Recent Developments in Hydrodynamics

Sangyong Jeon
McGill University
Montréal, Canada
Hydrodynamics theory talks and posters in SQM 2022
Spin Hydrodynamics & \( \Lambda \) polarization

- Andrea Palermo, Local equilibrium and Lambda polarization in high energy heavy ion collisions, Jun 15, 9:00 AM, GBR1
- Jinfeng Liao, Novel Effects of Rotational Polarization in Relativistic Nuclear Collisions, Jun 15, 12:10 PM, GBR1
- Matteo Buzzegoli, Polarization in heavy ion collisions: a theoretical review, Jun 16, 9:30 AM, Grand Ballroom
- Ondrej Lomicky, Global and local \( \Lambda \) polarization from 27 to 200 GeV from a 3D viscous hydrodynamic model, Jun 14, Poster Board: OTH-05, Other topics
- Larisa Bravina, Polarization of \( \Lambda \) and anti-\( \Lambda \) hyperons in heavy-ion collisions at intermediate energies in hydrodynamic and microscopic transport models, Jun 14, Poster, Light-flavor and Strangeness
- Matteo Buzzegoli, Pseudo-gauge dependence of spin polarization in heavy-ion collisions, Jun 14, Poster Board: OTH-01, Other topics
- Baochi Fu, Signatures of the spin Hall effect in hot and dense QCD matter Jun 14, Poster, Bulk matter phenomena
- Xiaowen Li, Time evolution of global polarization within an improved microscopic approach Jun 14, Poster, Light-flavor and Strangeness Speaker
Nonlinear Causality Conditions

Christopher Plumberg, Causality violations in realistic nuclear collision simulations, Jun 15, 11:30 AM, Sydney.

Hydro Initial States

Chun Shen, Collectivity and baryon junctions in ultra-peripheral heavy-ion collisions, Jun 14, 10:50 AM, GBR1.
Yuuka Kanakubo, Interplay between core and corona from small to large systems Jun 15, 10:50 AM, GBR3.

Fluctuations

Maneesha Sushama Pradeep, Freezing out critical fluctuations, Jun 14, 11:50 AM, GBR1
Dekrayat Almaalol, Baryon number, strangeness, and electric charge fluctuations in hydrodynamics at the LHC, Jun 14, Poster Board: BLK-26.

Critical point & BES

Xiang-Yu Wu, (3+1)-D viscous hydrodynamics CLVisc at finite net baryon density: identified particle spectra, anisotropic flows and flow fluctuations across BES energies, Jun 14, 9:40 AM, GBR1
Thiranat Bumnedpan, Modification of hadron multiplicity ratios at the chiral phase transition, Jun 14, Poster Board: BLK-20.
Noriyuki Sogabe, Exploring the criticality of QCD with effective field theory for fluctuating hydrodynamics, Jun 14, Poster Board: BLK-27, Bulk matter phenomena
Lattice EoS

Paolo Parotto, Resummed lattice QCD equation of state at finite baryon density: strangeness neutrality and beyond Jun 15, 9:20 AM, GBR1.

Longitudinal flow

Guang-You Qin, Asymmetric longitudinal flow decorrelations in proton-nucleus collisions, Jun 14, Poster, Bulk matter phenomena

Transport coefficients

Shubhalaxmi Rath, Charge and heat transport coefficients of a weakly magnetized hot and dense QCD medium, Jun 14, Poster Board: BLK-13, Bulk matter phenomena

Bayesian analysis

Dong Jo Kim, New constraints of QCD matter from improved Bayesian parameter estimation with the latest LHC data Jun 15, 11:50 AM, GBR3

Upgrades

Nikhil Hatwar, Effect of variation in relaxation time on elliptic flow in PbPb and AuAu collisions Jun 14, Poster Board: BLK-01, Bulk matter phenomena

Ankit Kumar Panda, Causal second order magnetohydrodynamics from kinetic theory using RTA approximation, Jun 14, Poster Board: OTH-04, Other topics

Iurii Karpenko, Multi-Fluid Hydrodynamics for RHIC BES/FAIR/NICA, Remade, Jun 14, Poster Board: BLK-15, Bulk matter phenomena
Can only cover selected topics (with apologies to those I won’t be able to cover)
What is hydrodynamics?

- Conservation laws + Constitutive relations + Thermodynamics
- Relevant conservation laws: Energy-momentum, charges (electric, baryon, etc), angular momentum
- Constitutive relations are needed because hydrodynamics is all about coarse-grained collective motions
- *New developments*: Is what we have been using the most general form of hydrodynamics?
Collective spin tensor
Meaning of $T^{\mu \nu}$

- Energy-momentum conservation laws

\[ \partial_\mu T^{\mu \nu} = 0 \]

- $T^{00}$: Energy density
- $T^{i0}$: The $i$-th component of the energy current
- $T^{0j}$: Momentum density
- $T^{ij}$: The $i$-th component of the momentum current

Questions we normally don’t ask:
- Should $T^{\mu \nu}$ be always symmetric?
- The total energy-momentum $P^\mu = \int d^3x T^{0\nu}$ is observable. But is $T^{\mu \nu}$ observable?
Is any current observable?

- Conservation law for a conserved quantity \( a \): \( \partial_\mu J^{\mu,a} = 0 \)
- One can always add

\[
J'^{\mu,a} = J^{\mu,a} + \partial_\lambda B^{\lambda\mu,a}
\]

provided that \( B^{\lambda\mu,a} = -B^{\mu\lambda,a} \).
This does not change the total conserved quantity

\[
Q^a = \int_V d^3 x \ J^{0,a} = \int_V d^3 x \ J'^{0,a}
\]

provide that \( B^{i0,a} \) vanishes at the boundary \( \partial V \)

- But it does change \( J^{\mu,a} \). Does this mean that conserved currents are not observable?
Is any current observable?

Is $J^{\mu,a}$ observable?

- Classical Microscopic Dynamics: Yes. Individual point particles carry well defined energy, momentum and other conserved charges.

$$J^{\mu,q}(t, x) = \sum_{s=\text{ptcls}} \frac{q_s}{p^\mu_s} \delta^{(3)}(x - x_s(t))$$

- Continuum Dynamics: Depends.

So where is the disjoint?

- Point-Particles vs Fields

Since the global charge is conserved, $\delta J^{0,q} = \nabla \cdot B^{0,q}$ can only shuffle the local charges around (with $B^{0,q} = \{ B^{i0,q} \}$).

$J^{\mu,a}$ in continuum dynamics is inherently ambiguous without further constraints.
Example: NR QM

Hamiltonian: $\hat{H} = -\frac{\nabla^2}{2m} + V(x)$

Two energy density definitions

$\epsilon_1 = \text{Re}(\psi^* \hat{H} \psi)$

$\epsilon_2 = \frac{1}{2m} \left( (\nabla \psi^*) \cdot (\nabla \psi) + \psi^* V \psi \right)$

$= \epsilon_1 - \frac{1}{2} \nabla \cdot \text{Im}(\psi^* \hat{v} \psi)$

Two energy current definitions

$Q_1 = \frac{1}{2} \text{Re} \left( (\hat{H} \psi)^* (\hat{v} \psi) + \psi^* \hat{v} \hat{H} \psi \right)$

$Q_2 = \text{Re} \left( (\hat{H} \psi)^* (\hat{v} \psi) \right)$

The probability density

$P_\psi = \psi^* \psi$

The probability current

$J = \frac{1}{2} \text{Re}(\psi^* \hat{v} \psi)$

The probability density is positive definite and measurable $\implies$ No freedom

Energy density may not be like that
Pseudo-gauge freedom

- Energy-momentum tensor is not unique

\[ T^\mu\nu = \Theta^\mu\nu + \partial_\lambda B^{\lambda\mu\nu} \text{ with } B^{\lambda\mu\nu} = -B^{\mu\lambda\nu} \]

- \( \partial_\mu \Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} = 0 \) since \( B^{\mu\lambda\nu} = -B^{\lambda\mu\nu} \)

- \( P^\nu = \int_V d^3x \Theta^{0\nu} = \int_V d^3x T^{0\nu} \) provided that all quantities vanish at the boundary \( \partial V \)

- One can define a symmetric energy-momentum tensor

\[ T_{\text{sym}}^{\mu\nu} = \frac{1}{2} (\Theta^{\mu\nu} + \Theta^{\nu\mu}) = \Theta^{\mu\nu} + \partial_\lambda B^{\lambda\mu\nu} \]

*If* the anti-symmetric part is a divergence

\[ \Theta^{\mu\nu} - \Theta^{\nu\mu} = -\partial_\lambda H^{\lambda\mu\nu} \]

\( B^{\lambda\mu\nu} \) is given by \( B^{\lambda\mu\nu} = \frac{1}{2} \left( H^{\lambda\mu\nu} + H^{\mu\nu\lambda} - H^{\nu\lambda\mu} \right) \). \textit{But this is a choice.}
Energy-momentum tensor

Q: When is $\Theta^{\mu\nu}$ not symmetric?
   
   Field-theoretical answer ($g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ for this part)
   
   Let $L(\varphi^A, \partial_\mu \varphi^A)$ be the Lagrangian for the fields $\varphi^A$.
   
   Conserved current for the translation symmetry:
   
   $$\Theta^{\mu\nu} = \Pi^\mu_A \partial_\nu \varphi^A - g^{\mu\nu} L(\varphi^A, \partial_\mu \varphi^A)$$ with $\Pi^\mu_A = \frac{\partial L}{\partial (\partial_\mu \varphi^A)}$
   
   Only for scalar ($S = 0$) fields, $\Theta^{\mu\nu} = \Theta^{\nu\mu}$. For $S \neq 0$, $\Theta^{\mu\nu} \neq \Theta^{\nu\mu}$.

A: When $S \neq 0$.

Total angular momentum tensor

$$\mathcal{L}^{\lambda\mu\nu} = \Theta^{\lambda\nu} x^\mu - \Theta^{\lambda\mu} x^\nu + S^{\lambda\mu\nu}$$

Total angular momentum conservation

$$0 = \partial_\lambda S^{\lambda\mu\nu} + \Theta^{\mu\nu} - \Theta^{\nu\mu}$$
Example: Spin 1

- Lagrangian: $\mathcal{L} = -\frac{1}{4} F_{\alpha \beta} F^{\alpha \beta}$ with $\partial_\alpha A^\alpha = 0$ and the EoM $\partial_\mu F^{\mu \nu} = 0$

- Energy momentum tensor:

$$\Theta^{\mu \nu} = - F_\lambda^{\mu} \partial^\nu A^\lambda - g^{\mu \nu} \mathcal{L}$$

- Anti-symmetric part:

$$\Theta^{\mu \nu} - \Theta^{\nu \mu} = - F_\lambda^{\mu} \partial^\nu A^\lambda + F_\lambda^{\nu} \partial^\mu A^\lambda$$

$$= - F_{\mu \lambda} \partial_\nu A^\lambda + F_{\nu \lambda} \partial_\mu A^\lambda$$

$$= - \partial_\lambda \left( - F^{\lambda \mu} A^\nu + F^{\lambda \nu} A^\mu \right)$$

$$= S^{\lambda \mu \nu}$$

- We can use $H^{\lambda \mu \nu} = S^{\lambda \mu \nu}$ to make the energy momentum tensor symmetric and gauge independent
How do we deal with the non-symmetric $\Theta^{\mu\nu}$ and the additional tensor $S^{\lambda\mu\nu}$ in hydrodynamics?

- Jinfeng Liao’s talk: Thermodynamically consistent treatment of 1st order dissipative hydrodynamics (also Phys. Rev. C 103, 044906 (2021) by Shi, Gale and Jeon)

How do we deal with the pseudo-gauge choice?

- Matteo Buzzegoli’s poster: The polarization pseudo-vector does depend on the pseudo-gauge
- If energy density depends on PG, so does pressure $\Rightarrow$ How does this change hydro evolution?
- Can Einstein equation $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$ help?

How do we deal with the initial state – hydro interface?

- IP-Glasma: $T_{\text{init}}^{\mu\nu}$ calculated using gauge-invariant symmetric energy-momentum tensor
- Kinetic theory (related) models: $T^{\mu\nu} \sim \int p \ f_{\text{init}}(x,p) p^\mu p^\nu$

How will the presence of $S^{\lambda\mu\nu}$ manifest itself? $\Rightarrow$ Polarization of spin 1/2 and spin 1 particles
Lambda polarization

[Palermo, Lomicky, Bravina, Buzzegoli]

- Hydrodynamics is carried out without separate $S_{\lambda \mu \nu}$ evolution

- Spin polarization pseudo-vector for an on shell 4-momentum $p$

$$S^\mu(p) = S^\mu_\omega(p) + S^\mu_\xi(p)$$

depends on $T$ and $u^\mu$. In particular, on the thermal vorticity (with $\beta^\mu = \beta u^\mu$)

$$\omega_{\mu \nu} = -(1/2)(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

and the thermal shear

$$\xi_{\mu \nu} = (1/2)(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu)$$

- Presence of the thermal shear term may solve the Lambda polarization puzzle
  - Caveat 1: Thermal shear can be eliminated by a suitable pseudo-gauge transformation
  - Caveat 2: Spin d.o.f. has not been considered in hydrodynamics so far
Nonlinear causality/stability conditions
The usual hydrodynamic decomposition of $T^{\mu\nu}$

- Symmetric energy-momentum tensor

$$T^{\mu\nu} = \epsilon u^{\mu} u^{\nu} + \Delta^{\mu\nu} (p(\epsilon) + \Pi) + \pi^{\mu\nu}$$

- Local energy density: $\epsilon$
- Flow vector: $T^{\mu\nu} u_{\nu} = -\epsilon u^{\mu}$
- Local 3-metric: $\Delta^{\mu\nu} = g^{\mu\nu} + u^{\mu} u^{\nu}$ with $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$
- Equation of state: $p = p(\epsilon)$
- Bulk pressure: $\Pi$
- Shear tensor: $\pi^{\mu\nu} = \pi^{\nu\mu}$, $u_{\mu} \pi^{\mu\nu} = 0 = \pi^{\mu}_{\mu}$
- A symmetric rank-2 tensor has 10 d.o.f. $\implies \epsilon, \Pi, u^{i}, \pi^{ij}$

In this context, local means co-moving frame where $u^{\mu} = (1, 0, 0, 0)$
Energy-momentum conservation laws

\[ 0 = D\varepsilon + (\varepsilon + p + \Pi)\nabla_\alpha u^\alpha + \pi_\mu^\alpha \sigma_\mu^\alpha \]

\[ 0 = (\varepsilon + p + \Pi)Du_\alpha + c_s^2 \nabla_\alpha \varepsilon + \nabla_\alpha \Pi + \Delta_\alpha^\beta \nabla_\mu \pi_\beta^\mu + \pi_\alpha^\mu Du_\mu \]

Relaxation equations

\[ \tau_\Pi D\Pi + \Pi = -\zeta \nabla_\alpha u^\alpha - \delta_\Pi \Pi \nabla_\alpha u^\alpha - \lambda_\Pi \pi^\mu \pi^\sigma \sigma_{\mu \sigma} \]

\[ \tau_\Pi \Delta_\alpha^\beta \Delta_\mu^\nu D\pi_\alpha^\beta + \pi^\mu \pi^\nu = -\eta \sigma^\mu \sigma^\nu - \delta_\pi \pi \pi^\mu \pi^\nu \nabla_\alpha u^\alpha - \tau_\pi \pi \pi^\mu \pi^\nu \sigma_\alpha^\beta \sigma_\mu^\nu \]

\[ D = u^\mu \partial_\mu, \quad \nabla_\alpha = \Delta_\alpha^\mu \partial_\mu \] are the local time and the space derivatives in the co-moving frame.

Navier-Stokes tensor: \( \sigma^\mu \sigma^\nu = 2\Delta_\alpha^\beta \partial_\alpha u^\beta \) with \( \Delta_\alpha^\beta = \frac{1}{2} \left( \Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\alpha^\nu \Delta_\beta^\mu - \frac{2}{3} \Delta^\mu \Delta_\alpha^\beta \right) \)
Linear Causality Conditions

- Linearize around $\epsilon_0, p_0, u_0 = 0, \pi^{ij} = 0, \Pi = 0$ to get
  \[
  0 = \partial_t \delta \epsilon + (\epsilon_0 + p_0) \partial_i \delta u^i \\
  0 = (\epsilon_0 + p_0) \partial_t \delta u^i + c_s^2 \partial^i \delta \epsilon + \partial_i \delta \Pi + \partial_j \delta \pi^{ji} \\
  0 = \tau \partial_t \delta \Pi + \zeta \partial_i \delta u^i + \delta \Pi \\
  0 = \tau \partial_t \delta \pi^{ij} + \eta \sigma^{ij} + \delta \pi^{ij}
  \]

  This can be re-arranged as
  \[
  E^0 \partial_t \Psi + E^j \partial_j \Psi = F(\Psi) \text{ where } \Psi = (\delta \epsilon, \delta u, \delta \pi^{ij}, \delta \Pi)
  \]

- Linear causality analysis: Let
  \[
  \Psi = e^{i k \cdot x - i \omega t} \Psi_0.
  \]

- The homogeneous part:
  \[
  \left(-E^0 \omega + E^j k_j\right) \Psi_0 = 0
  \]

- Dispersion relationships:
  \[
  \text{Det}(-E^0 \omega + E^j k_j) = 0
  \]

- Causal if
  \[
  \lim_{|k| \to \infty} \left| \frac{\partial \omega}{\partial k} \right| < 1
  \]
The expansion point \((\epsilon_0, p_0, u_0 = 0, \Pi = 0, \pi^{ij} = 0)\) implies that the expansion is around the local equilibrium and the fluid cell rest frame.

Do we \textit{need to do} an expansion?

- The second order hydrodynamic equations are a set of \textit{quasi-linear} PDEs. \(\implies\) Local stability conditions can be established \textit{without} assuming small deviations.
Re-organize the full non-linear energy-momentum conservation laws and relaxation equations as

\[ A^\alpha(\Psi) \nabla_\alpha \Psi = F(\Psi) \]

where \( \Psi = (\epsilon, u^\mu, \Pi, \pi^0, \pi^1, \pi^2, \pi^3) \) and \( F(\Psi) \) does not involve any derivatives.

Seek a solution of the form \( \Psi = \sin(\phi(t,x) + \phi_0)\Psi_0 \) for the homogeneous part. Equivalently,

\[ A^\alpha(\Psi) \partial_\alpha \phi = 0 \]

Highest order dispersion relationships come from the homogeneous part:
\[ \text{Det}(A^\alpha \partial_\alpha \phi) = 0 \]
Non-linear Causality Conditions

Solve \( \text{Det}(A^\alpha \partial_\alpha \phi) = 0 \) for \( \partial_t \phi = \Lambda(\partial_i \phi, \Psi) \)

- If \( \Lambda(\partial_i \phi, \Psi) \) is real for any \( \partial_i \phi \) \( \Rightarrow \) In the vicinity of \( \psi(t, x), \phi(t, x) \approx k \cdot \delta x - \omega(k, \Psi) \delta t \) for any \( k (\approx \nabla \phi) \)
  \( \Rightarrow \) Propagating mode

- If \( \Lambda(\partial_i \phi, \Psi) \) is complex, then the solution will either blow up or decay
  \( \Rightarrow \) Non-propagating mode

- The vector \( \xi^\mu = (\partial^t \phi, \partial^i \phi) \) plays the role of \( k^\mu \) \( \Rightarrow \) Causality demands \( \xi^\mu \) is space-like or light-like (e.g. \( \omega = v k \) with \( |v| < 1 \)).

- Linear causality conditions are generalized to include \( \Pi \) and \( \pi^{\mu\nu} \) (or its non-zero eigenvalues).
Why is this interesting?

- Violation of Causality/Stability conditions observed in early stages of hydro evolutions.
- Depends a lot on the initial conditions and the interface between the initial state evolution model (such as IP-Glasma) and the hydrodynamics phase.
- Violations occur mainly because $\Pi, \pi^{\mu\nu}$ are too large.
- Numerically: Can induce numerical instability $\Rightarrow$ Well tamed in current codes.
- Sensitivity studies should be carried out. Example: Full $T^{\mu\nu}$ vs $T^{\mu\nu}_{\text{ideal}}$ for the initial condition for hydro.
- Theoretically: Opens up an interesting new way to analyze causality and stability.
Hydrodynamics near critical point
Hydro with an order parameter

[Maneesha Sushama Pradeep’s talk]

- Near a critical point, we have an order parameter that scales with the correlation length. How do we deal with that in hydro?

- Energy-momentum tensor

\[ T^{\mu \nu} = \epsilon u^\mu u^\nu + (p^{(+) \!} + \Pi^{(+) \!}) \Delta^{\mu \nu} + \pi^{\mu \nu}_{(+)} \]

where the \((+)^{\!}\) quantities now depends on the order parameter Wigner function \(\phi_Q(x)\). This is Hydro+ (Stephanov and Yin, Phys. Rev. D 98, 036006 (2018)).

[Noriyuki Sogabe’s poster]

- Stochastic effective field theory treatment of the order parameter dynamics near the critical point
Why is this interesting?

- We have explored the small $\mu_B$ phase a lot
- Much to explore/study in the large $\mu_B$ phases
- Development of reliable EoS [Parotto's talk] and reliable hydrodynamics with multiple conserved currents [Wu's talk, Bumnedpan's poster, and Almaalol's poster] critical
New initial conditions
New developments in initial conditions

- Dynamic model of initial longitudinal energy-momentum distribution [Chun Shen’s talk]
- Dynamic model of energy-momentum deposition by partons [Yuuka Kanabkubo’s talk]

Why are these interesting?

- We need dynamic models for longitudinal structure of the initial condition (3D IP-Glasma is still in development) to confront the data
- Larger context of how to deal with out-of-equilibrium system in hydro
Summary and Outlook
Hydrodynamical description of Heavy Ion/QGP dynamics is “un-reasonably” effective. There are, however, improvements to be made.

- Proper treatment of angular momentum d.o.f.
- Out-of-equilibrium beyond 2nd order Israel-Stewart-like hydrodynamics
- Matching between QFT d.o.f. and Hydro $T_{\mu\nu}$
- Matching between Hydro $T_{\mu\nu}$ and particle d.o.f.
- Extension to low to medium energy range: Hydro with multiple conserved currents plus corresponding EoS
- Critical point!
- And many more
Exciting conference program ahead!
Backup Slides
Causal First Order Hydrodynamics
The usual hydrodynamic decomposition of \( T^{\mu \nu} \)

- A more general form of the symmetric energy-momentum tensor

\[
\Theta^{\mu \nu} = \mathcal{E} u^\mu u^\nu + \Delta^{\mu \nu} \mathcal{P} + \underbrace{Q^\mu u^\nu + Q^\nu u^\mu + \pi^{\mu \nu}}_{\text{symmetric and traceless}}
\]

where \( Q^\mu \) is a transverse vector \( (Q^\mu u_\mu = 0) \) and \( \pi^{\mu \nu} \) is a transverse tensor \( (u_\mu \pi^{\mu \nu} = 0) \)

- The “local energy density” is

\[
\mathcal{E} = u^\mu u^\nu \Theta^{\mu \nu}
\]

but \( \Theta^{\mu \nu} u_\nu = -\mathcal{E} u^\mu - Q^\mu \neq -\mathcal{E} u^\mu \).
First order dissipative hydrodynamics is acausal

The usual argument (Landau frame, or $Q^\mu = 0$)

- Get the energy density $\mathcal{E}$ from $T^{\mu\nu} u_\nu = -\mathcal{E} u^\mu$
- $\mathcal{E}$ is taken as the “local equilibrium” energy density $\implies \mathcal{P} = p(\mathcal{E}) + \Pi$
- Energy-momentum tensor

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \Delta^{\mu\nu}(p(\mathcal{E}) + \Pi) + \pi^{\mu\nu}$$

- $\Pi$ and $\pi^{\mu\nu}$ depends on first order derivatives of $\mathcal{E}, p, u^\mu$
- Namely,

$$\Pi = -\zeta \nabla_\mu u^\mu, \quad \pi^{\mu\nu} = -\eta \sigma^{\mu\nu}$$

- Gives the usual Navier-Stokes equations. Known to lead to acausal propagation.
Linear analysis of the Navier Stokes equation

- **Sound wave with attenuation**

\[ 0 = (\epsilon_0 + p_0) \partial_t^2 \delta \epsilon - c_s^2 (\epsilon_0 + p_0) \nabla^2 \delta \epsilon - \left( \zeta + \frac{4 \eta}{3} \right) \partial_t \nabla^2 \delta \epsilon \]

Causal and stable for small \(|k|\) if \(c_s^2 < 1\). For large \(|k|\), one branch is stable while the other is not stable.

- **Diffusion**

\[ 0 = (\epsilon_0 + p_0) \partial_t (\nabla \times u) - \eta \nabla^2 (\nabla \times u) \]

This part is acausal.

- **Technically due to:** (The number of \(\partial_t\) \neq (The number of \(\partial_i\))
First order dissipative hydrodynamics is acausal?

*Bemfica, Disconzi, Noronha, Kovtun (BDNK):* All quantities can get derivative corrections

Start with \( T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \Delta^{\mu\nu} \mathcal{P} + Q^\mu u^\nu + Q^\nu u^\mu + \pi^{\mu\nu} \)

Use

- **Available first order derivatives**
  - Scalar: \( \partial_\mu u^\mu = \nabla_\mu u^\mu, D\epsilon \)
  - Vector: \( Du^\mu, \partial_\mu \epsilon \)
  - Tensor: \( \partial^\mu u^\nu \)

- **Caveat:** \( u^\mu \) no longer has a clear interpretation

\[ D = u^\mu \partial_\mu, \; \nabla_\mu = \Delta^\nu_\mu \partial_\nu, \; \sigma^{\mu\nu} = \nabla^\mu u^\nu + \nabla_\nu u_\mu - \frac{2}{3} \Delta_{\mu\nu} \nabla_\alpha u^\alpha \]

- \( \mathcal{E} = \epsilon + \epsilon_1 D\epsilon + \epsilon_2 \nabla_\lambda u^\lambda \)
- \( \mathcal{P} = p(\epsilon) + \pi_1 D\epsilon + \pi_2 \nabla_\lambda u^\lambda \)
- \( Q^\mu = \theta_1 Du^\mu + \theta_2 \nabla^\mu \epsilon \propto (\epsilon + p) \delta u^\mu \)
- \( \pi^{\mu\nu} = -2\eta \sigma^{\mu\nu} \)
First order dissipative hydrodynamics is causal!

- Linearized equations of motion around $\epsilon_0, p_0 = p(\epsilon_0), u_0 = 0$
  - Time component
    
    \[ 0 = (\partial_t \epsilon + \epsilon_1 \partial_t^2 \epsilon + \epsilon_2 \partial_t \partial_k u^k) + (\epsilon_0 + p_0)(\partial_k u^k) + (\theta_1 \partial_k \partial_t u^k + \theta_2 \nabla^2 \epsilon) \]
  
  - Space component
    
    \[ 0 = (\epsilon_0 + p_0) \partial_t u^i + (\partial^i p + \pi_1 \partial_t \epsilon + \pi_2 \partial_i \partial_k u^k) + (\theta_1 \partial_t^2 u^i + \theta_2 \partial_t \partial_i \epsilon) \]
    \[ - \eta \left( \nabla^2 u^i + \frac{1}{3} \partial^i \partial_k u^k \right) \]

- The number of spatial derivatives is the same as the number of time derivatives
  \[ \implies \text{Hyperbolic equations} \implies \text{Detailed analysis by finding all dispersion relationships} \]

- Easiest one to work out
  
  \[ 0 = (\epsilon_0 + p_0) \partial_t(\nabla \times u) + \theta_1 \partial_t^2(\nabla \times u) - \eta \nabla^2(\nabla \times u) \]

Causality condition: $\theta_1 > \eta$
Caveats

- Initial condition: How do you find $\epsilon, u^\mu, D\epsilon, Du^\mu$ from a given $T_{\text{init}}^{\mu\nu}$?
- Decomposition: How do you find $\varepsilon, \mathcal{P}, u^\mu, Q^\mu, \pi^{\mu\nu}$ from a given $T^{\mu\nu}$?
- More unknown coefficients: How does one calculate the extra coefficients $\varepsilon_1, \varepsilon_2, \pi_1, \pi_2, \theta_1, \theta_2$?
- What does this all mean? Note

$$\epsilon(x^\mu + \varepsilon_1 u^\mu) = \epsilon(x^\mu) + \varepsilon_1 D\epsilon(x^\mu)$$

Hence, at least some of the effects are coming from evaluating quantities not exactly at the same $x^\mu$ but a bit further away in the direction of the fluid.