratios:
  - Constrain the hard-core radius
    - Fourth-order cumulants and excluded volume: $\chi_4^B/\chi_2^B, \chi_{31}^{BQ}/\chi_{11}^{BQ}, \chi_{31}^{BS}/\chi_{11}^{BS}$
  - Constrain hadronic spectrum
    - Extra states from baryon correlators: $\chi_{11}^{BQ}/\chi_2^B, \chi_{11}^{BS}/\chi_2^B$

V. Vovchenko et al, PLB (2017)
JMK, V. Koch, C. Ratti, V. Vovchenko, PRD (2021)
Include repulsive interactions for baryons & antibaryons, leading to a sum of pressures for each type of hadron: \( P = P^\text{id}_M + P^\text{ev}_B + P^\text{ev}_{\bar{B}} \)

The pressure in the excluded volume model yields a transcendental equation:

\[
P^\text{ev}_{B(\bar{B})} = \sum_{i \in B} \frac{m_i^2 T^2}{2\pi^2} K_2(m_i/T) \exp(\pm \mu_i/T) \exp \left( \frac{-b P^\text{ev}_{B(\bar{B})}}{T} \right),
\]

which can be solved by making use of the Lambert W function:

\[
P^\text{ev}_{B(\bar{B})} = \frac{T}{b} \text{W}[b \sum_{i \in B} \frac{m_i^2 T^2}{2\pi^2} K_2(m_i/T) \exp(\pm \mu_i/T)].
\]
The Hadronic Spectrum

- Pressure in HRG model depends on resonances included in the calculation:
  \[
  \frac{P}{T^4} = \frac{1}{VT^3} \sum_i \ln Z_i(T, V, \bar{\mu})
  \]

- **PDG2016**: 608 species
- **PDG2016+**: 738 species (all experimentally observed particles)
- **QM**: 1485 species (all states predicted by the Quark Model, updated in this work)

<table>
<thead>
<tr>
<th>Particle</th>
<th>J^P</th>
<th>Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>1/2^+</td>
<td>****</td>
</tr>
<tr>
<td>n</td>
<td>1/2^+</td>
<td>****</td>
</tr>
<tr>
<td>N(1860)</td>
<td>5/2^+</td>
<td>**</td>
</tr>
<tr>
<td>N(1875)</td>
<td>3/2^-</td>
<td>***</td>
</tr>
<tr>
<td>Δ(1232)</td>
<td>3/2^+</td>
<td>****</td>
</tr>
<tr>
<td>Δ(1750)</td>
<td>1/2^+</td>
<td>*</td>
</tr>
</tbody>
</table>

Note that PDG 2016 list provides a satisfactory description for most hadrons, and the PDG 2016 including all listed states (also the ones with six stars) arise from all possible combinations of different quark-antiquark or diquark structures. Moreover, all observables confirm the need for not yet detected, or at least not yet fully established, strangeness states. The full resonance spectra: the PDG 2016 including only the more positive and negative contributions (see the next section), and get differential information on the missing states, based on their strangeness content. The main result of this paper is a lattice determination of these partial pressures. This is a difficult sector is not a proper continuum extrapolation. In all cases, the solid lines correspond to the HRG model based on different lattices. For all other cases, the data are properly continuum extrapolated. In all cases, the solid lines correspond to the HRG model based on different lattices. For all other cases, the data are properly continuum extrapolated. In all cases, the solid lines correspond to the HRG model based on different lattices. For all other cases, the data are properly continuum extrapolated. In all cases, the solid lines correspond to the HRG model based on different lattices. For all other cases, the data are properly continuum extrapolated.
II. Confronting EV-HRG Susceptibilities with Lattice QCD
In the ideal HRG model, fourth-to-second order susceptibility ratios are predicted to be unity, while the EV-HRG model includes repulsive interaction terms:

$$b = 0.4 \pm 1 \text{ fm}^3$$

PDG2016

$$b = 0.4 \pm 1 \text{ fm}^3$$

Quark model

$$b = 0$$

4th-to-second-order ratios have:

- weak dependence on particle spectrum
In the ideal HRG model, fourth-to-second order susceptibility ratios are predicted to be unity, while the EV-HRG model includes repulsive interaction terms:

$$b = 0.4 - 1 \text{ fm}^3$$

PDG2016

+ PDG2016

Quark model ($b = 0.4 - 1 \text{ fm}^3$)

$$b = 0$$

31

4

31

BQ

130

140

150

160

170

0.0

0.2

0.4

0.6

0.8

1.0

T [MeV]

Sensitivity to excluded volume only

4th-to-second-order ratios have:

- weak dependence on particle spectrum

JMK, V. Koch, C. Ratti, V. Vovchenko, PRD (2021)
Lattice data: S. Borsyani et al (WB collaboration), JHEP (2018)
See also: D. Bollweg et al (HotQCD collaboration), PRD (2021)
In the ideal HRG model, fourth-to-second order susceptibility ratios are predicted to be unity, while the EV-HRG model includes repulsive interaction terms:

4th-to-second-order ratios have:

- weak dependence on particle spectrum
- the same EV corrections

where: \( \kappa_B(T, \mu_B, \mu_Q, \mu_S) = b \sum_{i \in B} \tilde{\phi}_i(T) \exp(\pm \mu_i/T) \)

\[
\frac{\chi_4^B}{\chi_2^B} = \frac{\chi_{31}^{BS}}{\chi_{11}^{BS}} = \frac{\chi_{31}^{BQ}}{\chi_{11}^{BQ}} = \frac{1 - 8 W(\kappa_B) + 6[W(\kappa_B)]^2}{[1 + W(\kappa_B)]^4} = 1 - 12 \kappa_B + O(\kappa_B^2)
\]
In these second order ratios, the EV parameter, \( b \), cancels exactly:

\[
\frac{\chi_{11}^{BQ}}{\chi_2^B} = \frac{\sum_{j \in \text{sectors}} B_j Q_j \tilde{\phi}_j(T)}{\sum_{j \in \text{sectors}} B_j^2 \tilde{\phi}_j(T)}
\]

\[\sum_{j \in \text{sectors}} Q_j \tilde{\phi}_j(T)\]

![Graph showing the ratios \(\chi_{11}^{BQ}/\chi_2^B\) and \(-\chi_{11}^{BS}/\chi_2^B\) against temperature \(T\) in MeV.](image)

**Figure:** Graphs showing the ratios \(\chi_{11}^{BQ}/\chi_2^B\) and \(-\chi_{11}^{BS}/\chi_2^B\) against temperature \(T\) in MeV.

**Legend:**
- PDG2016
- PDG2016 +
- Quark model
- WB 1910.14592 / 1507.04627

**Note:** The graphs illustrate the behavior of fourth-order susceptibilities as a function of temperature, highlighting the agreement between lattice data and theoretical predictions.

**References:**
- JMK, V. Koch, C. Ratti, V. Vovchenko, PRD (2021)
- See also: D. Bollweg et al (HotQCD collaboration), PRD (2021)
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\]
Further investigate strangeness-carrying quantities to study potential flavor hierarchy:

- Smaller b value preferred for strangeness susceptibilities with PDG2016+

JMK, V. Koch, C. Ratti, V. Vovchenko, PRD (2021)
Lattice data: P. Alba et al (WB collaboration), PRD (2017)
See also: D. Bollweg et al (HotQCD collaboration) PRD (2021)
Conclusions

➤ Improved HRG description of higher order cumulants with repulsive interactions

➤ Minimalistic extension of HRG constrains the excluded volume parameter and disentangles effects of particle spectrum with specific susceptibility ratios

➤ Best hadronic list: PDG2016+ list with a small $b$, QM list with a larger $b$

➤ Excluded volume parameter, $b$, shows a quark flavor hierarchy with smaller volumes for strangeness susceptibilities
Back-up Slides
Indications of Flavor Hierarchy

- Fluctuations of conserved charges
  - Fig. 5.6
  - Fig. 5.7

- More recently, these flavors hadronizing at different temperatures (see the discussion in Section 6.3.2).

- Fluctuations of electric charge. The black dots are the continuum extrapolated lattice QCD results.

- HRG prediction for the strange quark and for the light quarks.

- Correlator as a function of the temperature. The -diagonal fluctuations were extrapolated to finite chemical potential.

- Results for different second order -diagonal correlators are presented. We use a radial flow velocity profile of the form $v(r) = \frac{a}{r}$, where $a$ is the radial position in the thermal source, and $\beta$ is the exponent.

- The blast-wave model results are sensitive to the inclusion of more particles.

- Average transverse radial flow velocity $v_r$ at different RHIC energies and at different values of the temperature $T$. The spectra of multi-strange particles such as multi-strange hadrons in the fit would be constrained.

- In order to compare to the experimental results at RHIC, several diagonal and -diagonal correlators are presented.

- The continuum extrapolated lattice QCD results are particularly interesting and a lot of activity has been devoted to their extrapolation.

- Fluctuations of conserved charges and their results.

- From Ref. [32], the blue points are the -diagonal correlators.