The Disk Partition function in String Theory

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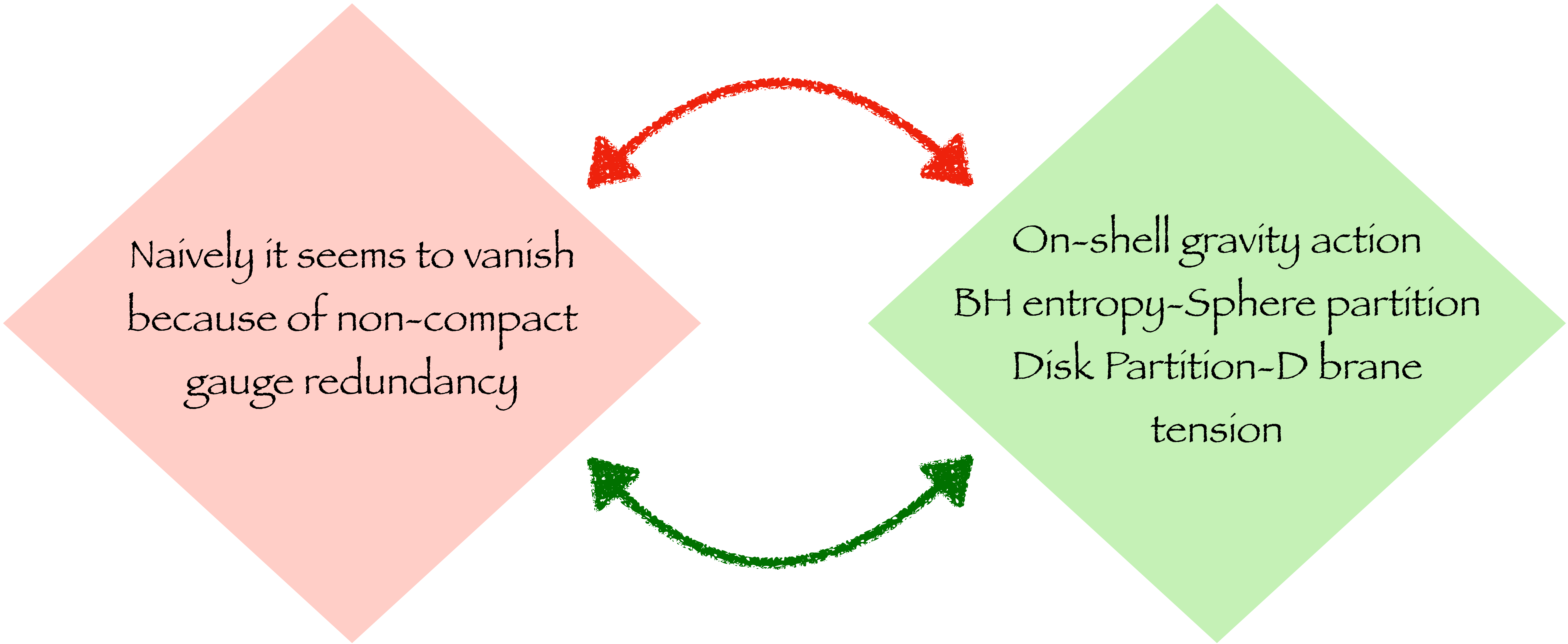
Based on 2105.08726 [hep-th]
Lorenz Eberhardt, **SP**

Why?

- Lower genus String partition functions are confusing and intiguing at the same time.



Unpacking the confusing element:

- Let us consider critical string (Bosonic)

$$S[g, X] = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2 z \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + \cdots$$

- We assume existence of one flat direction.

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2 z \ \sqrt{g} \ g^{z\bar{z}} 2 \partial X \bar{\partial} X$$

Unpacking the confusing element:

- We fix the conformal gauge.
- There is a combination of Weyl invariance and Diffeomorphism, which remains unfixed.
- For sphere, we have PSL (2,C) while for the Disk we have PSL(2,R)
- For three or higher point function, we use the gauge redundancy to fix the position of three vertex operators.
- Naively this implies that the lower point function vanishes.

Unpacking the confusing element:

- ~~Naively this implies that the lower point function vanishes.~~
- Two point String amplitude is non vanishing. [H.Erbin, J.Maldacena, D. Skliros]

<u>**We will focus on Zero point (DISK) case!**</u>

Gauge redundancy in Disk:

$$\mathrm{PSL}(2,\mathbb{R}) \equiv \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2 \qquad \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \qquad |a|^2 - |b|^2 = 1$$

- Action on the unit disk $\quad z \mapsto \dfrac{az+b}{\bar{b}z+\bar{a}}$

$$Z_{\mathrm{Disk}} = \frac{Z_{\mathrm{CFT}}}{\mathrm{Vol}(\mathrm{PSL}(2,\mathbb{R}))}$$

Apparent volume of PSL(2,R):

$$\mathrm{PSL}(2,\mathbb{R}) \equiv \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2 \qquad \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \qquad |a|^2 - |b|^2 = 1$$

- Volume $\int \mathrm{d}^2 a \ \mathrm{d}^2 b \ \delta(|a|^2 - |b|^2 - 1)$

$$a = e^{i\phi} \cosh x$$

$$b = e^{i\psi} \sinh x$$

$$x \in [0, \infty)$$

$$\phi \in [0, 2\pi) \, , \psi \in [0, \pi)$$

Apparent volume of PSL(2,R):

$$\mathrm{PSL}(2,\mathbb{R}) \equiv \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$$

$$\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$$

$$|a|^2 - |b|^2 = 1$$

- Volume $$2\pi^2 \int_0^\infty \mathrm{d}x \;\; \cosh x \sinh x$$

DIVERGENT

Regularized volume of PSL(2,R):

- Volume $2\pi^2 \int_0^\infty \mathrm{d}x \;\; \cosh x \sinh x$

Subtract off the DIVERGENCE [Liu, Polchinski 1989]

$$V_{reg} = \lim_{x_* \to \infty} \left(\int_{M_*} \mathrm{d}^3 V \; \sqrt{g} - \frac{1}{2} \int_{\partial M_*} \mathrm{d}^2 a \sqrt{h} \right)$$

Regularized volume of PSL(2,R):

$$-\pi^2/2$$

[Liu, Polchinski 1989]

- GOAL: Obtain the above doing honest QFT calculation.

Gauge Fixing- Faddeev Popov

- Proceeds via choosing a gauge condition and inserting the following

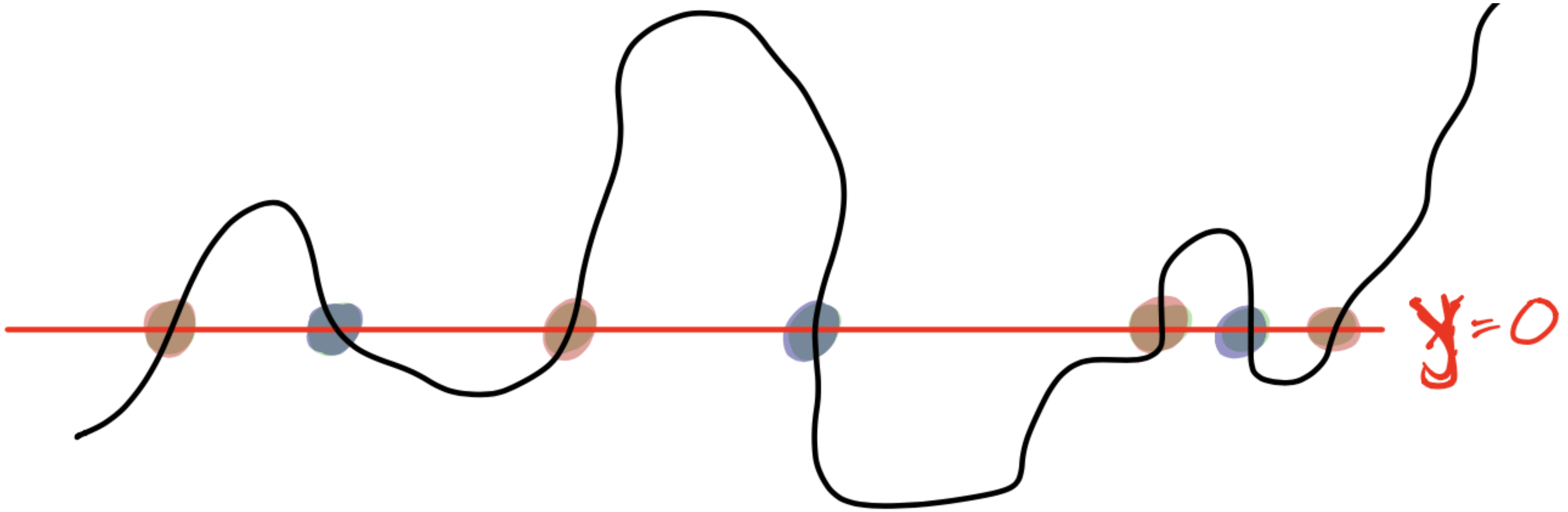
$$1 = \int \mathrm{d}g \; \Delta(X^g)\delta(F(X^g))$$

- Key Assumption: Gauge orbit intersects the Gauge fixing surface once and only once.

Gauge Fixing- Toy Example

$$1 = \int \mathrm{d}g \; \Delta(X^g)\delta(F(X^g))$$

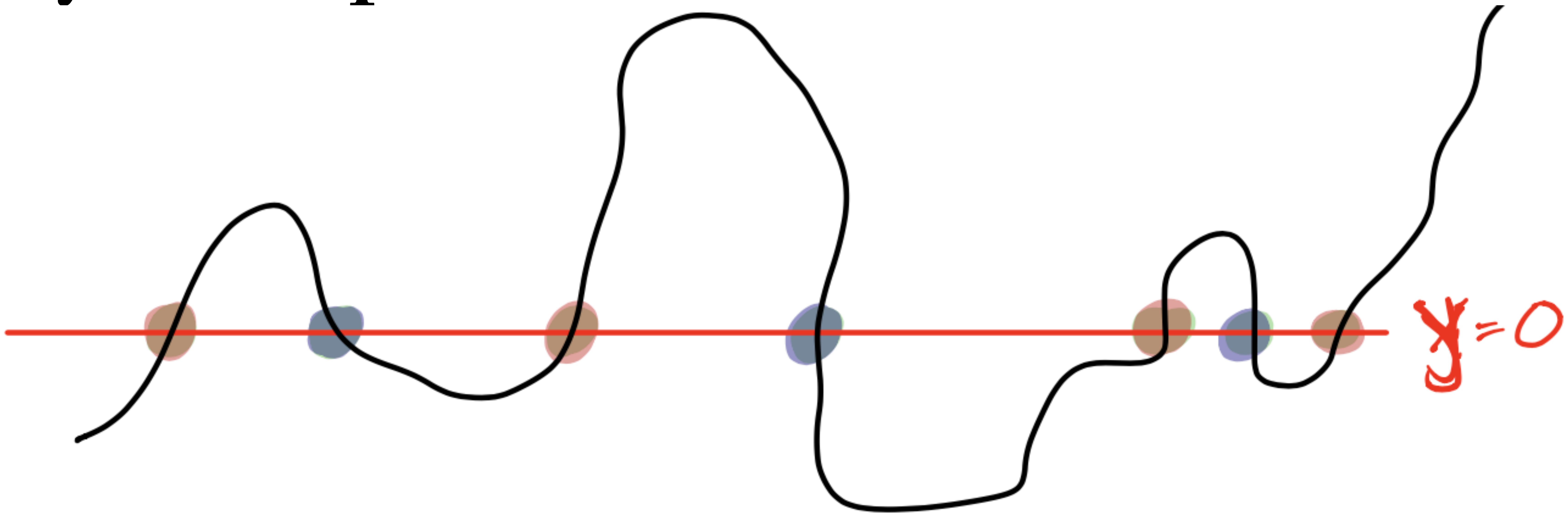
$$\#\text{Roots} = \int_{-\infty}^{\infty} \mathrm{d}x \; |f'(x)|\delta(f(x))$$



Gauge Fixing- Toy Example

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$

$$1 = \int_{-\infty}^{\infty} \mathrm{d}x \; f'(x)\delta(f(x))$$



- Take Home Lesson: One can get rid of absolute value of Jacobian under suitable condition.

Gauge Fixing-PSL(2,R)

$$ds^2 = \frac{4dzd\bar{z}}{(1+|z|^2)^2}$$

- Choose the upper hemisphere metric

$$X = \sum_{\ell,m} X_{\ell,m} Y_{\ell,m}$$

- Expand in appropriate spherical Harmonics consistent with boundary condition

$$X_{2,\pm 1} = 0$$ Gauge fixing condition For Dirichlet

$$X_{1,\pm 1} = 0$$ Gauge fixing condition For Neumann

Gauge Fixing-PSL(2,R)

$X_{2,\pm 1} = 0$ Gauge fixing condition For Dirichlet

$X_{1,\pm 1} = 0$ Gauge fixing condition For Neumann

$$\gamma_\alpha = \frac{1}{\sqrt{1-\alpha^2}} \begin{bmatrix} 1 & \alpha \\ \bar{\alpha} & 1 \end{bmatrix}$$

- Takes away from the Gauge fixing surface.
- Remaining U(1) preserves it.

- Next GOAL: to find the signed intersection number.

$$\int \frac{4 \mathrm{d}^2 z}{(1+|z|^2)^2} \; X(\gamma_\alpha^{-1}(z,\bar{z})) \quad Y^*_{2,1}(z,\bar{z}) = 0$$

Gauge Fixing-PSL(2,R)

$$\int \frac{4\mathrm{d}^2 z}{\left(1+|z|^2\right)^2} \; X(\gamma_\alpha^{-1}(z,\bar{z})) \quad Y^*_{2,1}(z,\bar{z}) = 0$$

- Given any configuration X, can we choose a Gauge transformation to reach the gauge fixing surface?

$$X(z) \rightarrow X(\gamma_\alpha^{-1}(z)) \text{ and then Project onto modes}$$

Gauge Fixing-PSL(2,R)

$$\int \frac{4\mathrm{d}^2 z}{(1+|z|^2)^2} \; X(\gamma_\alpha^{-1}(z,\bar{z})) \quad Y^*_{2,1}(z,\bar{z}) = 0$$

- The signed intersection number is related to Winding number, which is -1

- After the dust settles, we have

$$-\int \frac{\pi \mathrm{d}^2\alpha}{(1-|\alpha|^2)^2} \; \mathrm{Jac}(\mathrm{Det}(\mathrm{Gauge}))\delta(F^\alpha(x)) = 1$$

Inserting Signed FP determinant

$$-\int \frac{\pi \mathrm{d}^2\alpha}{(1-|\alpha|^2)^2}\ \mathrm{Jac}(\mathrm{Det}(\mathrm{Gauge}))\delta(F^\alpha(x)) = 1$$

- Without Gauge fixing, CFT path integral is nothing but the Gaussian integral over $X_{\ell,m}$

- Two mode integral gets omitted due to delta function and Jacobian introduces a polynomial of finite number of modes.

$$\pi J_{\mathrm{D}}(X) = \frac{64}{7}\left[(\mathrm{Im}\, X_{3,2})^2 + (\mathrm{Re}\, X_{3,2})^2\right] - \frac{16}{5}\sqrt{\frac{3}{7}} X_{1,0}X_{3,0} - \frac{2}{5}(X_{1,0})^2 - \frac{96}{35}(X_{3,0})^2$$

Disk Partition function

$$Z_{\text{Disk}} = \frac{Z_{\text{CFT}}}{-\pi^2/2}$$

- We have two more different methods to arrive at the same result.
- In all cases, D-Brane tension can be found using $e^{-T_p \text{Vol}(D_p)} = e^{Z_{\text{Disk}}}$

Open Problem-Sphere partition function

- The idea of Gauge fixing by considering the spherical harmonic modes apparently fails for the sphere partition function. Is there any first principle calculation ?

- Can we compute the sphere partition function with AdS_3 target space and reproduce the boundary conformal anomaly ? (w.i.p with L.Eberhardt and R.Mahajan)

THANK YOU