Recent Developments in Hydrodynamics

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Hydrodynamics theory talks and posters in SQM 2022

Spin Hydrodynamics & Λ polarization

- Andrea Palermo, Local equilibrium and Lambda polarization in high energy heavy ion collisions, Jun 15, 9:00 AM, GBR1
- Jinfeng Liao, Novel Effects of Rotational Polarization in Relativistic Nuclear Collisions, Jun 15, 12:10 PM, GBR1
- Matteo Buzzegoli, Polarization in heavy ion collisions: a theoretical review, Jun 16, 9:30 AM, Grand Ballroom
- Ondrej Lomicky, Global and local Λ polarization from 27 to 200 GeV from a 3D viscous hydrodynamic model, Jun 14, Poster Board: OTH-05, Other topics
- Larisa Bravina, Polarization of Λ and anti-Λ hyperons in heavy-ion collisions at intermediate energies in hydrodynamic and microscopic transport models, Jun 14, Poster, Light-flavor and Strangeness
- Matteo Buzzegoli, Pseudo-gauge dependence of spin polarization in heavy-ion collisions, Jun 14, Poster Board: OTH-01, Other topics
- Baochi Fu, Signatures of the spin Hall effect in hot and dense QCD matter Jun 14, Poster, Bulk matter phenomena
- Xiaowen Li, Time evolution of global polarization within an improved microscopic approach Jun 14, Poster, Light-flavor and Strangeness Speaker

Nonlinear Causality Conditions

- Christopher Plumberg, Causality violations in realistic nuclear collision simulations, Jun 15, 11:30 AM, Sydney.

Hydro Initial States

- Chun Shen, Collectivity and baryon junctions in ultra-peripheral heavy-ion collisions, Jun 14, 10:50 AM, GBR1.
- Yuuka Kanakubo, Interplay between core and corona from small to large systems Jun 15, 10:50 AM, GBR3.

Fluctuations

- Maneesha Sushama Pradeep, Freezing out critical fluctuations, Jun 14, 11:50 AM, GBR1
- Dekrayat Almaalol, Baryon number, strangeness, and electric charge fluctuations in hydrodynamics at the LHC, Jun 14, Poster Board: BLK-26.

Critical point & BES

- Xiang-Yu Wu, (3+1)-D viscous hydrodynamics CLVisc at finite net baryon density: identified particle spectra, anisotropic flows and flow fluctuations across BES energies, Jun 14, 9:40 AM, GBR1
- Thiranat Bumnedpan, Modification of hadron multiplicity ratios at the chiral phase transition, Jun 14, Poster Board: BLK-20.
- Noriyuki Sogabe, Exploring the criticality of QCD with effective field theory for fluctuating hydrodynamics, Jun 14, Poster Board: BLK-27, Bulk matter phenomena

Lattice EoS

- Paolo Parotto, Resummed lattice QCD equation of state at finite baryon density: strangeness neutrality and beyond Jun 15, 9:20 AM, GBR1.

Longitudinal flow

- Guang-You Qin, Asymmetric longitudinal flow decorrelations in proton-nucleus collisions, Jun 14, Poster, Bulk matter phenomena

Transport coefficients

- Shubhalaxmi Rath, Charge and heat transport coefficients of a weakly magnetized hot and dense QCD medium, Jun 14, Poster Board: BLK-13, Bulk matter phenomena

Bayesian analysis

- Dong Jo Kim, New constraints of QCD matter from improved Bayesian parameter estimation with the latest LHC data Jun 15, 11:50 AM, GBR3

Upgrades

- Nikhil Hatwar, Effect of variation in relaxation time on elliptic flow in PbPb and AuAu collisions Jun 14, Poster Board: BLK-01, Bulk matter phenomena
- Ankit Kumar Panda, Causal second order magnetohydrodynamics from kinetic theory using RTA approximation, Jun 14, Poster Board: OTH-04, Other topics
- Iurii Karpenko, Multi-Fluid Hydrodynamics for RHIC BES/FAIR/NICA, Remade, Jun 14, Poster Board: BLK-15, Bulk matter phenomena

Plan

- Can only cover selected topics (with apologies to those I won't be able to cover)

What is hydrodynamics?

- Conservation laws $+$ Constitutive relations $+$ Thermodynamics
- Relevant conservation laws: Energy-momentum, charges (electric, baryon, etc), angular momentum
- Constitutive relations are needed because hydrodynamics is all about *coarse-grained* collective motions
- *New developments:* Is what we have been using the most general form of hydrodynamics?

Collective spin tensor

Meaning of $T^{\mu\nu}$

- Energy-momentum conservation laws

$$\partial_\mu T^{\mu\nu} = 0$$

  - T^{00}: Energy density
  - T^{i0}: The i-th component of the energy current
  - T^{0j}: Momentum density
  - T^{ij}: The i-th component of the momentum current
- Questions we normally don't ask:
  - Should $T^{\mu\nu}$ be always symmetric?
  - The total energy-momentum $P^\mu = \int d^3x T^{0\nu}$ is observable.

    But is $T^{\mu\nu}$ *observable*?

Is any current observable?

- Conservation law for a conserved quantity a: $\partial_\mu J^{\mu,a} = 0$
- One can always add

$$J'^{\mu,a} = J^{\mu,a} + \partial_\lambda B^{\lambda\mu,a}$$

provided that $B^{\lambda\mu,a} = -B^{\mu\lambda,a}$.
This does not change the total conserved quantity

$$Q^a = \int_V d^3x\, J^{0,a} = \int_V d^3x\, J'^{0,a}$$

provide that $B^{i0,a}$ vanishes at the boundary ∂V

- But it *does* change $J^{\mu,a}$. Does this mean that conserved currents are *not* observable?

Is any current observable?

Is $J^{\mu,a}$ observable?

- Classical Microscopic Dynamics: Yes. Individual point particles carry well defined energy, momentum and other conserved charges.

$$J^{\mu,q}(t,\mathbf{x}) = \sum_{s=ptcls} \frac{q_s}{p_s^0} p_s^\mu \delta^{(3)}(\mathbf{x} - \mathbf{x}_s(t))$$

- Continuum Dynamics: Depends.

So where is the disjoint?

- Point-Particles vs Fields
- Since the global charge is conserved, $\delta J^{0,q} = \nabla \cdot \mathbf{B}^{0,q}$ can only shuffle the local charges around (with $\mathbf{B}^{0,q} = \left\{ B^{i0,q} \right\}$).
- $J^{\mu,a}$ in continuum dynamics is inherently ambiguous *without further constraints*.

Example: NR QM

- Hamiltonian: $\hat{H} = -\frac{\nabla^2}{2m} + V(\mathbf{x})$
- Two energy density definitions

$$\begin{aligned} \epsilon_1 &= \mathrm{Re}(\psi^* \hat{H}\psi) \\ \epsilon_2 &= \frac{1}{2m}\left((\nabla\psi^*)\cdot(\nabla\psi) + \psi^* V\psi\right) \\ &= \epsilon_1 - \frac{1}{2}\nabla\cdot\mathrm{Im}\left(\psi^*\hat{\mathbf{v}}\psi\right) \end{aligned}$$

- Two energy current definitions

$$\begin{aligned} \mathbf{Q}_1 &= \frac{1}{2}\mathrm{Re}\left((\hat{H}\psi)^*(\hat{\mathbf{v}}\psi) + \psi^*\hat{\mathbf{v}}\hat{H}\psi\right) \\ \mathbf{Q}_2 &= \mathrm{Re}\left((\hat{H}\psi)^*(\hat{\mathbf{v}}\psi)\right) \end{aligned}$$

- The probability density

$$P_\psi = \psi^*\psi$$

- The probability current

$$\mathbf{J} = \frac{1}{2}\mathrm{Re}(\psi^*\hat{\mathbf{v}}\psi)$$

- The probability density is *positive definite and measurable* $\Longrightarrow$ No freedom
- Energy density may not be like that

Pseudo-gauge freedom

- Energy-momentum tensor is not unique

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda B^{\lambda\mu\nu} \text{ with } B^{\lambda\mu\nu} = -B^{\mu\lambda\nu}$$

  - $\partial_\mu \Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} = 0$ since $B^{\lambda\mu\nu} = -B^{\mu\lambda\nu}$
  - $P^\nu = \int_V d^3x\, \Theta^{0\nu} = \int_V d^3x\, T^{0\nu}$ *provided that* all quantities vanish at the boundary ∂V

- One can define a symmetric energy-momentum tensor

$$T^{\mu\nu}_{sym} = \frac{1}{2}(\Theta^{\mu\nu} + \Theta^{\nu\mu}) = \Theta^{\mu\nu} + \partial_\lambda B^{\lambda\mu\nu}$$

if the anti-symmetric part is a divergence

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = -\partial_\lambda H^{\lambda\mu\nu}$$

$B^{\lambda\mu\nu}$ is given by $B^{\lambda\mu\nu} = \frac{1}{2}\left(H^{\lambda\mu\nu} + H^{\mu\nu\lambda} - H^{\nu\lambda\mu}\right)$. *But this is a choice.*

Energy-momentum tensor

Q: When is $\Theta^{\mu\nu}$ not symmetric?

- Field-theoretical answer ($g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ for this part)
  - Let $L(\varphi^A, \partial_\mu \varphi^A)$ be the Lagrangian for the fields φ^A.
  - Conserved current for the translation symmetry:

$$\Theta^{\mu\nu} = \Pi^\mu_A \partial^\nu \varphi^A - g^{\mu\nu} L(\varphi^A, \partial_\mu \varphi^A) \;\; \text{with} \;\; \Pi^\mu_A = \frac{\partial L}{\partial(\partial_\mu \varphi^A)}$$

    Only for scalar ($S = 0$) fields, $\Theta^{\mu\nu} = \Theta^{\nu\mu}$. For $S \neq 0$, $\Theta^{\mu\nu} \neq \Theta^{\mu\nu}$.
  - Energy momentum is conserved: $\partial_\mu \Theta^{\mu\nu} = 0$. But for $S \neq 0$, $\partial_\nu \Theta^{\mu\nu} \neq 0$

A: When $S \neq 0$.

- Total angular momentum tensor

$$\mathcal{L}^{\lambda\mu\nu} = \underbrace{\Theta^{\lambda\nu} x^\mu - \Theta^{\lambda\mu} x^\nu}_{\mathbf{L}=\mathbf{x}\times\mathbf{p}} + \underbrace{\mathcal{S}^{\lambda\mu\nu}}_{\text{spin d.o.f.}}$$

  Total angular momentum conservation

$$0 = \partial_\lambda \mathcal{S}^{\lambda\mu\nu} + \Theta^{\mu\nu} - \Theta^{\nu\mu}$$

Example: Spin 1

- Lagrangian: $\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$ with $\partial_\alpha A^\alpha = 0$ and the EoM $\partial_\mu F^{\mu\nu} = 0$
- Energy momentum tensor:

$$\Theta^{\mu\nu} = -F^{\mu}_{\ \lambda} \partial^\nu A^\lambda - g^{\mu\nu} \mathcal{L}$$

- Anti-symmetric part:

$$\begin{aligned} \Theta^{\mu\nu} - \Theta^{\nu\mu} &= -F^{\mu}_{\ \lambda} \partial^\nu A^\lambda + F^{\nu}_{\ \lambda} \partial^\mu A^\lambda \\ &= -F^{\mu\lambda} \partial_\lambda A^\nu + F^{\nu\lambda} \partial_\lambda A^\mu \\ &= -\partial_\lambda \underbrace{\left(-F^{\lambda\mu} A^\nu + F^{\lambda\nu} A^\mu \right)}_{\mathcal{S}^{\lambda\mu\nu}} \end{aligned}$$

- We can use $H^{\lambda\mu\nu} = \mathcal{S}^{\lambda\mu\nu}$ to make the energy momentum tensor symmetric and guage independent

Issues

- How do we deal with the non-symmetric $\Theta^{\mu\nu}$ and the additional tensor $\mathcal{S}^{\lambda\mu\nu}$ *in hydrodynamics?*
  - Jinfeng Liao's talk: Thermodynamically consistent treatment of 1st order dissipative hydrodynamics (also Phys. Rev. C 103, 044906 (2021) by Shi, Gale and Jeon)
- How do we deal with the pseudo-gauge choice?
  - Matteo Buzzegoli's poster: The polarization pseudo-vector *does* depend on the pseudo-gauge
  - If energy density depends on PG, so does pressure $\Longrightarrow$ How does this change hydro evolution?
  - Can Einstein equation $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$ help?
- How do we deal with the initial state – hydro interface?
  - IP-Glasma: $T^{\mu\nu}_{\rm init}$ calculated using gauge-invariant symmetric energy-momentum tensor
  - Kinetic theory (related) models: $T^{\mu\nu} \sim \int_p f_{\rm init}(x,p) p^\mu p^\nu$
- How will the presence of $\mathcal{S}^{\lambda\mu\nu}$ manifest itself? $\Longrightarrow$ Polarization of spin 1/2 and spin

Λ polarization

[Palermo, Lomicky, Bravina, Buzzegoli]

- Hydrodynamics is carried out without separate $\mathcal{S}^{\lambda\mu\nu}$ evolution
- Spin polarization pseudo-vector for an on shell 4-momentum p

$$S^\mu(p) = S^\mu_\varpi(p) + S^\mu_\xi(p)$$

depends on T and u^μ. In particular, on the thermal vorticity (with $\beta^\mu = \beta u^\mu$)

$$\varpi_{\mu\nu} = -(1/2)(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)$$

and the thermal shear

$$\xi_{\mu\nu} = (1/2)(\partial_\mu\beta_\nu + \partial_\nu\beta_\mu)$$

- Presence of the thermal shear term may solve the Λ polarization puzzle
  - Caveat 1: Thermal shear can be eliminated by a suitable pseudo-gauge transformation
  - Caveat 2: Spin d.o.f. has not been considered in hydrodynamics so far

Nonlinear causality/stability conditions

The usual hydrodynamic decomposition of $T^{\mu\nu}$

- Symmetric energy-momentum tensor

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + \Delta^{\mu\nu}(p(\epsilon) + \Pi) + \pi^{\mu\nu}$$

- Local energy density: ϵ
- Flow vector: $T^{\mu\nu} u_\nu = -\epsilon u^\mu$
- Local 3-metric: $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ with $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$
- Equation of state: $p = p(\epsilon)$
- Bulk pressure: Π
- Shear tensor: $\pi^{\mu\nu} = \pi^{\nu\mu}, u_\mu \pi^{\mu\nu} = 0 = \pi^\mu_\mu$
- A symmetric rank-2 tensor has 10 d.o.f. $\Longrightarrow \epsilon, \Pi, u^i, \pi^{ij}$

In this context, local means co-moving frame where $u^\mu = (1, 0, 0, 0)$

2nd order Dissipative Hydrodynamics

- Energy-momentum conservation laws

$$0 = D\epsilon + (\epsilon + p + \Pi)\nabla_\alpha u^\alpha + \pi_\mu^\alpha \sigma_\alpha^\mu$$
$$0 = (\epsilon + p + \Pi) D u_\alpha + c_s^2 \nabla_\alpha \epsilon + \nabla_\alpha \Pi + \Delta_\alpha^\beta \nabla_\mu \pi_\beta^\mu + \pi_\alpha^\mu D u_\mu$$

- Relaxation equations

$$\tau_\Pi D\Pi + \Pi = -\zeta \nabla_\alpha u^\alpha - \delta_{\Pi\Pi} \Pi \nabla_\alpha u^\alpha - \lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu}$$
$$\tau_\pi \Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} + \pi^{\mu\nu} = -\eta \sigma^{\mu\nu} - \delta_{\pi\pi} \pi^{\mu\nu} \nabla_\alpha u^\alpha - \tau_{\pi\pi} \pi_\alpha^{\langle\mu} \sigma^{\nu\rangle\alpha} - \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu}$$

$D = u^\mu \partial_\mu$, $\nabla_\alpha = \Delta_\alpha^\mu \partial_\mu$ are the local time and the space derivatives in the co-moving frame.

Navier-Stokes tensor: $\sigma^{\mu\nu} = 2\Delta^{\mu\nu}_{\alpha\beta} \partial^\alpha u^\beta$ with $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\alpha^\nu \Delta_\beta^\mu - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right)$

Linear Causality Conditions

- Linearize around $\epsilon_0, p_0, \mathbf{u}_0 = 0, \pi^{ij} = 0, \Pi = 0$ to get

$$\begin{aligned} 0 &= \partial_t \delta\epsilon + (\epsilon_0 + p_0)\partial_i \delta u^i \\ 0 &= (\epsilon_0 + p_0)\partial_t \delta u^i + c_s^2 \partial^i \delta\epsilon + \partial^i \delta\Pi + \partial_j \delta\pi^{ji} \\ 0 &= \tau_\Pi \partial_t \delta\Pi + \zeta \partial_i \delta u^i + \delta\Pi \\ 0 &= \tau_\pi \partial_t \delta\pi^{ij} + \eta\sigma^{ij} + \delta\pi^{ij} \end{aligned}$$

This can be re-arranged as

$$E^0 \partial_t \Psi + E^j \partial_j \Psi = F(\Psi) \text{ where } \Psi = (\delta\epsilon, \delta\mathbf{u}, \delta\pi^{ij}, \delta\Pi)$$

- Linear causality analysis: Let $\Psi = e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}\Psi_0$.
- The homogeneous part: $\left(-E^0\omega + E^j k_j\right)\Psi_0 = 0$
- Dispersion relationships: $\mathrm{Det}(-E^0\omega + E^j k_j) = 0$
- Causal if $\lim_{|\mathbf{k}|\to\infty} \left|\frac{\partial\omega}{\partial\mathbf{k}}\right| < 1$

Is this the most general analysis?

- The expansion point $(\epsilon_0, p_0, \mathbf{u}_0 = 0, \Pi = 0, \pi^{ij} = 0)$ implies that the expansion is around the local equilibrium and the fluid cell rest frame.
- Do we *need to do* an expansion?
  - The second order hydrodynamic equations are a set of *quasi-linear* PDE $\Longrightarrow$ Local stability conditions can be established *without* assuming small deviations

Non-linear Causality Conditions

[Christopher Plumberg's talk]

- Re-organize the full non-linear energy-momentum conservation laws and relaxation equations as

$$A^{\alpha}(\Psi)\nabla_{\alpha}\Psi = F(\Psi)$$

  where $\Psi = (\epsilon, u^{\mu}, \Pi, \pi^{0\mu}, \pi^{1\mu}, \pi^{2\mu}, \pi^{3\mu})$ and $F(\Psi)$ does not involve any derivatives.

- Seek a solution of the form $\Psi = \sin(\phi(t, x) + \phi_0)\Psi_0$ for the homogeneous part. Equivalently,

$$A^{\alpha}(\Psi)\partial_{\alpha}\phi = 0$$

- Highest order dispersion relationships come from the homogeneous part: $\mathrm{Det}(A^{\alpha}\partial_{\alpha}\phi) = 0$

Non-linear Causality Conditions

Solve $\mathrm{Det}(A^\alpha \partial_\alpha \phi) = 0$ *for* $\partial_t \phi = \Lambda(\partial_i \phi, \Psi)$

- If $\Lambda(\partial_i \phi, \Psi)$ is real for any $\partial_i \phi \implies$ In the vicinity of $\Psi(t, \mathbf{x})$, $\phi(t, \mathbf{x}) \approx \mathbf{k}\cdot\delta\mathbf{x} - \omega(\mathbf{k}, \Psi)\delta t$ for any $\mathbf{k}$ $(\approx \nabla\phi)$
  $\implies$ Propagating mode
- If $\Lambda(\partial_i \phi, \Psi)$ is complex, then the solution will either blow up or decay
  $\implies$ Non-propagating mode
- The vector $\xi^\mu = (\partial^t \phi, \partial^i \phi)$ plays the role of $k^\mu \implies$ Causality demands ξ^μ is space-like or light-like (e.g. $\omega = vk$ with $|v| < 1$).
- Linear causality conditions are generalized to include Π and $\pi^{\mu\nu}$ (or its non-zero eigenvalues)

Why is this interesting?

- Violation of Causality/Stability conditions observed in early stages of hydro evolutions
- Depends a lot on the initial conditions *and* the interface between the initial state evolution model (such as IP-Glasma) and the hydrodynamics phase
- Violations occur mainly because $\Pi, \pi^{\mu\nu}$ are too large
- Numerically: Can induce numerical instability $\Longrightarrow$ Well tamed in current codes
- Sensitivity studies should be carried out. Example: Full $T^{\mu\nu}$ vs $T^{\mu\nu}_{\text{ideal}}$ for the initial condition for hydro.
- Theoretically: Opens up an interesting new way to analyze causality and stability

Hydrodynamics near critical point

Hydro with an order parameter

[Maneesha Sushama Pradeep's talk]

- Near a critical point, we have an order parameter that scales with the correlation length. How do we deal with that in hydro?
- Energy-momentum tensor

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + (p_{(+)} + \Pi_{(+)})\Delta^{\mu\nu} + \pi^{\mu\nu}_{(+)}$$

where the $(+)$ quantities now depends on the order parameter Wigner function $\phi_Q(x)$. This is Hydro$+$ (Stephanov and Yin, Phys. Rev. D 98, 036006 (2018)).

[Noriyuki Sogabe's poster]

- Stochastic effective field theory treatment of the order parameter dynamics near the critical point

Why is this interesting?

- We have explored the small μ_B phase a lot
- Much to explore/study in the large μ_B phases
- Development of reliable EoS [Parotto's talk] and reliable hydrodynamics with multiple conserved currents [Wu's talk, Bumnedpan's poster, and Almaalol's poster] critical

New initial conditions

New developments in initial conditions

- Dynamic model of initial longitudinal energy-momentum distribution [Chun Shen's talk]
- Dynamic model of energy-momentum deposition by partons [Yuuka Kanabkubo's talk]

Why are these interesting?

- We need dynamic models for longitudinal structure of the initial condition (3D IP-Glasma is still in development) to confront the data
- Larger context of how to deal with out-of-equilibrium system in hydro

Summary and Outlook

Summary

Hydrodynamical description of Heavy Ion/QGP dynamics is “un-reasonably” effective.
There are, however, improvements to be made.

- Proper treatment of angular momentum d.o.f.
- Out-of-equilibrium beyond 2nd order Israel-Stewart-like hydrodynamics
- Matching between QFT d.o.f. and Hydro $T^{\mu\nu}$
- Matching between Hydro $T^{\mu\nu}$ and particle d.o.f.
- Extension to low to medium energy range: Hydro with multiple conserved currents plus corresponding EoS
- Critical point!
- And many more

Exciting conference program ahead!

Backup Slides

Causal First Order Hydrodynamics

The usual hydrodynamic decomposition of $T^{\mu\nu}$

- A more general form of the symmetric energy-momentum tensor

$$\Theta^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \Delta^{\mu\nu}\mathcal{P} + \underbrace{\mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \pi^{\mu\nu}}_{\text{symmetric and traceless}}$$

  where $\mathcal{Q}^\mu$ is a transverse vector ($\mathcal{Q}^\mu u_\mu = 0$) and $\pi^{\mu\nu}$ is a transverse tensor ($u_\mu \pi^{\mu\nu} = 0$)

- The "local energy density" is

$$\mathcal{E} = u^\mu u^\nu \Theta^{\mu\nu}$$

  but $\Theta^{\mu\nu} u_\nu = -\mathcal{E} u^\mu - \mathcal{Q}^\mu \neq -\mathcal{E} u^\mu$.

First order dissipative hydrodynamics is acausal

The usual argument (Landau frame, or $\mathcal{Q}^{\mu} = 0$)

- Get the energy density $\mathcal{E}$ from $T^{\mu\nu} u_{\nu} = -\mathcal{E} u^{\mu}$
- $\mathcal{E}$ is taken as the “local equilibrium” energy density $\Longrightarrow \mathcal{P} = p(\mathcal{E}) + \Pi$
- Energy-momentum tensor

$$T^{\mu\nu} = \mathcal{E} u^{\mu} u^{\nu} + \Delta^{\mu\nu}(p(\mathcal{E}) + \Pi) + \pi^{\mu\nu}$$

  - Π and $\pi^{\mu\nu}$ depends on first order derivatives of $\mathcal{E}, p, u^{\mu}$
  - Namely,

$$\Pi = -\zeta \nabla_{\mu} u^{\mu}, \quad \pi^{\mu\nu} = -\eta \sigma^{\mu\nu}$$

- Gives the usual Navier-Stokes equations. Known to lead to acausal propagation.

Linear analysis of the Navier Stokes equation

- Sound wave with attenuation

$$0 = (\epsilon_0 + p_0)\partial_t^2 \delta\epsilon - c_s^2(\epsilon_0 + p_0)\nabla^2 \delta\epsilon - \left(\zeta + \frac{4\eta}{3}\right)\partial_t \nabla^2 \delta\epsilon$$

Causal and stable for small $|\mathbf{k}|$ if $c_s^2 < 1$. For large $|\mathbf{k}|$, one branch is stable while the other is not stable

- Diffusion

$$0 = (\epsilon_0 + p_0)\partial_t(\nabla \times \mathbf{u}) - \eta\nabla^2(\nabla \times \mathbf{u})$$

This part is acausal

- Technically due to: (The number of ∂_t) $\neq$ (The number of ∂_i)

First order dissipative hydrodynamics is acausal?

Bemfica, Disconzi, Noronha, Kovtun (BDNK): All quantities can get derivative corrections

Start with $T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \Delta^{\mu\nu}\mathcal{P} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \pi^{\mu\nu}$

Use

- Available first order derivatives
  - Scalar: $\partial_\mu u^\mu = \nabla_\mu u^\mu$, $D\epsilon$
  - Vector: Du^μ, $\partial_\mu \epsilon$
  - Tensor: $\partial^\mu u^\nu$

to get

- $\mathcal{E} = \epsilon + \varepsilon_1 D\epsilon + \varepsilon_2 \nabla_\lambda u^\lambda$
- $\mathcal{P} = p(\epsilon) + \pi_1 D\epsilon + \pi_2 \nabla_\lambda u^\lambda$
- $\mathcal{Q}^\mu = \theta_1 Du^\mu + \theta_2 \nabla^\mu \epsilon \propto (\epsilon + p)\delta u^\mu$
- $\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$

- Caveat: u^μ no longer has a clear interpretation

$D = u^\mu \partial_\mu$, $\nabla_\mu = \Delta^\nu_\mu \partial_\nu$, $\sigma^{\mu\nu} = \nabla^\mu u^\nu + \nabla_\nu u_\mu - \frac{2}{3}\Delta_{\mu\nu}\nabla_\alpha u^\alpha$

First order dissipative hydrodynamics is causal!

- Linearized equations of motion around $\epsilon_0, p_0 = p(\epsilon_0), \mathbf{u}_0 = 0$
  - Time component

$$0 = \left(\partial_t \epsilon + \varepsilon_1 \partial_t^2 \epsilon + \varepsilon_2 \partial_t \partial_k u^k\right) + (\epsilon_0 + p_0)\left(\partial_k u^k\right) + \left(\theta_1 \partial_k \partial_t u^k + \theta_2 \nabla^2 \epsilon\right)$$

  - Space component

$$\begin{aligned} 0 &= (\epsilon_0 + p_0)\,\partial_t u^i + \left(\partial^i p + \pi_1 \partial^i \partial_t \epsilon + \pi_2 \partial^i \partial_k u^k\right) + \left(\theta_1 \partial_t^2 u^i + \theta_2 \partial_t \partial^i \epsilon\right) \\ &\quad - \eta\left(\nabla^2 u^i + (1/3)\partial^i \partial_k u^k\right) \end{aligned}$$

- The number of spatial derivatives is the same as the number of time derivatives $\Longrightarrow$ Hyperbolic equations $\Longrightarrow$ Detailed analysis by finding all dispersion relationships
- Easiest one to work out

$$0 = (\epsilon_0 + p_0)\,\partial_t(\nabla \times \mathbf{u}) + \theta_1 \partial_t^2(\nabla \times \mathbf{u}) - \eta \nabla^2(\nabla \times \mathbf{u})$$

Causality condition: $\theta_1 > \eta$

Caveats

- Initial condition: How do you find $\epsilon, u^\mu, D\epsilon, Du^\mu$ from a given $T^{\mu\nu}_{\text{init}}$?
- Decomposition: How do you find $\mathcal{E}, \mathcal{P}, u^\mu, Q^\mu, \pi^{\mu\nu}$ from a given $T^{\mu\nu}$?
- More unknown coefficients: How does one calculate the extra coefficients $\varepsilon_1, \varepsilon_2, \pi_1, \pi_2, \theta_1, \theta_2$?
- What does this all mean? Note

$$\epsilon(x^\mu + \varepsilon_1 u^\mu) = \epsilon(x^\mu) + \varepsilon_1 D\epsilon(x^\mu)$$

Hence, at least some of the effects are coming from evaluating quantities not exactly at the same x^μ but a bit further away in the direction of the fluid.