Second Order Kalman filtering

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Ref: https://www.diva-portal.org/smash/get/diva2:511843/FULLTEXT01.pdf

Kalman filtering in second order

Update of the state vector:

$$\boldsymbol{x}_k = \boldsymbol{x}_k^{k-1} + \mathbf{K}_k\left(\boldsymbol{m}_k - \mathbf{H}_k \boldsymbol{x}_k^{k-1}\right).$$

Kalman gain matrix:

$$\mathbf{K}_k = \mathbf{C}_k^{k-1}\mathbf{H}_k^{\mathrm{T}}\left(\mathbf{V}_k + \mathbf{H}_k \mathbf{C}_k^{k-1}\mathbf{H}_k^{\mathrm{T}}\right)^{-1}$$

Update of the covariance matrix:

$$\mathbf{C}_k = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{C}_k^{k-1}.$$

The updated measurement covariance S element

$$\begin{aligned} S_k^{lm} = & (\nabla_x h_l(\hat{\delta}_{k|k-1}))^T P_{k|k-1} \nabla_x h_m(\hat{\delta}_{k|k-1}) \\ & + (\nabla_e h_l(\hat{\delta}_{k|k-1}))^T R_k \nabla_e h_m(\hat{\delta}_{k|k-1}) \\ & + \frac{1}{2}\operatorname{tr}\left(\nabla_x^2 h_l(\hat{\delta}_{k|k-1}) P_{k|k-1} \nabla_x^2 h_m(\hat{\delta}_{k|k-1}) P_{k|k-1}\right) \\ & + \frac{1}{2}\operatorname{tr}\left(\nabla_e^2 h_l(\hat{\delta}_{k|k-1}) R_k \nabla_e^2 h_m(\hat{\delta}_{k|k-1}) R_k\right). \qquad (6) \end{aligned}$$

The first order

The second order

Hessian matrix in approximation ?

$$
\begin{aligned}
&\frac{1}{2}\,\mathrm{tr}\Big(\nabla_x^2 h_l(\hat{\delta}_{k|k-1})P_{k|k-1}\nabla_x^2 h_m(\hat{\delta}_{k|k-1})P_{k|k-1}\Big)\\
&\frac{1}{2}\,\mathrm{tr}\Big(\nabla_e^2 h_l(\hat{\delta}_{k|k-1})R_k\nabla_e^2 h_m(\hat{\delta}_{k|k-1})R_k\Big). \qquad (6)
\end{aligned}
$$

- Suppose P is the covariance of x, H is the Hessian matrix, then:

$$
\begin{aligned}
\mathrm{tr}(\mathcal{H}_l\mathcal{P}\mathcal{H}_m\mathcal{P}) = \lim_{\alpha\to 0}\frac{1}{\alpha^4 n_x^2}\Bigg(&\sum_{i=1}^{n_x}\Big(z_l^{(i)}+z_l^{(-i)}-2z_l^{(0)}\Big)\\
&\Big(z_m^{(i)}+z_m^{(-i)}-2z_m^{(0)}\Big)+\frac{1}{4}\sum_{i=1}^{n_x}\sum_{j=1,j\neq i}^{n_x}\Big(z_l^{(ij)}+z_l^{(-ij)}\\
&-z_l^{(i)}-z_l^{(-i)}-z_l^{(j)}-z_l^{(-j)}+2z_l^{(0)}\Big)\Big(z_m^{(ij)}+z_m^{(-ij)}\\
&-z_m^{(i)}-z_m^{(-i)}-z_m^{(j)}-z_m^{(-j)}+2z_m^{(0)}\Big)\Bigg). \qquad (12d)
\end{aligned}
$$

Z^0, Z^i, Z^{ij} ... are the values of h(x) evaluated at x^0, x^i, x^{ij} … (central and sigma points of x)

How to choose sigma points?

One approach: Singular value decomposition

Let $\hat{x} = \mathrm{E}(x)$ and $\mathcal{P} = \mathrm{cov}(x)$. Using the singular value decomposition, the symmetric $n_x \times n_x$ covariance matrix can be written as

$$\mathcal{P} = \mathcal{U}\mathcal{S}\mathcal{U}^T = \sum_{i=1}^{n_x} s_i u_i u_i^T. \quad (10)$$

Here, s_i is the i-th singular value of $\mathcal{P}$ and u_i the i-th column of the unitary matrix $\mathcal{U}$.

Next, a set of sigma points around $\hat{x}$, as used in the unscented transformation [3], can be chosen systematically. With hindsight that the conventional set comprising $2n_x+1$ vectors in x will not suffice for our purpose, the set is extended by adding $n_x^2 - n_x$ distinct sigma points. The set members are constructed according to:

$$x^{(0)} = \hat{x}, \quad (11a)$$

$$x^{(\pm i)} = \hat{x} \pm \alpha\sqrt{n_x s_i} u_i, \quad i = 1, \ldots, n_x, \quad (11b)$$

$$x^{(\pm ij)} = \hat{x} \pm \alpha\sqrt{n_x}(\sqrt{s_i} u_i + \sqrt{s_j} u_j), \quad i, j = 1, \ldots, n_x, \quad i \neq j. \quad (11c)$$

$2n_x^2$ sigma points. For free track parameter, 2*8*8 = 128 points?
Or 2*8 = 16 using just Eq. (11b) might be enough?