

Initial Value Problem and Causality

in String-inspired Non-local Field Theory

Atakan Firat, MIT (WIP, w/ Erbin, Zwiebach)

Outline: 1. Introduction & Setup

2. Redefining the Purely Time-dependent Theory

3. Rolling Tachyons

4. Redefining the Covariant Theory

5. Dispersion Relations

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1. Introduction & Setup

Goal: Understanding issues of initial value formulation & causality in string (field) theory.

⇒ Hard! Instead consider a toy model that shares the non-locality feature of SFT.

• Inspired by tachyon in open SFT, we consider the theory:

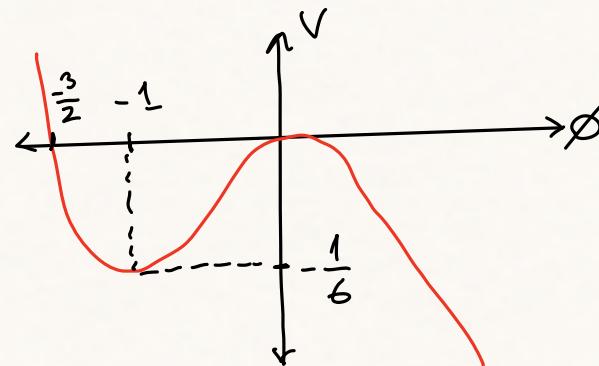
$$L = \frac{1}{2} \phi (\partial^2 + 1) \phi + \frac{1}{3} (e^{\xi^2 \partial^2} \phi)^2$$

ξ : Non-locality parameter

$$= \frac{1}{2} \phi (\partial^2 + 1) \phi + \frac{1}{3} \phi^3 + \xi^2 \phi^2 \partial^2 \phi + \mathcal{O}(\xi^4)$$

→ Everything would be dimensionless.

→ Potential:



$$V(\phi) = -\frac{1}{2}\phi^2 - \frac{1}{3}\phi^3$$

2. Redefining the Purely-time dependent theory

- Assume the field has no spatial dependence: $\phi = \phi(t)$
- The theory is

$$L = -\frac{1}{2}\phi(\partial_t^2 - 1)\phi + \frac{1}{3}(e^{-\xi^2\partial_t^2}\phi)^3 = L_0 + \xi^2 L_2 + \xi^4 L_4 + \mathcal{O}(\xi^6),$$

where

$$L_0 = -\frac{1}{2}\phi\partial_t^2\phi + \frac{1}{2}\phi^2 + \frac{1}{3}\phi^3 = -\frac{1}{2}\phi\partial_t^2\phi - V(\phi),$$

$$L_2 = -\phi^2\partial_t^2\phi,$$

$$L_4 = \frac{1}{2}\phi^2\partial_t^4\phi + \phi(\partial_t^2\phi)^2.$$

- Now imagine performing the field redefinition

$$\phi \rightarrow \phi + \delta\phi = \phi + \xi^2\delta\phi_2 + \mathcal{O}(\xi^4). \quad \rightarrow \delta\phi \text{ is not infinitesimal!}$$

This would shift L as

$$L \rightarrow L_0 + \xi^2 \left[-\delta\phi_2(\partial_t^2\phi + V'(\phi)) - \phi^2\partial_t^2\phi \right] + \mathcal{O}(\xi^4).$$

By choosing, we see

$$\begin{aligned} \delta\phi_2 = -\phi^2 &\Rightarrow L \rightarrow L_0 + \xi^2\phi^2V'(\phi) + \mathcal{O}(\xi^4) \\ &= L_0 + \underbrace{\xi^2(-\phi^3 - \phi^4)}_{\text{L derivative int. is eliminated!}} + \mathcal{O}(\xi^4). \end{aligned}$$

Q: Is it possible to put the theory of the form

$$L \rightarrow -\frac{1}{2}\phi\partial_t^2\phi - \tilde{V}(\phi; \xi^2),$$

perturbatively in ξ^2 , where $\tilde{V}(\phi; \xi^2)$ contains no derivative:

$$\tilde{V}(\phi; \xi^2) = -\frac{1}{2}\phi^2 + \left(-\frac{1}{3} + \xi^2\right)\phi^3 + \xi^2\phi^4 + \mathcal{O}(\xi^4),$$

after performing $\phi \rightarrow \phi + \delta\phi = \phi + \xi^2\delta\phi_2 + \xi^4\delta\phi_4 + \mathcal{O}(\xi^6)$ suitably?

A: Yes!!

Sketch of the proof

Proceed via induction:

- 1 Suppose all-higher derivative interactions has been eliminated up to $\mathcal{O}(\xi^{2k-2})$ and perform $\phi \rightarrow \phi + \xi^{2k} \delta\phi_{2k}$ which would induce

$$L \rightarrow L_0 + (\text{d}\xi\text{-free terms of order } < \xi^{2k}) \\ + \xi^{2k} [-\delta\phi_{2k} (\partial_t^2 \phi + V'(\phi)) + L'_{2k}] + \mathcal{O}(\xi^{2k+2}).$$

↑ prime denotes this would involve terms induced by lower orders - not just L'_{2k} !

Clearly, if $\partial_t^2 \phi \times (\phi, \partial\phi) \in L'_{2k}$, one can pick

$$\delta\phi_{2k} = X(\phi, \partial\phi) \Rightarrow \partial_t^2 \phi \times (\phi, \partial\phi) \in L'_{2k} \\ \downarrow \\ -V'(\phi) X(\phi, \partial\phi) \in L'_{2k}. \quad \begin{array}{l} \# \text{ of } \partial_t \text{ in } X \\ \text{has been} \\ \text{reduced!} \end{array}$$

- 2 Show a generic term in L'_{2k} can be brought into form $\partial_t^2 \phi \times (\phi, \partial\phi)$ after int-by-parts so that it can get eliminated algorithmically. A generic term in L'_{2k} is

$$T = (\partial_t^{k_1} \phi) (\partial_t^{k_2} \phi) \dots (\partial_t^{k_r} \phi) (\partial_t \phi)^r \phi^s.$$

$k_i, r, s \in \mathbb{Z}_{\geq 0}$
 $3 \leq k_1 \leq \dots \leq k_r$
 l : index
 k_r : the lowest order

Integrate by parts:

$$T \simeq -(\partial_t^{k_1-1} \phi) \partial_t [(\partial_t^{k_2} \phi) \dots (\partial_t^{k_r} \phi) (\partial_t \phi)^r \phi^s] \\ = -\partial_t^{k_1-1} \phi \partial_t [(\partial_t^{k_2} \phi) \dots (\partial_t^{k_r} \phi)] - V'(\phi) \\ + r (\partial_t^{k_1-1} \phi) (\partial_t^{k_2} \phi) \dots (\partial_t^{k_r} \phi) (\partial_t \phi)^{r-1} (\partial_t^2 \phi) \phi^s \\ + s (\partial_t^{k_1-1} \phi) (\partial_t^{k_2} \phi) \dots (\partial_t^{k_r} \phi) (\partial_t \phi)^{r+1} \phi^{s-1}$$

contains $\partial_t^2 \phi$ so can be reduced!
Lonest order has reduced

Doing this recursively can reduce $k_1=3$, after which one can replace $\partial_t^{2k-1}\phi = \partial_t^2\phi$ in front with $-V'(\phi)$.

→ This shows index l can be reduced using field redefinitions and most generic term can be brought to a form:

$$T^1 = (\partial_t\phi)^{2p} \phi^q, \quad p, q \in \mathbb{Z}_{\geq 0}.$$

After int-by-parts:

$$\begin{aligned} T^1 &\simeq -\phi \partial_t [(\partial_t\phi)^{2p-1} \phi^q] \\ &= -(2p-1) \partial_t^2 \phi (\partial_t\phi)^{2p-2} \phi^{q+1} - q \frac{(\partial_t\phi)^{2p} \phi^q}{T^1}, \\ \Rightarrow T^1 &\simeq -\frac{(2p-1)}{1+q} \cancel{\partial_t^2 \phi} (\partial_t\phi)^{2p-2} \phi^{q+1} \\ &\rightarrow -\frac{2p-1}{1+q} (\partial_t\phi)^{2p-2} (\phi^{q+2} + \phi^{q+3}). \end{aligned}$$

⇒ T^1 can be reduced to $f(\phi)$ after field redefinitions!

⇒ By induction, non-local theory in $0+1$ -dims can be turned (perturbatively-in- ξ^2) into standard two- ∂_t theory with potential!

The Result

Doing this algorithmically we find the resulting potential is

$$\begin{aligned} \tilde{V}(\phi; \xi^2) &= -\frac{1}{2} \phi^2 + \left[-\frac{1}{3} + \xi^2 - \frac{3}{2} \xi^4 + 2 \xi^6 + \dots \right] \phi^3 \\ &\quad + \left[\xi^2 - \frac{19}{3} \xi^4 + \frac{419}{18} \xi^6 + \dots \right] \phi^4 \\ &\quad + \left[-\frac{16}{3} \xi^4 + \frac{517}{9} \xi^6 + \dots \right] \phi^5 + \left[\frac{118}{3} \xi^6 + \dots \right] \phi^6 + \mathcal{O}(\phi^7). \end{aligned}$$

with

$$\begin{aligned} \delta\phi &= -\xi^2 \phi^2 + \xi^4 \left[\frac{3}{2} \phi^2 + \frac{13}{3} \phi^3 + (\partial_t\phi)^2 + 2\phi \partial_t^2 \phi \right] \\ &\quad + \xi^6 \left[2\phi^2 + \frac{178}{9} \phi^3 + \frac{91}{3} \phi^4 + \frac{4}{3} (\partial_t\phi)^2 + \frac{46}{3} \phi (\partial_t\phi)^2 + \frac{8}{3} \phi (\partial_t^2\phi) \right. \\ &\quad \left. + 18\phi^2 (\partial_t^2\phi) + \frac{4}{3} (\partial_t^2\phi)^2 + \frac{4}{3} (\partial_t\phi) (\partial_t^3\phi) + \frac{4}{8} \phi (\partial_t^4\phi) \right] + \mathcal{O}(\xi^8). \end{aligned}$$

Remarks:

1) This potential is not unique! Consider the variation

$$\phi \rightarrow \phi + \xi^{2k} c [(\partial_t \phi)^2 + V(\phi)], \quad c \in \mathbb{R}.$$

The Lagrangian is going to shift by

$$L \rightarrow L + \xi^{2k} c \underbrace{[(\partial_t \phi)^2 + V(\phi)] [\partial_t^2 \phi + V'(\phi)]}_{(\partial_t \phi)^2 (\partial_t^2 \phi) = \frac{1}{3} \partial_t \phi^3} + \mathcal{O}(\xi^{2k+2})$$

$$V(\phi) \partial_t^2 \phi + (\partial_t \phi)^2 V'(\phi) = \partial_t [V(\phi) \partial_t \phi]$$

these are total derivatives - ignore!

$$\simeq L + \xi^{2k} c V(\phi) V'(\phi) + \mathcal{O}(\xi^{2k+2})$$

$$= L + \xi^{2k} c \left[\frac{1}{2} \phi^3 + \frac{5}{6} \phi^4 + \frac{1}{3} \phi^5 \right] + \mathcal{O}(\xi^{2k+2}).$$

In fact this can be generalized:

$$\delta \phi = \xi^{2k} c_{n,k} \sum_{p=0}^n \frac{(2n-1)!!}{(2n-1-2p)!!} (\partial_t \phi)^{2n-2p} \frac{V(\phi)^p}{p!} \quad (-1)!! = 1$$

$$c_{n,k} \in \mathbb{R}$$

$$L \rightarrow L + \xi^{2k} c_{n,k} \frac{(2n-1)!!}{n!} V(\phi)^n V'(\phi) + \mathcal{O}(\xi^{2k+2}).$$

Perturbatively we can add these ambiguities at each order and deal with higher-order terms like any other higher-derivative term.

→ This set of field redefinitions seem to cover all ambiguities of the potential of this sort!

→ This provides a choice for the form of the potential by adjusting const. One choice is the one that makes odd-power terms polynomial in ξ^2 as:

$$\begin{aligned}\tilde{V}(\phi; \xi^2) = & -\frac{1}{2}\phi^2 + \left[-\frac{1}{3} + \xi^2\right]\phi^3 \\ & + \left[\xi^2 - \frac{23}{6}\xi^4 + \frac{112}{9}\xi^6 - \frac{409}{9}\xi^8 + \dots\right]\phi^4 \\ & + \left[-\frac{13}{3}\xi^4 + \frac{370}{9}\xi^6 - \frac{2356}{9}\xi^8\right]\phi^5 \\ & + \left[\frac{92}{3}\xi^6 - \frac{7444}{15}\xi^8 + \dots\right]\phi^6 \\ & + \left[-\frac{13532}{45}\xi^8 + \dots\right]\phi^7 + O(\phi^8).\end{aligned}$$

We will call this canonical form.

[2] The depth at the critical point ϕ_* of the potential $\tilde{V}(\phi; \xi^2)$ is independent of ξ^2 and is equal to the value in $V(\phi)$, i.e.

$$\tilde{V}(\phi_*; \xi^2) = V(-1) = -\frac{1}{6}.$$

This is easy to see, since $s\phi = g(\phi) + h(\phi, \partial\phi)$ only the variation of $V(\phi)$ by $g(\phi)$ would not contain any derivative i.e.

$$\tilde{V}(\phi; \xi^2) = V(\phi + g(\phi)),$$

Showing the depth shouldn't change. (minimum shifts from -1 though). This is reassuring since this depth is observable

3. Rolling Tachyons

*Equation of motion for our non-local theory is (in $D+1-\text{dim}$)

$$(\partial_t^2 - 1) \phi = e^{-\xi^2 \partial_\phi^2} (e^{-\xi^2 \partial_\phi^2} \phi).$$

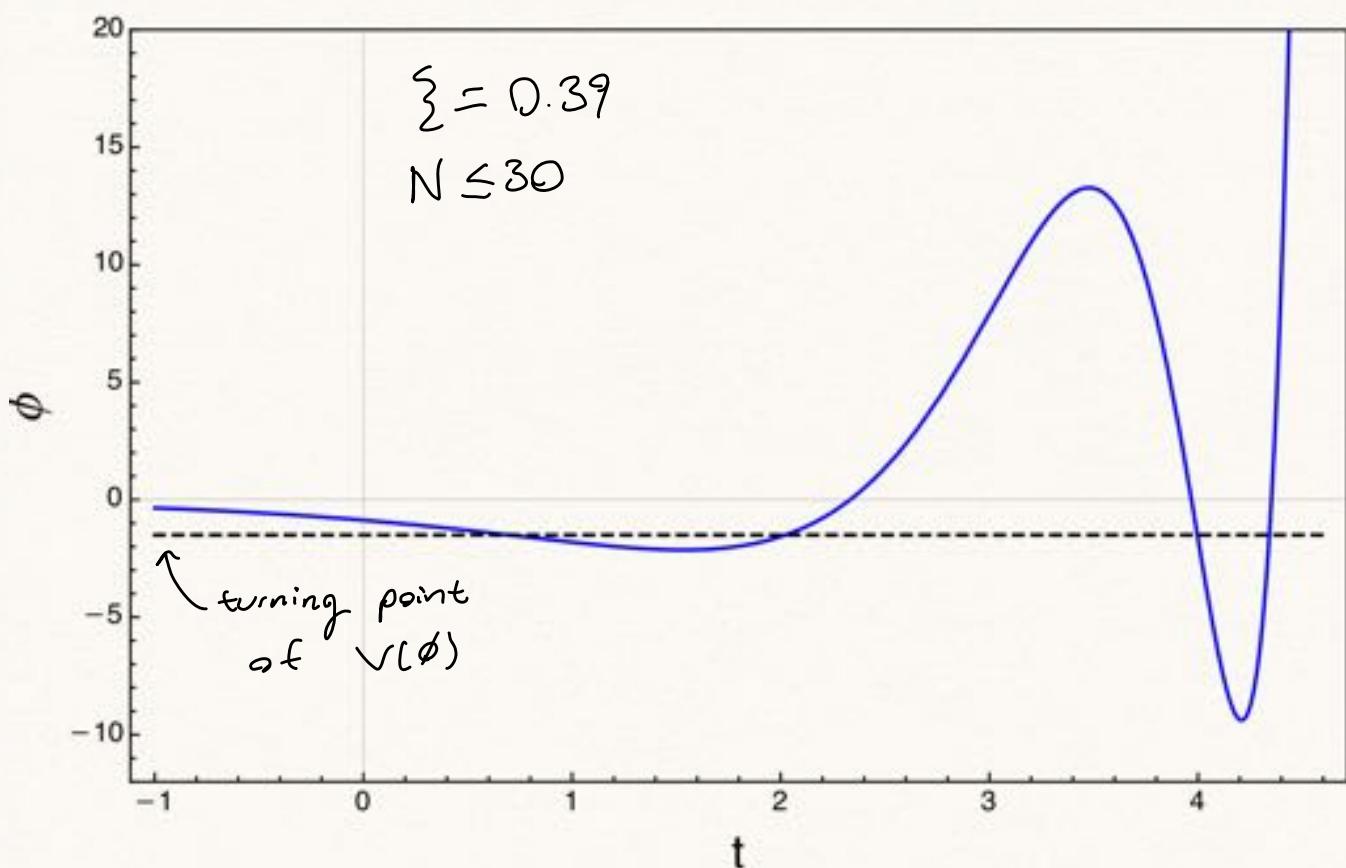
Tachyon can be rolled from $\phi=0$ at $t=-\infty$ to $\phi=-1$ by the ansatz:

$$\phi(t) = \sum_{n=1}^{\infty} b_n e^{nt} = -e^t + b_2 e^{2t} + b_3 e^{3t} + \dots$$

$\uparrow \frac{1}{3} e^{-6\xi^2} \quad \uparrow -\frac{1}{12} e^{-20\xi^2}$

and b_n can be found recursively after substitution.

This gives unbounded oscillations for ϕ :



→ It is suspected this solution converges for all ξ^2

* Another way to approach rolling is to consider solution perturbative - in - ξ^2 , but exact in t . So make the ansatz

$$\phi = \phi_0(t) + \xi^2 \phi_2(t) + \xi^4 \phi_4(t) + \mathcal{O}(\xi^6).$$

Substituting to EOM, order-by-order we get:

$$\mathcal{O}(\xi^0): (\partial_t^2 - 1)\phi_0 = \phi_0^2,$$

$$\mathcal{O}(\xi^2): (\partial_t^2 - 1)\phi_2 = 2\phi_0\phi_2 - 2(\partial_t\phi_0)^2 - 4\phi_0(\partial_t^2\phi_0),$$

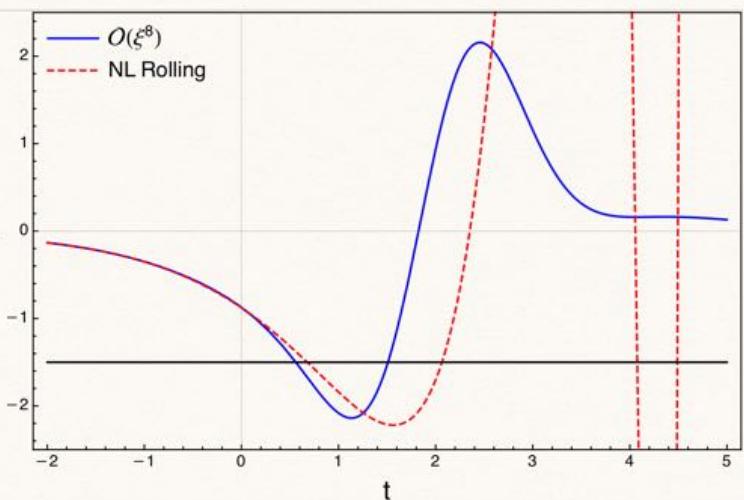
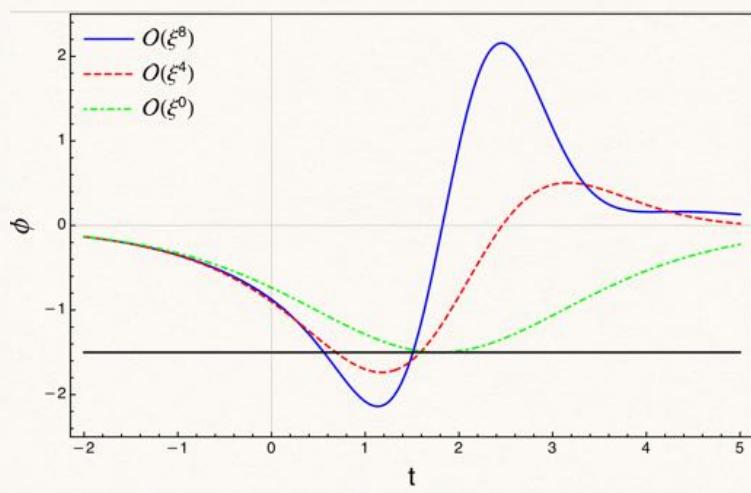
First equation is just rolling in local theory so it can be solved by energy conservation, exactly - in - t,

$$\phi_0(t) = -\frac{36e^t}{(6+e^t)^2} = -e^t + \frac{1}{3}e^{2t} - \frac{1}{12}e^{3t} + \dots$$

Notice $\phi_0(t \rightarrow -\infty) \rightarrow -e^t$. The rest can be solved order-by-order in ξ^2 since they are linear, second-order ODEs in the function we're solving. Solution would be

$$\phi(t) = -\frac{36e^t}{(6+e^t)^2} + \frac{432e^{2t}(e^t-6)}{(6+e^t)^4} \xi^2 + \mathcal{O}(\xi^4).$$

This solution looks like ($\xi = 0.4$)



It is expected including all orders in ξ^2 would produce the full non-local rolling.

* Now let us field redefine our rolling solutions using $\phi = \phi' + \delta\phi(\phi', \partial\phi')$, where $\delta\phi$ is field redefinition to reach canonical form. Here

ϕ : Original solution

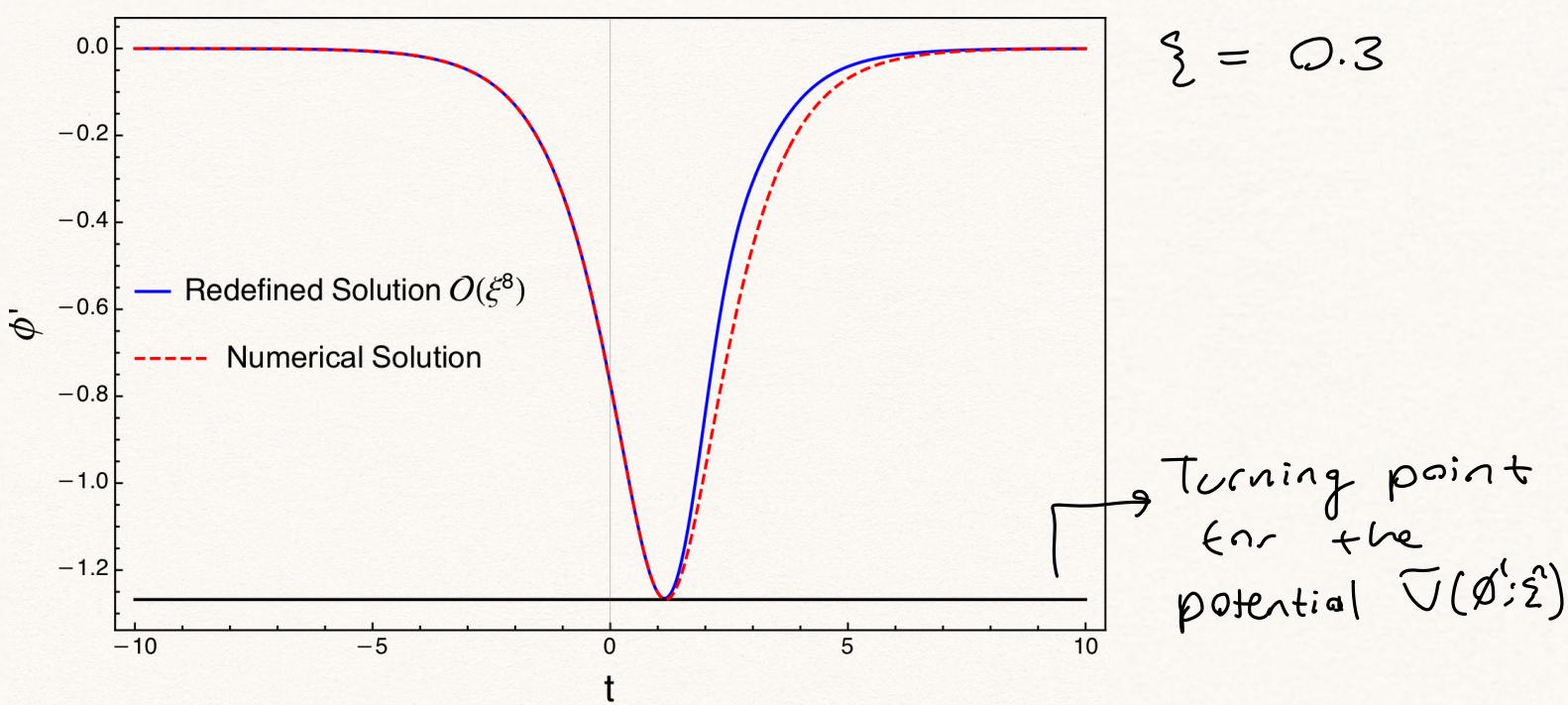
ϕ' : Redefined solution

The result we get is:

$$\phi'(t) = -\frac{36e^t}{(6+e^t)^2} + \frac{432e^{2t}(e^t-3)}{(6+e^t)^4}\xi^2 + \mathcal{O}(\xi^4).$$

after (perturbatively) inverting $\phi = \phi' + \delta\phi(\phi', \partial\phi')$.

This looks like, compared to the usual rolling in the potential $\tilde{V}(\phi'; \xi^2)$:



Remarks:

- 1 **No overshooting or osc!** This result shows that oscillations in the original formulation are due to using "wrong" field variable and unphysical.
- 2 Including more orders in ξ^2 improves the fit between the solutions. So it seems the full rolling solution would map to the rolling solution of $\tilde{V}(\phi', \xi^2)$ computed to all orders.
- 3 This works for any choice of potential we have, and using their field redefinition we can map the NL rolling solution to their rolling solutions.

4. Redefining the Covariant theory

Back to

$$\begin{aligned} L &= \frac{1}{2} \phi (\partial^2 + 1) \phi + \frac{1}{3} (e^{\xi^2 \partial^2} \phi)^2 \\ &= \frac{1}{2} \phi (\partial^2 + 1) \phi + \frac{1}{3} \phi^3 + \xi^2 \phi^2 \partial^2 \phi + O(\xi^4). \end{aligned}$$

Similar arguments show we can redefine this to

$$\begin{aligned} L \rightarrow \tilde{L} &\simeq \left[\frac{1}{2} \phi \partial^2 \phi + \frac{1}{2} \phi^2 + \frac{1}{3} \phi^3 \right] - \xi^2 [\phi^3 + \phi^4] \\ &\quad + \xi^4 \left[\frac{3}{2} \phi^3 + \frac{19}{3} \phi^4 + \frac{16}{3} \phi^5 \right] \\ &\quad - \xi^6 \left[\left(\frac{3}{2} \phi^3 + \frac{178}{9} \phi^4 + \frac{472}{9} \phi^5 + \frac{112}{3} \phi^6 \right) + \frac{8}{3} (\partial \phi)^4 \right] \\ &\quad + \xi^8 [\dots + \beta (\partial \phi)^2 \partial^2 (\partial \phi)^2] + O(\xi^{10}), \quad \beta = -\frac{4}{3} \end{aligned}$$

Two issues:

- It is not possible to redefine $(\partial \phi)^4 = \partial^\mu \phi \partial_\mu \phi \partial_\nu \phi \partial^\nu \phi$ to a potential term, analogous formula doesn't work by contractions of Lorentz indices.
 \Rightarrow This is first-order in derivatives, so can keep it while having an initial value formulation!
- It is not possible to redefine the higher derivative term

$$(\partial \phi)^2 \partial^2 (\partial \phi)^2 = (\partial^\mu \phi \partial_\mu \phi) \partial^2 (\partial^\nu \phi \partial_\nu \phi)$$

covariantly, i.e. can't be written as $\partial^\mu \phi (\dots)$!

→ However, by breaking manifest Lorentz covariance we can redefine this term so that it doesn't contain higher time derivatives. The result is

$$\begin{aligned} \tilde{L} \simeq & \left[\frac{1}{2} \phi \partial^2 \phi + \frac{1}{2} \phi^2 + \frac{1}{3} \phi^3 \right] - \xi^2 [\phi^3 + \phi^4] \\ & + \xi^4 \left[\frac{3}{2} \phi^3 + \frac{19}{3} \phi^4 + \frac{16}{3} \phi^5 \right] \\ & - \xi^6 \left[\left(\frac{3}{2} \phi^3 + \frac{178}{9} \phi^4 + \frac{472}{9} \phi^5 + \frac{112}{3} \phi^6 \right) + \frac{8}{3} (\partial \phi)^4 \right] \\ & + \xi^8 \left[\dots - \frac{2\beta}{5} \phi^7 - \frac{7\beta}{5} \phi^6 - \frac{5}{3} \beta \phi^5 - \frac{2\beta}{3} \phi^4 \right. \\ & \quad \left. - 2\beta (2\phi+1) (\partial \phi)^4 + 2\beta (\partial \phi)^2 ((\nabla^2 \phi)^2 + (\partial_i \partial_j \phi)^2) \right. \\ & \quad \left. - 2(\nabla \dot{\phi})^2 + 2(\phi^2 + \phi) \partial^2 \phi \right] + \mathcal{O}(\xi^{10}). \end{aligned}$$

Alternatively, we can focus on light-cone and eliminate any $x^\pm = \frac{1}{2}(x^0 \pm x^1)$ derivatives at the expense of keeping non-locality in $x^- = \frac{1}{2}(x^0 - x^1)$. The result would be

$$\begin{aligned} \tilde{L} \simeq & \left[\frac{1}{2} \phi \partial^2 \phi + \frac{1}{2} \phi^2 + \frac{1}{3} \phi^3 \right] - \xi^2 [\phi^3 + \phi^4] \\ & + \xi^4 \left[\frac{3}{2} \phi^3 + \frac{19}{3} \phi^4 + \frac{16}{3} \phi^5 \right] \\ & - \xi^6 \left[\left(\frac{3}{2} \phi^3 + \frac{178}{9} \phi^4 + \frac{472}{9} \phi^5 + \frac{112}{3} \phi^6 \right) \right. \\ & \quad \left. + \frac{8}{3} (\nabla_T \phi)^4 - \frac{16}{3} (\partial_T \phi)(\nabla_T \phi)^2 \frac{1}{2} (\nabla^2 \phi + \phi + \phi^2) \right. \\ & \quad \left. + \frac{8}{3} (\partial_T \phi)^2 \left[\frac{1}{2} (\nabla_T^2 \phi + \phi + \phi^2) \right]^2 \right] + \mathcal{O}(\xi^8). \end{aligned}$$

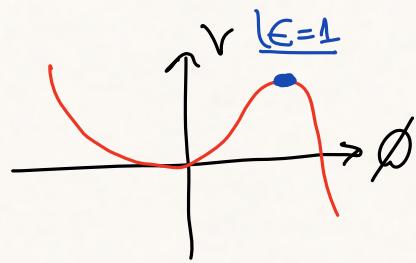
→ The arguments can be repeated to all orders in ξ^2 !

5. Dispersion Relations

We linearize eom around non-trivial vacuum for both normal ($\epsilon = 1$) and tachyonic ($\epsilon = -1$) case:

$$[-\partial^2 + \epsilon - 2\epsilon e^{2\xi^2 \partial^2}] \delta\phi = 0$$

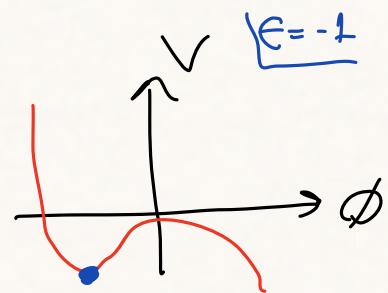
$$\xi^2 + \epsilon - 2\epsilon e^{-2\xi^2 k^2} = 0$$



and refractive index is

$$n(\omega)^2 = \frac{(\vec{k})^2}{\omega^2} = 1 - \frac{m_{\text{eff}}^2}{\omega^2},$$

$$m_{\text{eff}}^2 = \epsilon - \frac{1}{2\xi^2} W(4\epsilon\xi^2 e^{2\epsilon\xi^2}).$$

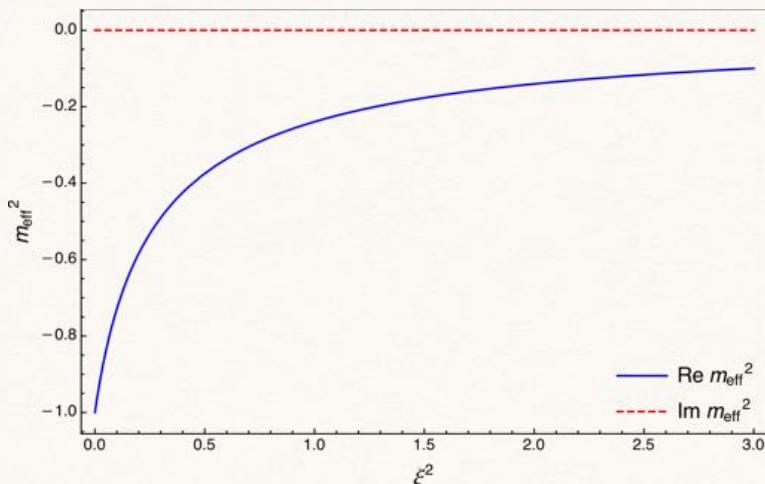


Here $W(z)$ is Lambert W -function, satisfying

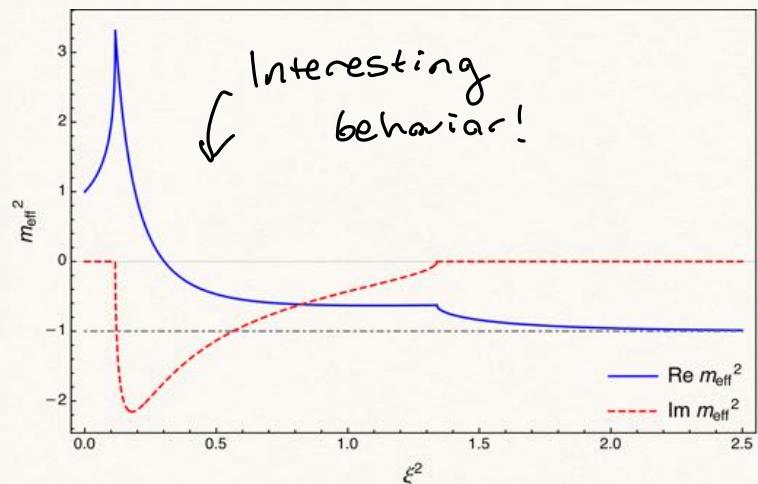
$$We^w = z \iff w = W(z).$$

Effective masses looks like:

$\epsilon = 1$



$\epsilon = -1$



In both cases

$$V_{wf} = \lim_{\omega \rightarrow \infty} n(\omega)^{-1} = 1.$$

where V_{wf} is wave-front velocity, which should be bounded by 1 in causal theories. Then this shows signal propagation around non-trivial backgrounds should be causal, despite non-localities!

6. Conclusion & Open questions

* We see it is possible to field-redefine a non-local theory to a standard two-derivative theory (but with some caveats). This "tame's" oscillations of rolling solutions of this NL theory.

→ Open issues:

① Everything we did was perturbative in ξ^2 . Is it converging & invertible in all orders in ξ^2 ?

② Is there a way to work non-perturbatively in ξ^2 ? Relatedly, is there a closed-form expression for $\tilde{V}(\phi; \xi^2)$, possibly after fixing the ambiguity of the potential?

3 Is there a formulation where Lorentz symmetry and initial value formulation is manifest at the same time?

4 How does quantum case work? What is the relation between this and other approaches for causality (e.g. microcausality, Bogoliubov's condition...)

$$\frac{\delta}{\delta g(x)} \left(\frac{\delta S(g)}{\delta g(y)} S^*(g) \right) = 0 \quad \text{for } x \in y.$$

5 Are the results here lifts up to SFT?