Two-loop master integrals for quarkonium physics at NNLO

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Introduction

Part I: Quarkonium phenomenology at NLO

Part II: Two-loop master integrals and form-factors

Part III: Decay of pseudo-scalar to di-photon at NNLO accuracy
Introduction: What is a Quarkonium?

- bound state of heavy quark and its anti-quark, e.g. Charmonium (charm quark) and Bottomonium (bottom quark)
- similar to positronium bound state $e^+e^-$
- potential energy
  - $V_{\text{QED}} = -\alpha_{\text{em}}/r$
  - $V_{\text{QCD}} = -\frac{4}{3}\alpha_s/r + kr$
- Toponium bound state is unstable due to high mass of top quark, decays via weak interaction
- for light quarks there is mixing between $(u,d,s)$ quarks due to low mass difference resulting into mesons such as $\pi$-meson, the $\rho$-meson and the $\eta$-meson
- first application of asymptotic freedom

Motivation: Why study Quarkonia?

- charmonium production allows us to probe QCD at its interplay between the perturbative and non-perturbative regimes
- deeper understanding of confinement (production mechanism)
- access to spin/momentum distribution of gluons in protons → use quarkonia to constrain the gluon PDFs in the proton (see Part I)
- it is interesting to assess the convergence of perturbative expansion in $\alpha_s$ where $\alpha_s(m_c) \sim 0.34$ and $\alpha_s(m_b) \sim 0.22$ (see Part III)
Part I

Quarkonium phenomenology at NLO

Quarkonia - three different models

- Colour-Evaporation Model
  - quark and anti-quark colours are summed up at amplitude squared level (evaporation)
  - no spin-projection

- Colour-Singlet Model
  - quark and anti-quark pair are in color-singlet state
  - heavy quark spins projected on final bound state
  - leading Fock state in NRQCD

- Colour-Octet Mechanism (NRQCD)
  - quark and anti-quark pair are in color-octet state
  - heavy quark spins projected on final bound state
  - higher Fock states in NRQCD, higher \( v \)-order
the $\eta_c$ - a good gluon probe

- $\eta_c$ is a gluon probe at low scales at $M_{\eta_c} = 3$ GeV
- is a pseudo-scalar particle and simplest of all quarkonia as far as computation of hadro-production

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- $\eta_c$ cross section computation known
  - at NLO since 1992 in collinear factorisation
    
  - at LO since 2012 and at NLO since 2013 in TMD factorisation
    
problem of negative cross-sections - $\eta_c$ and $J/\psi$ at NLO

comparison of $\eta_c$ (left) and $J/\psi$ (right) differential cross-sections at NLO with different scale choices of $\mu_R$ and $\mu_F$ with CTEQ6M

$\eta_c$ at NLO - historical development

- J. Kühn & E. Mirkes compute pseudo-scalar toponium cross-section at NLO in 1992
- G. Schuler publishes his Review in 1994
  - confirms result by J. Kühn & E. Mirkes
  - points out issues with negative cross-sections at high energies
  - demonstrates that for some PDF choices there is strong/weak scale dependence
- M. Mangano comes to same conclusions as G. Schuler in his 1997 Proceedings
- A. Petrelli et al. confirm result by J. Kühn & E. Mirkes in 1998
- I confirm that everybody above was correct ;-)}
partonic high-energy limit
The partonic high-energy limit is defined as taking $\hat{\sigma}$ at $\hat{s} \to \infty$ or equivalently $z \to 0$ with $z = \frac{M_Q^2}{\hat{s}}$.

**Definition**

\[
\lim_{z \to 0} \hat{\sigma}_{ab}^{NLO}(z) = C_{ab} \frac{\alpha_s}{\pi} \hat{\sigma}_0^{LO} \left( \log \frac{M_Q^2}{\mu_F^2} + \hat{A} \right)
\]  
(1)
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- apart specific colour factor, the structure is same for different partonic channels $ab$ if form-factor is resolved
  - $C_{gg} = 2C_A$, $C_{qg} = C_F$, $C_{\gamma g} = C_A$
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\( \hat{A} \) is a constant determined by the form-factor resolution.

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  \[ C_{gg} = 2C_A, \quad C_{qg} = C_F, \quad C_{\gamma g} = C_A \]

- for \( 1S_0^{[1,8]} \):
  
  \[ \hat{A} = A_{gg} = A_{qg} = -1 \]
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- for $\mathbf{1S}_0^{[1,8]}$: $\hat{A} = A_{gg} = A_{qg} = -1$

- for $\mu_F = M_Q$, this limit is negative $\to \propto - \frac{\alpha_s}{\pi} \hat{\sigma}_{0}^{\text{LO}}$

- this limit is the most dominant contribution for flat gluon PDFs at low $x$. If PDFs are not steep enough, the large-$\hat{s}$ region dominates and the hadronic cross-section becomes negative
**Effect of PDF**

K-factor of $\eta_c$ production with $n_f=3$, $\mu_r=m_c$, $\mu_f=2m_c$, $y=0$

K-factor at $y=0$ as a function of energy and with different PDF choices. Scale choice used $\mu_R = m_c = 1.5\text{GeV}$, $\mu_F = 2m_c = 3\text{GeV}$. 

- **CT14nlo_NF3**
- **NNPDF31sx_nlo_as_0118**
- **NNPDF31sx_nlonllx_as_0118**
- **MRS(A')**, $g(x) \sim 1/x^{1.30037}$
- **MRS(G)**, $g(x) \sim 1/x^{1.14215}$
Recap of NLO calculation & origin of negative numbers

LO + virtual corrections: \( g(k_1) + g(k_2) \rightarrow \eta_Q(P) \) process:

\[
\begin{align*}
\text{LO:} & \quad g(k_1) + g(k_2) \\
\text{Virtual corrections:} & \quad \text{Diagram}
\end{align*}
\]
Recap of NLO calculation & origin of negative numbers

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- in general, interference terms \( 2\Re(\mathcal{M}^{(0)}\mathcal{M}^{(1)\dagger}) \) may give rise to negative contributions but for \( \eta_Q \) it is positive
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- in general, interference terms $2\Re\left(\mathcal{M}^{(0)}\mathcal{M}^{(1)\dagger}\right)$ may give rise to negative contributions but for $\eta_Q$ it is positive
- but no contribution at $z \rightarrow 0$ as threshold only at $z = 1$ (fixed) $\rightarrow$ virtual corrections are irrelevant for discussion that follows
Recap of NLO calculation & origin of negative numbers

$z$-dependence only present in real corrections:

$g(k_1) + g(k_2) \rightarrow \eta_Q(P) + g(k_3)$ process

- real corrections are perfect square $|M^{(\text{Real})}|^2 \geq 0$
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$$\bar{\sigma}_{gg, z \neq 1}^{\text{NLO}}(z) = \int d\hat{t} \frac{\bar{\sigma}^{\text{NLO}, z \neq 1}_{gg}}{d\hat{t}}$$ (2)
Recap of NLO calculation & origin of negative numbers

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\[
\sigma_{gg, z \neq 1}^{\text{NLO}}(z) = - \frac{1}{\epsilon_{\text{IR}}} \frac{\alpha_s}{\pi} \left( \frac{4\pi \mu_R^2}{M_Q^2} \right) \epsilon \Gamma(1 + \epsilon) \hat{\sigma}_0^{\text{LO}} z P_{gg}(z) + 2 C_A \frac{\alpha_s}{\pi} \hat{\rho}_0^{\text{LO}} A_{gg}(z) \tag{2}
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• with \( \epsilon_{IR} \to 0^- \), \( \sigma_{NLO, z \neq 1}^{gg} \geq 0 \) for all \( 0 \leq z < 1 \)
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• \( \lim_{z \to 0} \bar{A}_{gg}(z) = A_{gg} \)
mismatch between partonic cross-section and PDF

\[ \lim_{z \to 0} \hat{\sigma}_{ab}^{NLO} = C_{ab} \frac{\alpha_s}{\pi} \hat{\sigma}_0^{LO} \left( \log \frac{M_Q^2}{\mu_F^2} + \hat{A} \right) \]

• Problem: \( \hat{A} \) is process-dependent and thus cannot be compensated in a global manner via the process-independent DGLAP equations.
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\[ \lim_{z \to 0} \hat{\sigma}^{NLO}_{ab} = C_{ab} \frac{\alpha_s}{\pi} \hat{\sigma}^{LO}_0 \left( \log \frac{M^2_Q}{\mu_F^2} + \hat{A} \right) \]

- If \( \left( \log \frac{M^2_Q}{\mu_F^2} + \hat{A} \right) < 0 \) then over-subtraction from the AP-CT in \( \overline{\text{MS}} \)-scheme

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  \( \rightarrow \) should be compensated through steeper gluon PDFs via the *universal* DGLAP equations
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- Problem: \( \hat{A} \) is *process-dependent* and thus cannot be compensated in a global manner via the *process-independent* DGLAP equations
  → mismatch between PDFs and \( \hat{\sigma} \)!
a new scale prescription for \( \mu_F \)

- We have mismatch because \( A_{gg} \) and \( A_{qg} \) are process-dependent while DGLAP evolution is process-independent.
a new scale prescription for $\mu_F$

- We have mismatch because $A_{gg}$ and $A_{qg}$ are *process-dependent* while DGLAP evolution is *process-independent*.
- But with $\hat{A} = A_{gg} = A_{qg}$ we have an opportunity
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- Keeping this in mind, we propose a new scale prescription for $\mu_F$,

\[ \mu_F = \hat{\mu}_F = M e^{\hat{A}/2} \]

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such that \( \left( \log \frac{M_Q^2}{\mu_F^2} + \hat{A} \right) = 0 \quad \rightarrow \quad \lim_{z \to 0} \hat{\sigma}_{ab}^{NLO}(z) = 0. \]
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$$\mu_F = \hat{\mu}_F = M e^{\hat{A}/2}$$

such that $\left( \log \frac{M^2}{\mu_F^2} + \hat{A} \right) = 0 \quad \Rightarrow \quad \lim_{z \to 0} \hat{\sigma}_{ab}^{\text{NLO}}(z) = 0$.

for $\eta_Q$ we have $\hat{\mu}_F = \frac{M}{\sqrt{e}} = \begin{cases} 1.82\text{GeV} & \text{for } \eta_c \text{ with } M = 3\text{GeV} \\ 5.76\text{GeV} & \text{for } \eta_b \text{ with } M = 9.5\text{GeV} \end{cases}$

scale choice for $\eta_Q$ are within typical bounds $[\frac{M}{2}, 2M]$
PDFs at low scales

- for $\eta_c$, the new scale prescription is $\hat{\mu}_F = 1.82\text{GeV}$
PDFs at low scales

- for $\eta_c$, the new scale prescription is $\hat{\mu}_F = 1.82 \text{GeV}$
- due to low scale close to mass of charm quark, there is not much evolution $\rightarrow$ PDFs are close to initial parametrisation and thus not well constrained due to lack of data
PDFs at low scales

- luminosity plots, \( \frac{d\sigma^{\text{LO}}}{dy} \propto \frac{M^2}{s} \frac{\partial^2 \mathcal{L}}{\partial \tau \partial y} \)

PDF4LHC15

Adapted from a plot generated with APFEL 2.7.1 Web

NNPDF30 constrained from J/ψ exclusive photoproduction

\( \mu_F = 1.5 \text{ GeV} \)

\( \mu_F = 3.0 \text{ GeV} \)

\( \mu_F = 6.0 \text{ GeV} \)
PDFs at low scales

- luminosity plots, $\frac{d\sigma^{\text{LO}}}{dy} \propto \frac{M^2}{s} \frac{\partial^2 \mathcal{L}}{\partial \mathcal{L}}$

NNPDFNLL

Adapted from a plot generated with APFEL 2.7.1 Web

NNPDF30 constrained from J/ψ exclusive photoproduction

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Two-loop master integrals
CERN TH QCD 17/52
PDFs at low scales

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**JR14NLO08VF**

- Adapted from a plot generated with APFEL 2.7.1 Web
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Two-loop master integrals

CERN TH QCD 17 / 52
PDFs at low scales

\[ x g(x, \mu_F) \]

- For \( \eta_c \), the new scale prescription is \( \hat{\mu}_F = 1.82 \text{GeV} \).
- Due to low scale close to mass of charm quark, there is not much evolution, PDFs are close to initial parametrisation and thus not well constrained due to lack of data.

\[ x \quad g(x, \mu_F) \]

\( \mu_F = 1.55 \text{ GeV} \)

\( \mu_F = 3 \text{ GeV} \)

\( x \quad g(x, \mu_F) \)

Adapted from a plot generated with APFEL 2.7.1 Web

NNPDF30 constrained from \( J/\psi \) exclusive photoproduction

\( x \quad g(x, \mu_F) \)

→ use \( \eta_c \) and \( J/\psi \) data/predictions to perform PDF fits from first principles
\[ \frac{d\sigma}{dy} \] \( K \)-factor for \( \eta_c \)-production at \( y = 0 \)

- new scale choice is green curve → stability over energy range and approaching \( K \sim 1 \) at large \( \sqrt{s} \)
- other scale choices are not stable, give negative results or deviate from \( K = 1 \)
- bump for PDF4LHC15 (left plot) and at \( \mu_F = \frac{M}{2} \) is entirely due to weird PDF shape with bump/dip shown in previous slide
\[ \frac{d\sigma}{dy} \] K-factor for \( \eta_b \)-production at \( y = 0 \)

- new scale choice is green curve \( \rightarrow \) stability over energy range and approaching \( K \sim 1 \) at large \( \sqrt{s} \)
- other scale choices are not stable, deviate from \( K = 1 \)
- bump for PDF4LHC15 is now absent as the PDFs have evolved from initial parametrisation at around 1.7 GeV to scale of \( \eta_b \) with [4.75, 19] GeV
Part I: Prospects

- used a new scale prescription $\hat{\mu}_F$ based on partonic high-energy limit to cure negative numbers for charmonium production...
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• in this context interesting to study resummation effects...

→ need to compute double-virtual contribution which involves two-loop master integrals with massive propagators (see Part II)
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- reduction of scale uncertainties for production and decay at NNLO?
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  → need to compute double-virtual contribution which involves two-loop master integrals with massive propagators (see Part II)
Part II

Two-loop master integrals
Form-factors

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  - \( \gamma\gamma \leftrightarrow \eta Q \left( [1, 1]_0 \right) \)
  - \( gg \leftrightarrow \eta Q \left( [1, 1]_0 \right) \)
  - \( \gamma g \leftrightarrow [8] \)
  - \( gg \leftrightarrow [8] \)
  - \( \gamma\gamma \leftrightarrow \text{para-Positronium} \)
Form-factors

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  - $\gamma\gamma \leftrightarrow \eta Q \left(1 S^1_0\right)$
  - $gg \leftrightarrow \eta Q \left(1 S^1_0\right)$
  - $\gamma g \leftrightarrow 1 S^8_0$
  - $gg \leftrightarrow 1 S^8_0$
  - $\gamma\gamma \leftrightarrow \text{para-Positronium}$

- form-factors applicable to both production and decay
Form-factors

• We will compute the two-loop form-factors analytically for the pseudo-scalar states in different channels that contribute at NNLO accuracy
  • $\gamma\gamma \leftrightarrow \eta Q \left( 1S_{0}^{[1]} \right)$
  • $gg \leftrightarrow \eta Q \left( 1S_{0}^{[1]} \right)$
  • $\gamma g \leftrightarrow 1S_{0}^{[8]}$
  • $gg \leftrightarrow 1S_{0}^{[8]}$
  • $\gamma\gamma \leftrightarrow \text{para-Positronium}$

• form-factors applicable to both production and decay
• in the past form-factors have been computed only in numerical form
Form-factors

- We will compute the two-loop form-factors analytically for the pseudo-scalar states in different channels that contribute at NNLO accuracy
  - $\gamma\gamma \leftrightarrow \eta_Q \left( 1S_0^{[1]} \right)$
  - $gg \leftrightarrow \eta_Q \left( 1S_0^{[1]} \right)$
  - $\gamma g \leftrightarrow 1S_0^{[8]}$
  - $gg \leftrightarrow 1S_0^{[8]}$
  - $\gamma\gamma \leftrightarrow \text{para-Positronium}$

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\[ \text{para-Positronium} \rightarrow \gamma\gamma \quad [\text{A. Czarnecki, K. Melnikov, A. Yelkhovsky, Phys.Rev.A 61 (2000) 052502}] \]
Amplitude generation & partial fraction

\[ \gamma(k_1)\gamma(k_2) \rightarrow Q(p_1)\overline{Q}(p_2) \]  \hspace{1cm} (5)

- heavy quark momenta are equal \( p = p_1 = p_2 \)
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The fact that the two heavy-quark momenta are equal allows us to simplify some integrals beforehand via the procedure of partial fractioning.

Example

Feynman diagram:
Amplitude generation & partial fraction

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\[
I_{\text{Coul.}} = \int d^D q_1 \frac{1}{D_1 D_2 D_3 D_4} = \]

(6)
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Denominators are linearly dependent: \( D_4 = \frac{1}{2} (D_1 + D_3) \)
Amplitude generation & partial fraction

Example

Feynman diagram:

\[ I_{\text{Coul.}} = \int \frac{dD_1}{D_1} \frac{dD_2}{D_2} \frac{dD_3}{D_3} \frac{dD_4}{D_4} = \int \frac{dD_1}{D_1} \frac{dD_2}{D_2} \frac{dD_2}{D_2} \frac{dD_3}{D_3} - \int \frac{dD_1}{D_1} \frac{dD_2}{D_2} \frac{dD_4}{D_4} \]

\[ (7) \]

\[ k_1 \quad k_2 \quad p \]

\[ (8) \]

Melih A. Ozcelik (IJCLab)

Two-loop master integrals

CERN TH QCD
Example

Feynman diagram:

\[ I_{\text{Coul.}} = \int d^D q_1 \frac{1}{D_1 D_2 D_3 D_4} = \int d^D q_1 \frac{2}{D_1 D_2 D_3^2} - \int d^D q_1 \frac{1}{D_2 D_3^2 D_4} \] (7)
Amplitude generation & partial fraction

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(7)

\[ = 2 - \quad (8) \]
Amplitude generation & partial fraction

- partial fraction allows us to simplify integrals, 4-point function $\rightarrow$ 3-point function
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- Apply partial fractioning to all two-loop diagrams and perform tensor integral decomposition.
Amplitude generation & partial fraction

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- at higher loop orders, many denominators are involved
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- partial fractioning can then be performed with $Apart$-package
  

- apply partial fractioning to all two-loop diagrams and perform tensor integral decomposition

- reduce integrals to master integrals via IBP with FIRE
  
Topologies and master integrals

Some examples of topologies:

1. 
   \[ \begin{align*}
   &1 \\
   &2 \quad 5 \\
   &3 \quad 4 \\
   &7 \quad 6
   \end{align*} \]

2. 
   \[ \begin{align*}
   &1 \\
   &2 \quad 5 \\
   &3 \quad 4 \\
   &6 \quad 7
   \end{align*} \]

3. 
   \[ \begin{align*}
   &3 \quad 2 \\
   &5 \quad 4 \\
   &1 \quad 6 \quad 7
   \end{align*} \]

4. 
   \[ 4m^2 \]

5. 
   \[ 4m^2 \]

6. 
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7. 
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Topologies and master integrals

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Example

\[ 4m^2 = \] (9)
In order to derive this identity, let’s for the moment consider that $p^2 = -xm^2,$
Partial fraction revisited

In order to derive this identity, let's for the moment consider that \( p^2 = -xm^2 \),

\[
\frac{1}{D_1 D_2 D_3} = \frac{D_1}{p} + \frac{D_3}{p} + (1 + x)m^2
\]  

we have that \( D_2 = \frac{1}{2} D_1 + \frac{1}{2} D_3 + (1 + x)m^2 \)
In order to derive this identity, let’s for the moment consider that $p^2 = -xm^2$, 

$$\frac{1}{D_1 D_2 D_3} = \frac{1}{(1 + x)m^2} \left( \frac{1}{D_1 D_3} - \frac{1}{2} \frac{1}{D_1 D_2} - \frac{1}{2} \frac{1}{D_2 D_3} \right)$$ (10)
Partial fraction revisited

In order to derive this identity, let’s for the moment consider that $p^2 = -x m^2$.

Example

$$\frac{1}{D_1 D_2 D_3} = \frac{1}{(1 + x) m^2} \left( \frac{1}{D_1 D_3} - \frac{1}{2} \frac{1}{D_1 D_2} - \frac{1}{2} \frac{1}{D_2 D_3} \right)$$

(10)
Partial fraction revisited

Example

\[(1 + x) m^2 \frac{1}{D_1 D_2 D_3} = \left( \frac{1}{D_1 D_3} - \frac{1}{2} \frac{1}{D_1 D_2} - \frac{1}{2} \frac{1}{D_2 D_3} \right) \]

\[(1 + x) m^2 = \left( \frac{2p}{D_1 D_2} - \frac{1}{2} \frac{p}{D_1 D_3} - \frac{1}{2} \frac{p}{D_2 D_3} \right) \]

(11)
Partial fraction revisited

Example

\[
(1 + x) m^2 \frac{1}{D_1 D_2 D_3} = \left( \frac{1}{D_1 D_3} - \frac{1}{2} \frac{1}{D_1 D_2} - \frac{1}{2} \frac{1}{D_2 D_3} \right)
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Let’s now investigate the limit \( x \to -1 \)
Partial fraction revisited

Example

\[(1 + x) m^2 \frac{1}{D_1 D_2 D_3} = \left( \frac{1}{D_1 D_3} - \frac{1}{2} \frac{1}{D_1 D_2} - \frac{1}{2} \frac{1}{D_2 D_3} \right)\]

\[(1 + x) m^2 = \left( \begin{array}{c}
\frac{2p}{D_1 D_3} \\
\frac{p}{D_1} \\
\frac{p}{D_2 D_3}
\end{array} \right) - \frac{1}{2} \left( \begin{array}{c}
\frac{p}{D_1 D_2} \\
\frac{p}{D_1} \\
\frac{p}{D_2 D_3}
\end{array} \right) - \frac{1}{2} \left( \begin{array}{c}
\frac{p}{D_1 D_2} \\
\frac{p}{D_1} \\
\frac{p}{D_2 D_3}
\end{array} \right) \]

\[(11)\]

Let’s now investigate the limit \(x \to -1\)

\[0 = \left( \frac{1}{D_1 D_3} - \frac{1}{2} \frac{1}{D_1 D_2} - \frac{1}{2} \frac{1}{D_2 D_3} \right) \bigg|_{x=-1}\]

\[(12)\]
Partial fraction revisited

Identity

\[
\frac{1}{D_1 D_3} \bigg|_{x=-1} = \frac{1}{2} \frac{1}{D_1 D_2} \bigg|_{x=-1} + \frac{1}{2} \frac{1}{D_2 D_3} \bigg|_{x=-1}
\]

(11)
Partial fraction revisited

Identity

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\]

(11)

can derive additional identities...
Equivalence relations

Identity

\[ 4m^2 = \frac{1}{2} + \frac{1}{2} \]  \hspace{1cm} (12)

Example

\[ 4m^2 = \]  \hspace{1cm} (13)
Topologies and master integrals

- Appearance of around $\sim 77$ master integrals
- These are seemingly independent, however we find some interesting equivalence relations among these
- Analytical results for most of the integrals in these topologies are not available in the literature
Example: Topology 2

- some topologies would occur also for open $t\bar{t}$-production
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\begin{center}
\begin{tikzpicture}
\draw[thick] (0,0) -- (1,1) -- (1,2) -- (0,0);
\draw[thick] (0,0) -- (-1,0) -- (-1,1) -- (0,0);
\draw[thick] (0,0) -- (0,-1) -- (0,0);
\draw[thick] (0,1) -- (0,2);
\draw[thick] (0,-1) -- (0,-2);
\node at (0,0) {1};
\node at (1,1) {2};
\node at (1,2) {6};
\node at (-1,0) {3};
\node at (0,-1) {4};
\node at (0,1) {5};
\node at (0,2) {7};
\end{tikzpicture}
\end{center}

- has been considered for open $t\bar{t}$-production at general kinematics

[M. Becchetti et al, JHEP 08 (2019) 071]
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1 \\
2 \\
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\end{array} \]

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- However, these analytical results are not applicable at threshold kinematics $\hat{s} = 4m_Q^2$ due to prefactors which scale as $1/\sqrt{\hat{s} - 4m_Q^2}$ and thus would naively diverge.
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- In addition, it turns out that some integral in this topology encounter a drop in weight because of these prefactors $\rightarrow$ need to compute these integrals to one additional order higher in $\epsilon$ which otherwise would correspond to weight $w = 5$ in general kinematics.
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\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (-1,-1) {2};
  \node (3) at (-1,-2) {3};
  \node (4) at (-2,-2) {4};
  \node (5) at (-2,-1) {5};
  \node (6) at (-2,0) {6};
  \node (7) at (-2,-1.5) {7};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \draw (4) -- (5);
  \draw (5) -- (6);
  \draw (6) -- (7);
\end{tikzpicture}
\end{center}

has been considered for open $t\bar{t}$-production at general kinematics

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- In addition, it turns out that some integral in this topology encounter a drop in weight because of these prefactors $\rightarrow$ need to compute these integrals to one additional order higher in $\epsilon$ which otherwise would correspond to weight $w = 5$ in general kinematics.

- $\rightarrow$ computed nearly entire topology family via direct integration at threshold
Topologies and master integrals

- Appearance of around \( \sim 77 \) master integrals
- These are seemingly independent, however we find some interesting equivalence relations among these
- Analytical results for most of the integrals are not available in the literature
- will compute most of these integrals via the method of direct integration
• Appearance of around $\sim 77$ master integrals
• These are seemingly independent, however we find some interesting equivalence relations among these
• Analytical results for most of the integrals are not available in the literature
• will compute most of these integrals via the method of direct integration
• functions that will appear include *Multiple Polylogarithms* (MPLs) and *elliptic Multiple Polylogarithms* (eMPLs)
Multiple Polylogarithms (MPLs)

\[ G(a_1, ..., a_n; z) = \int_0^z dt \frac{1}{t - a_1} G(a_1, ..., a_n; t) \]

\[ G(a_n; t) = \int_0^z dt' \frac{1}{t' - a_n} G(; t') \text{ where } G(; t') = 1 \]

\[ G(0; t) = \log t \]

- weight of function corresponds to number of indices \( w = n \)
Multiple Polylogarithms (MPLs)

\[ G(a_1, \ldots, a_n; z) = \int_0^z dt \frac{1}{t - a_1} G(a_1, \ldots, a_n; t) \]  \hspace{1cm} (14)

\[ G(a_n; t) = \int_0^t dt' \frac{1}{t' - a_n} G(; t') \text{ where } G(; t') = 1 \]  \hspace{1cm} (15)

\[ G(0; t) = \log t \]  \hspace{1cm} (16)

- weight of function corresponds to number of indices \( w = n \)
- \( m \)-loop amplitude usually exhibits functions up to weight of \( w = 2m \)
  \( \rightarrow \) will be useful as cross-check of amplitude
Multiple Polylogarithms (MPLs)

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- weight of function corresponds to number of indices \( w = n \)
- \( m \)-loop amplitude usually exhibits functions up to weight of \( w = 2m \) 
  \( \rightarrow \) will be useful as cross-check of amplitude
- MPLs satisfy the shuffle algebra
- numerical evaluation can be achieved with GiNaC-interface  
  [Vollinga, Weinzierl]
elliptic Multiple Polylogarithms (eMPLs)

\[ E_4\left( \frac{n_1 \cdots n_m}{c_1 \cdots c_m}; x, \vec{q_r} \right) = \int_0^x dt \, \psi_{n_1}(c_1, t, \vec{q_r}) \, E_4\left( \frac{n_2 \cdots n_m}{c_2 \cdots c_m}; x, \vec{q_r} \right) \]  

(17)

\[ E_4\left( \frac{1}{c}; x, \vec{q_r} \right) = G(\vec{c}; x) \]  

(18)

- \( \vec{q_r} \) are the branch points of the elliptic curve defined by \( y^2 = P_4(x) = (x - q_{r,1}) \cdots (x - q_{r,4}) \)
elliptic Multiple Polylogarithms (eMPLs)

elliptic Multiple Polylogarithms (eMPLs) [Brown, Levin; Broedel, Duhr, Dulat, Penante, Tancredi, ...]

\[
E_4 \left( \frac{n_1 \ldots n_m}{c_1 \ldots c_m}; x, \vec{q_r} \right) = \int_0^x dt \, \psi_{n_1} \left( c_1, t, \vec{q_r} \right) E_4 \left( \frac{n_2 \ldots n_m}{c_2 \ldots c_m}; x, \vec{q_r} \right) \tag{17}
\]

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E_4 \left( \frac{\vec{1}}{\vec{c}}; x, \vec{q_r} \right) = G(\vec{c}; x) \tag{18}
\]

- \( \vec{q_r} \) are the branch points of the elliptic curve defined by 
  \[ y^2 = P_4 \left( x \right) = \left( x - q_{r,1} \right) \ldots \left( x - q_{r,4} \right) \]
- \( \psi_{n_1} \left( c_1, t, \vec{q_r} \right) \) are the elliptic kernels
elliptic Multiple Polylogarithms (eMPLs)

\[ E_4 \left( \frac{n_1 \ldots n_m}{c_1 \ldots c_m}; x, \vec{q}_r \right) = \int_0^x dt \, \psi_{n_1} (c_1, t, \vec{q}_r) \, E_4 \left( \frac{n_2 \ldots n_m}{c_2 \ldots c_m}; x, \vec{q}_r \right) \]  \hspace{1cm} (17)

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- \( \vec{q}_r \) are the branch points of the elliptic curve defined by \( y^2 = P_4(x) = (x - q_{r,1}) \ldots (x - q_{r,4}) \)
- \( \psi_{n_1} (c_1, t, \vec{q}_r) \) are the elliptic kernels
  - e.g. \( \psi_0 (0, t, \vec{q}_r) = \frac{c_4}{y} \) where \( c_4 = \frac{1}{2} \sqrt{(q_{r,1} - q_{r,3})(q_{r,2} - q_{r,4})} \)
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E_4\left( \frac{n_1}{c_1}, \ldots, \frac{n_m}{c_m}; x, \vec{q}_r \right) = \int_0^x dt \, \psi_{n_1}(c_1, t, \vec{q}_r) \, E_4\left( \frac{n_2}{c_2}, \ldots, \frac{n_m}{c_m}; x, \vec{q}_r \right)
\]

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  - e.g. \(\psi_1(c, t, \vec{q}_r) = \frac{1}{t - c}\)
- define weight as \(w = \sum_{i}^{m} |n_i|\) and length as \(l = m\)
elliptic Multiple Polylogarithms (eMPLs)

elliptic Multiple Polylogarithms (eMPLs) [Brown, Levin; Broedel, Duhr, Dulat, Penante, Tancredi, ...]

\[ E_4\left( \frac{n_1 \cdots n_m}{c_1 \cdots c_m}; x, \vec{q}_r \right) = \int_0^x dt \psi_{n_1}(c_1, t, \vec{q}_r) E_4\left( \frac{n_2 \cdots n_m}{c_2 \cdots c_m}; x, \vec{q}_r \right) \quad (17) \]

\[ E_4\left( \frac{1}{c}; x, \vec{q}_r \right) = G(\vec{c}; x) \quad (18) \]

- \( \vec{q}_r \) are the branch points of the elliptic curve defined by
  \[ y^2 = P_4(x) = (x - q_{r,1}) \cdots (x - q_{r,4}) \]
- \( \psi_{n_1}(c_1, t, \vec{q}_r) \) are the elliptic kernels
  - e.g. \( \psi_0(0, t, \vec{q}_r) = \frac{c_4}{y} \) where \( c_4 = \frac{1}{2} \sqrt{(q_{r,1} - q_{r,3})(q_{r,2} - q_{r,4})} \)
  - e.g. \( \psi_1(c, t, \vec{q}_r) = \frac{1}{t-c} \)
- define weight as \( w = \sum_i^m |n_i| \) and length as \( l = m \)
- satisfies shuffle algebra
Direct Integration

Feynman integral can be represented via two graph polynomials $U$ and $F$ which are the first and second Symanzik polynomial respectively.

$$I = (-1)^a (e^{\epsilon \gamma E})^h \Gamma \left( a - h \frac{D}{2} \right) \int_0^\infty dx_1 \ldots \int_0^\infty dx_m \delta(1 - \Delta_H) \times$$
$$\times \prod_{i=1}^m \left( \frac{x_i^{a_i - 1}}{\Gamma(a_i)} \right) \frac{U^{a-(h+1)} \frac{D}{2}}{F^{a-h} \frac{D}{2}}$$

(19)
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- each $x_i$ corresponds to a edge in a graph
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\]

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\[
\times \prod_{i=1}^m \left(\frac{x_i^{a_i-1}}{\Gamma(a_i)}\right) \frac{\mathcal{U}^{a-(h+1)}\frac{D}{2}}{\mathcal{F}^{a-h\frac{D}{2}}} \tag{19}
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- the second Symanzik polynomial $F$ distinguishes between massive and massless propagators
  - each massless propagator/edge contributes linearly to $F$
  - each massive propagator/edge contributes quadratically to $F$
- need to integrate out each single edge $x_i$; one done via Cheng-Wu delta function $\delta(1 - \Delta_H)$. 
Direct Integration

- Before each edge integration, the integrand needs to be fibrated and brought into the canonical form where the integration variable $t$ appears only in the argument rather than the indices, i.e. $G(\vec{a}; t) \rightarrow$ can be computationally a very heavy exercise.
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• However, in general for integrals involving elliptic kernels, there is no full automation and one has to perform it manually
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- some features implemented in PolyLogTools and EllipticPolyLogTools

[Panzer]

[Duhr, Dulat]
We now discuss briefly different cases that we have to consider:

- **linear reducibility**: an order of integration variables can be found where the integration kernels are all linear → MPL functions
- **elliptic linear reducibility**: an order of integration variables where all integration kernels up to the last integration are linear. The last integration contains a square-root $\sqrt{y}$ and introduces the elliptic kernels → eMPLs
- **elliptic next-to-linear reducibility**: an order where all integration kernels up to the second-last integration are linear. This case requires rationalisation of square-roots for the second-last integration → eMPLs

Since massive propagators introduce the quadratic variables, the more massive propagators a diagram exhibit, the more difficult it is to evaluate.
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These topologies involve elliptics

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- need rigorous systematic approach in eliminating all such spurious terms.
- two elliptic curves present in amplitude, one is sunrise elliptic curve and second is associated to light-by-light contribution (3rd diagram above)

sunrise elliptic curve:
elliptic next-to-linear reducibility & 2nd delta function

The following integral is elliptic next-to-linear reducible,

Example

- integrals in our list required the rationalisation of square-roots at the second-last integration. Can use RationalizeRoots package for this,

[Besier, Wasser, Weinzierl]
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![Diagram of an integral]

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Alternative new approach:

- there is the idea to introduce a 2nd delta function similarly to Cheng-Wu delta function, which introduces a new fictitious edge
- this approach can make the integral elliptic linear reducible $\rightarrow$ no rationalisation necessary anymore

[Besier, Wasser, Weinzierl]
Analytics and Numerics

We have computed all integrals analytically via direct integration and validated these numerically versus pySecDec.

- Many MPL integrals can be reduced to functions involving indices of 6th root of unity.
- For the high-precision numerics (>1000 digits) involving the MPLs, we can make use of the GiNaC package.
- For the eMPLs however, if these functions can be represented by iterated Eisenstein integrals which allow fast convergence, one can produce high-precision numerics (200 digits) [Duhr, Tancredi, JHEP 02 (2020) 105].
- For the other eMPLs where the convergence is rather slow, we make use of the differential equation approach and solve the system numerically via series expansion approach. Automated in DiffExp-package. [Hidding, arXiv:2006.05510]

→ Obtain high-precision numerics up to 200 digits for all master integrals needed for form-factors.
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Form-factors

Now ready to plug in analytics and numerics for the form-factors. Validation of results,

- compare to known numerical results for $\gamma\gamma \leftrightarrow \eta_Q$ case
Form-factor $\gamma\gamma \leftrightarrow \eta Q$

Bare amplitude contains only UV singularities and the Coulomb singularity which is specific to quarkonium physics

$$\left(\frac{\alpha_{s}^{\text{bare}}}{\pi}\right)^2 A^{(2)} = A^{(0)} S^2_{\epsilon} \left(\frac{\alpha_{s}^{\text{bare}}}{\pi}\right)^2 \left(\frac{1}{m^2}\right)^{2\epsilon} \left(\frac{1}{\epsilon^4} c_{-4} + \frac{1}{\epsilon^3} c_{-3} + \frac{1}{\epsilon^2} c_{-2} + \frac{1}{\epsilon} c_{-1} + c_0\right)$$

We have the following pole structure,

$$c_{-4} = 0$$
$$c_{-3} = 0$$
$$c_{-2} = \frac{3}{32} C_F^2 - \frac{1}{8} C_F T_F n_l$$
$$c_{-1} = C_F^2 \left(\frac{-39}{32} - \frac{\pi^2}{16} + \frac{3}{4} \log 2\right) + C_A C_F \left(-\frac{205}{96} - \frac{\pi^2}{96}\right)$$
$$+ C_F T_F n_l \left(\frac{17}{24} - \frac{\pi^2}{24}\right) + C_F T_F n_h \left(\frac{7}{8} - \frac{\pi^2}{24}\right),$$

some of the poles contain elliptic functions which however cancel. Can be made explicit with PSLQ fit. Similar for other form-factors.
Form-factor $\gamma\gamma \leftrightarrow \eta_Q$

Renormalisation of Coulomb singularity will introduce NRQCD scale $\mu_{\text{NRQCD}}$

$$\tilde{A}_{\text{ren}} = A_{\text{ren}}^{(0)} \left[ 1 + \left( \frac{\alpha_s^{(m_l)}}{\pi} \right) K^{(1)} + \left( \frac{\alpha_s^{(m_l)}}{\pi} \right)^2 K^{(2)} \right],$$

$$K^{(1)} = C_F \left( \frac{\pi^2}{8} - \frac{5}{2} \right) = -1.2662994498638302510 C_F,$$

$$K^{(2)} = C_F^2 c_1 + C_F C_A c_2 + C_F T_F n_h c_3 + C_F T_F n_l c_4 + K_{\text{lbl}}^{(2)} \left[ \frac{\beta_0}{4} \log \left( \frac{\mu_{\text{NRQCD}}^2}{m^2} \right) + K_{\text{Coulomb}}^{(2)} \right] \log \left( \frac{\mu_{\text{NRQCD}}^2}{m^2} \right),$$

$$K_{\text{Coulomb}}^{(2)} = -C_F^2 \frac{\pi^2}{2} - C_F C_A \frac{\pi^2}{4}.$$

$$K_{\text{lbl}}^{(2)} = C_F T_F n_h c_5 + C_F T_F n_l \sum_{i} \frac{e_i^2}{e_Q^2} c_6.$$
Form-factor $\gamma\gamma \leftrightarrow \eta Q$

$$K^{(2)} = C_F^2 c_1 + C_F C_A c_2 + C_F T_F n_h c_3 + C_F T_F n_i c_4 + K_{\text{lbl}}^{(2)} + \frac{K^{(1)} \beta_0}{4} \log \left( \frac{\mu^2}{m^2} \right) + K_{\text{Coulomb}}^{(2)} \log \left( \frac{\mu_{\text{NRQCD}}^2}{m^2} \right)$$

$c_1 = -21.107897967310671456611138$

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\]

\[
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- full analytical results and improved precision up to 200 digits accuracy
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- full analytical results and improved precision up to 200 digits accuracy
- interesting to note that the Abelian corrections $C_F^2$ are large and negative
Form-factors

Now ready to plug in analytics and numerics for the form-factors. Validation of results,

- compare to known numerical results for $\gamma\gamma \leftrightarrow \eta_Q$ case → find full agreement
- for the new form-factors, validation is based on universal IR pole structure → amplitudes are manifestly finite after UV and IR renormalisation

[Catani; Becher, Neubert]
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- regular Abelian corrections ($C_F^2$, $C_F T_F n_{h/1}$) are identical for all form-factors → further confirmation of the results
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- QED corrections to para-Positronium result can be obtain by setting

\[
C_A \to 0, \quad C_F \to 1, \quad T_F \to 1
\]  (20)

Part III

Decay of pseudo-scalar to di-photon at NNLO accuracy
Decay width to di-photon

- form-factors are applicable to both exclusive and inclusive decay to di-photon
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  - at LO $\eta_Q \to \gamma\gamma$
Decay width to di-photon

- form-factors are applicable to both exclusive and inclusive decay to di-photon
  - at LO $\eta_Q \rightarrow \gamma\gamma$
  - at NLO $\eta_Q \rightarrow \gamma\gamma$ at one-loop

- NLO contribution $\eta Q \rightarrow \gamma\gamma g$ vanishes because of colour-structure and charge conjugation
- up to NLO both exclusive and inclusive decay are identical

- at NNLO $\eta Q \rightarrow \gamma\gamma g$ is non-vanishing and (most likely) finite based on dipole mechanism approach
- we consider here exclusive decay and will investigate the uncertainty with respect to $\mu_R$ and $\mu_{\text{NRQCD}}$ and compare with existing data
Decay width to di-photon

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  - at LO $\eta_Q \rightarrow \gamma\gamma$
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  - NNLO contribution $\eta_Q \rightarrow \gamma\gamma gg$ is non-vanishing and (most likely) finite based on dipole mechanism approach
Decay width to di-photon

- form-factors are applicable to both exclusive and inclusive decay to di-photon
  - at LO $\eta_Q \rightarrow \gamma\gamma$
  - at NLO $\eta_Q \rightarrow \gamma\gamma$ at one-loop
  - NLO contribution $\eta_Q \rightarrow \gamma\gamma g$ vanishes because of colour-structure and charge conjugation
    → up to NLO both exclusive and inclusive decay are identical
  - at NNLO $\eta_Q \rightarrow \gamma\gamma$ at two-loop
  - NNLO contribution $\eta_Q \rightarrow \gamma\gamma gg$ is non-vanishing and (most likely) finite based on dipole mechanism approach
- we consider here exclusive decay and will investigate the uncertainty with respect to $\mu_R$ and $\mu_{NRQCD}$ and compare with existing data
Exclusive decay width to di-photon

\[ \Gamma_{\eta Q \rightarrow \gamma \gamma} = \Gamma_0 \left[ 1 + \frac{\alpha_s}{\pi} \Gamma_1 + \left(\frac{\alpha_s}{\pi}\right)^2 \Gamma_{2,Q} \right] \] (21)

\[ \Gamma_1 = -3.3767985329702 \] (22)

\[ \Gamma_{2,c} = -110.17929296929 \] (23)

\[ -7.5977966991830 \log \left( \frac{\mu^2_R}{m^2} \right) \] (24)

\[ -37.285172181893 \log \left( \frac{\mu^2_{\text{NRQCD}}}{m^2} \right) \] (25)
Exclusive decay width to di-photon

- variation of scales
Exclusive decay width to di-photon

• variation of scales
  • three scales in quarkonium physics: $m v^2$, $m v$, $m$

\[ \Gamma_{\text{NLO}} \eta_c \rightarrow \gamma \gamma = \Gamma_0 \times \left[ 0.737 + 0.032 - 0.044 \right] \text{keV} \] (26)

\[ \Gamma_{\text{NNLO}} \eta_c \rightarrow \gamma \gamma = \Gamma_0 \times \left[ 0.28 + 0.11 - 0.17 \right] \text{keV} \] (27)

• large NNLO corrections due to Abelian $C_2^F$ coefficient!

• $\mu_R$ uncertainty has contrary to expectation not reduced from NLO to NNLO

• latest experimental prediction [Particle Data Group] $\Gamma_{\text{exp}} \eta_c \rightarrow \gamma \gamma = (5.06 \pm 0.34) \text{keV} \] (28)

• NNLO result is closer to experimental value than NLO result → importance of higher order corrections
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Melih A. Ozcelik (IJCLab)
Exclusive decay width to di-photon

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\[
\Gamma_{\text{NLO}} \eta_c \to \gamma \gamma = \Gamma_0 \times \left[ 0.737 + 0.432 - 0.62 \right] = (10.34 + 0.45 - 0.62) \text{ keV (26)}
\]

\[
\Gamma_{\text{NNLO}} \eta_c \to \gamma \gamma = \Gamma_0 \times \left[ 0.28 + 0.11 - 0.17 \pm 0.16 \right] = (3.9 + 1.6 - 2.3 \pm 2.2) \text{ keV (27)}
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Summary

Part I:

• a new scale prescription to cure the issue of negative cross-section and the mismatch between hard part and PDF
• PDFs are unconstrained at low scales → can use charmonium calculations/data in global PDF fits

Part II:

• two-loop double-virtual contributions involve massive Feynman integrals
• used cutting-edge techniques to compute all relevant master integrals analytically for the form-factor discussed and produced high-precision numerics

Part III:

• shown that NNLO corrections to $\eta_c$ decay to di-photon are large and turn out to be important as these are closer to experimental data than the NLO results
• however $\mu_R$ uncertainties have contrary to expectation not reduced from NLO to NNLO
Thank you for attention!
partonic high-energy limit

\[
\lim_{z \to 0} \hat{\sigma}_{gg}^{NLO}(z) = 2 C_A \frac{\alpha_s}{\pi} \hat{\sigma}_{0}^{LO} \left( \log \frac{M_Q^2}{\mu_F^2} + A_{gg} \right)
\]  (29)
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for \( \eta_Q^{[1,8]}(1 S_0^{[1,8]}) \):   \[ A_{gg} = A_{qg} = -1 \]
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for $\eta_Q^{[1,8]} (1 S_0^{[1,8]})$: \hspace{1cm} $A_{gg} = A_{qg} = -1$

for $1 P_{0}^{[1,8]}$: \hspace{1cm} $A_{gg} = A_{qg} = -\frac{43}{27}$

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for \( \tilde{H}^0 \) \( \left( \frac{m_Q}{M_{\tilde{H}}} = 1 \right): \) \( A_{gg} = A_{qg} = 1.61 \)

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- the \( \mu_F^2 \) dependence is universal while the quantity \[ A_{gg} = A_{qg} \] is process-dependent
Fictitious-H⁰ production for M_H=3 GeV and m_Q=1.5 GeV in the loop computed with ggHiggs and PDF4LHC30

(ξ_R,ξ_F)=(0.93,0.93)        (ξ_R,ξ_F)=(0.5,0.5)
(ξ_R,ξ_F)=(1.0,1.0)          (ξ_R,ξ_F)=(2.0,1.0)
(ξ_R,ξ_F)=(1.0,2.0)          (ξ_R,ξ_F)=(2.0,2.0)
(ξ_R,ξ_F)=(1.0,0.5)          (ξ_R,ξ_F)=(0.5,1.0)
negative cross-sections - open $c\bar{c}$ production at N2LO

open $c\bar{c}$ production at NLO/N2LO, comparison with different PDFs (ABM12, PDF4LHC15)  

Quarkonia - three different models

- **Colour-Evaporation Model**
  - quark and anti-quark colours are summed up at amplitude squared level (evaporation)
  - no spin-projection

- **Colour-Octet Model**
  - quark and anti-quark pair are in color-octet state
  - heavy quark spins projected on final bound state
  - higher Fock states in NRQCD, higher $v$-order

- **Colour-Singlet Model**
  - quark and anti-quark pair are in color-singlet state
  - heavy quark spins projected on final bound state
  - leading Fock state in NRQCD
 gluon-gluon channel

\[\hat{\sigma}_{gg}(s, \hat{s}, \mu_R, \mu_F) = \frac{\alpha_s^2(\mu_R)\pi^2}{96m_c^5} |R(0)|^2 \delta(1 - z)\]

\[+ \frac{\alpha_s^3(\mu_R)\pi}{1152m_c^5} |R(0)|^2 \left[ \left(-44 + 7\pi^2 + 54 \log \left(\frac{\mu_R^2}{\mu_F^2}\right)\right)
\right.\]

\[\left. + 72 \log \left(1 - \frac{4m_c^2}{s}\right) \left(\log \left(1 - \frac{4m_c^2}{s}\right) - \log \left(\frac{\mu_F^2}{4m_c^2}\right)\right)\right] \delta(1 - z)\]

\[+ 6 \left(24 \left(\log \left(1 - z\right)\right)\right) \rho (1 - (1 - z) z)^2\]

\[+ 12 \left(\frac{1}{1 - z}\right) \rho \frac{\log (z)}{(1 - z)(1 + z)^3} (1 - z^2 (5 + z (2 + z + 3z^3 + 2z^4)))\]

\[- \left(\frac{1}{1 - z}\right) \rho \frac{1}{(1 + z)^2} (12 + z^2 (23 + z (24 + 2z + 11z^3)))\]

\[+ 12 (1 + z^3)^2 \log \left(\frac{z\mu_F^2}{4m_c^2}\right)\right]\]

, where \(z = 4m_c^2/\hat{s}\) and \(\rho = 4m_c^2/s\)

(30)
quark-antiquark channel

\[
\hat{\sigma}_{q\bar{q}}(\hat{s}, \mu_R) = \frac{16\alpha_s^3(\mu_R)\pi}{81m_c} |R(0)|^2 \frac{(\hat{s} - 4m_c^2)}{\hat{s}^3}
\]

quark-gluon channel

\[
\hat{\sigma}_{qg}(\hat{s}, \mu_R, \mu_F) = \frac{\alpha_s^3(\mu_R)\pi}{72m_c^5\hat{s}^2} |R(0)|^2 \left(8m_c^4 + 4m_c^2\hat{s} - \hat{s}^2 + 2 \left(8m_c^4 - 4m_c^2\hat{s} + \hat{s}^2\right) \log \left(1 - \frac{4m_c^2}{\hat{s}}\right) + \hat{s} \left(-4m_c^2 + \hat{s}\right) \log \left(\frac{4m_c^2}{\hat{s}}\right) - \left(8m_c^4 - 4m_c^2\hat{s} + \hat{s}^2\right) \log \left(\frac{\mu_F^2}{\hat{s}}\right)\right)
\]

(32)
problem of negative cross-sections - $J/\psi, \ ^1S^0_{[8]}$ at NLO

comparison of $J/\psi \ ^1S^0_{[8]}$ differential cross-section at NLO with different choices of $\mu_R$ and $\mu_F$ with CTEQ6M [Y. Feng, J.-P. Lansberg, J.X. Wang, Eur.Phys.J. C75 (2015) no.7, 313]
Schuler 1994 - structure of partonic cross-section

- Let’s define $z = M^2/\hat{s}$ and $\tau_0 = M^2/s$
- LO partonic cross-section and virtual corrections ($2 \rightarrow 1$ process) have $\delta(1 - z)$ function while real corrections ($2 \rightarrow 2$) are complicated functions of $z$
- Negative contributions come from real corrections
- Idea is to use simple toy-models for gluon PDFs and convolute with partonic cross-section; different $z$-terms will contribute differently at hadronic level
Asymptotic ($\tau_0 = M^2/s \to 0$) behaviour of the proton-proton or proton-antiproton cross section for various forms of the gluon-gluon subprocess ($z = M^2/$\hat{s} = $\tau_0/\tau$) and two extreme choices of the gluon distribution function. Taken from G. Schuler, Review, 1994

<table>
<thead>
<tr>
<th>$\hat{\sigma}_{gg}(z, M^2)$</th>
<th>$xg(x) \to 1$</th>
<th>$xg(x) \to 1/\sqrt{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(1-z)$</td>
<td>$\ln \left(\frac{1}{\tau_0}\right)$</td>
<td>$\frac{1}{\sqrt{\tau_0}} \ln \left(\frac{1}{\tau_0}\right)$</td>
</tr>
<tr>
<td>$z^k$</td>
<td>$\frac{1}{k} \ln \left(\frac{1}{\tau_0}\right)$</td>
<td>$\frac{2}{(2k+1)\sqrt{\tau_0}} \ln \left(\frac{1}{\tau_0}\right)$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\frac{1}{2} \ln^2 \left(\frac{1}{\tau_0}\right)$</td>
<td>$\frac{2}{\sqrt{\tau_0}} \ln \left(\frac{1}{\tau_0}\right)$</td>
</tr>
<tr>
<td>$\ln^k \left(\frac{1}{z}\right)$</td>
<td>$\frac{1}{(k+1)(k+2)} \ln^{k+2} \left(\frac{1}{\tau_0}\right)$</td>
<td>$\frac{k!2^{k+1}}{\sqrt{\tau_0}} \ln \left(\frac{1}{\tau_0}\right)$</td>
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toymodel $g(x) = 1/x$: real corrections dominate at high energies;
toymodel $g(x) = 1/x^{1.5}$: all contributions have same energy scaling