# The Primitive Domain for Analytic Off-shell Correlators in SFT

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# Based on the work

"Analyticity of off-shell Green's functions in superstring field theory"
JHEP **01** (2021) 010, Ritabrata Bhattacharya, RM

- In complex external momenta variables, Feynman loop diagrams those with no massless internal propagator in closed SFT are analytic on a domain – work of *de Lacroix, Erbin, and Sen (LES)* form 2018. We prove an analytic extension of the LES domain to a larger domain
- Explicit applications of our formula to 2-, 3-, 4-point functions show: the extension equals the well-known primitive domain. In case of 5-point functions, we do the analysis for limited subcases

will start by posing a question. then, to answer this, will discuss our proposed extension and its applications

# Green's functions in Local QFTs: some old results

Assumptions –
1. Commutator of local operators vanishes outside lightcone,
2. Existence of a complete system of (physical) states with $+$ve energy

– put constraints on position space correlators

$\implies$ primitive analyticity of off-shell $p$-space Green's functions in LQFTs without massless states [Bogolyubov et al. book]

Here, off-shell $p$-space correlators are defined as **Fourier Transforms** of position space correlators

# Primitive analyticity

Analyticity of $G(p_1, \ldots, p_n)$ on primitive domain in complex external $D$-momenta

Define, $\quad P_I = \sum_{a \in I} p_a, \quad I \subsetneq \{1, \ldots, n\} \setminus \emptyset$

A collection of points $(p_1, \ldots, p_n)$ satisfying 2 conditions –

C1. external momentum conservation: $p_1 + \cdots + p_n = 0$

C2. For each $I$,

if $\mathrm{Im}\ P_I \neq 0$, then $\mathrm{Im}\ P_I$ must be timelike,
if $\mathrm{Im}\ P_I = 0$, then $-P_I^2 < M_I^2$, where $M_I$ is the threshold mass for producing any (multi-particle) state in the collision of particles carrying total momentum $P_I$

– Primitive domain (PD) $\subsetneq \mathbb{C}^{(n-1)D}$

# Consequences

PD $\not\ni$ on-shell external momenta

Holomorphic extension of primitive domain includes on-shell external momenta $\implies$ can compute physical amplitudes

Properties of a subregion of PD $\implies$ crossing symmetry of amplitudes for $2 \to 2$ scattering [Bros, Epstein, Glaser (1965)]

Analytic extension of the same subregion leads to JLD domain in one $P_I$ variable keeping other independent variables real at forward lightcone [Bros, Messiah, Stora (1961)]

These results are due to property of PD, irrespective of functions that are analytic on it

# Green's functions in closed SFT: recent results

Action directly written in momentum space. Has non-local vertices. No position space construction known

Work with perturbative expansion of $G(\overbrace{p_1, \ldots, p_n}^{p})$. Consider any Feynman diagram $\mathrm{F}(p)$ with $n$ amputated external legs and which $\not\supset$ any massless internal propagator

$\mathrm{F}(p)$ is analytic on a domain defined as

C1, C2, and C3: each $\mathrm{Im}\ p_a$ lies on a 2d Lorentzian plane $p^0 - p^1$ if $\mathrm{Im}\ p_a \neq 0$

– de Lacroix, Erbin, Sen (2018)

PD $\supsetneq$ LES domain $\implies$ crossing symmetry, JLD domain

# Green's functions in closed SFT: recent results

Complex matrix $\tilde{\Lambda}$, s.t. $\tilde{\Lambda}^T \eta \tilde{\Lambda} = \eta =$ Minkowski metric in $\mathbb{R}^D$

Define its action on $p$ as $\tilde{\Lambda} p \equiv (\tilde{\Lambda} p_1, \ldots, \tilde{\Lambda} p_n)$ – complex Lorentz transformation

2 corollaries from the LES paper:

Cor1. $\mathrm{F}(p)$ remains analytic at $\tilde{\Lambda} p$, if $p \in$ LES domain

Cor2. at each $p \in$ LES domain, we can allow a small open neighbourhood where $\mathrm{F}(p)$ remains analytic

Q. Are $\mathrm{F}(p)$ analytic on the full primitive domain?

# Outline of our strategy

$\mathrm{F}(p)$ is analytic on extended LES domain (LESD+Cor1+Cor2). No further input from SFT

Known that, PD $\supset$ union of a family of tube domains $\mathcal{T}_\lambda$, $\lambda$: some index

Primitive tube: $\mathcal{T}_\lambda = \mathbb{R}^{(n-1)D} + i\mathcal{C}_\lambda$, $\mathcal{C}_\lambda$: cone in $\mathbb{R}^{(n-1)D}$

In $\mathcal{T}_\lambda$, identify an open tube (call: LES tube) from (LESD+Cor1+Cor2)

We prove that LES tube admits non-trivial holomorphic extension in $\mathcal{T}_\lambda$ for $n \geq 3$, due to *Bochner*

possibly obtaining full of $\mathcal{T}_\lambda$? explicitly analyse when $n = 3, 4, 5$

# Primitive tube, $\mathcal{T}_\lambda$

A specific way to divide space $\mathbb{R}^{n-1}$ into finite no. of cells

– each of which defines a sign-valued map: $\lambda(I),\ I \subsetneq \{1, \ldots, n\} \setminus \emptyset$

$$\begin{aligned}\mathcal{T}_\lambda &= \left\{ p \equiv (p_1, \ldots, p_n) :\ \sum_{a=1}^{n} p_a = 0,\ \lambda(I)\mathrm{Im}\ P_I \in V^+ \quad \forall I \right\} \\ &= \mathbb{R}^{(n-1)D} \ +\ i \underbrace{\left\{ \mathrm{Im}\ p \in \mathbb{R}^{(n-1)D} :\ \lambda(I)\mathrm{Im}\ P_I \in V^+ \quad \forall I \right\}}_{\equiv\, \mathcal{C}_\lambda}\end{aligned} \tag{1}$$

$V^+$: open forward lightcone in $\mathbb{R}^D$; $\mathcal{C}_\lambda$: called the base of $\mathcal{T}_\lambda$ [Araki, Burgoyne '60; Bros, Epstein, Glaser '64; Lassalle '74; Bros, Lassalle '75]

For $n = 3$, # of cells $= 6$; for $n = 4$ it is 32; for $n = 5$ it is 370, etc

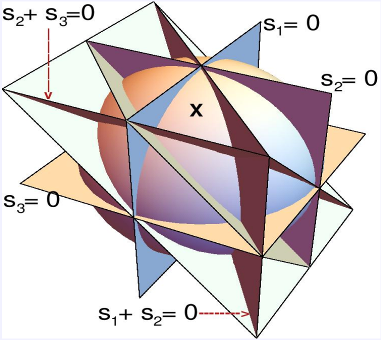
# $\lambda(I)$: more details

$$\mathbb{R}^{n-1} = \left\{ s \equiv (s_1, \ldots, s_n) : \sum_{a=1}^{n} s_a = 0 \right\} ;$$

$$S_I \equiv \sum_{a \in I} s_a, \quad I \subsetneq \{1, \ldots, n\} \setminus \emptyset$$

Family of planes $\{S_I = 0\}$ divides $\mathbb{R}^{n-1}$ into open convex cones with common appex at origin. $S_I = 0$, $S_{X \setminus I} = 0$ are same plane. Each such cone is called a cell, $\gamma_\lambda$. Inside a cell each $S_I$ is of definite sign $\lambda(I)$

E.g., for $n = 4$, $\mathbb{R}^3$ is devided into 32 cells. One such cell has been marked in the figure



# LES tube, $\cup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$

$\vec{\theta}$: set of $D-2$ angles specifying a 2d Lorentzian plane $p^0 - p^{\vec{\theta}}$, $\vec{\theta} = 0$ specifying $p^0 - p^1$ plane

$$\mathcal{T}_\lambda^{\vec{\theta}} = \Big\{ p \equiv (p_1, \ldots, p_n) : \forall a \ \mathrm{Im}\ p_a \in \ p^0 - p^{\vec{\theta}} \text{ plane; } \sum_{a=1}^n p_a = 0, \\ \text{and } \lambda(I)\mathrm{Im}\ P_I \in V^+ \quad \forall I \Big\}$$

$$= \mathbb{R}^{(n-1)D} + i\mathcal{C}_\lambda^{\vec{\theta}} \quad ,$$

$$\mathcal{C}_\lambda^{\vec{\theta}} = \Big\{ (\mathrm{Im}\ p_1, \ldots, \mathrm{Im}\ p_n) \text{ on manifold } \sum_{a=1}^n \mathrm{Im}\ p_a = 0 \text{ such that} \\ \forall a \ \mathrm{Im}\ p_a \in \ p^0 - p^{\vec{\theta}} \text{ plane and } \lambda(I)\mathrm{Im}\ P_I \in V^+ \quad \forall I \Big\} \tag{2}$$

$\mathcal{C}_\lambda^{\vec{\theta}} \subset \mathbb{R}^{(n-1)D}$. $\mathcal{T}_\lambda^{\vec{\theta}}$ can be obtained by acting a real rotation on $\mathcal{T}_\lambda^{\vec{\theta}=0}$.
Analyticity of $\mathrm{F}(p)$ on $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ – guaranteed by Cor1

# Extension of LES tube

$\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ is given by $\mathbb{R}^{(n-1)D} + i(\bigcup_{\vec{\theta}} \mathcal{C}_\lambda^{\vec{\theta}}) \subset \mathcal{T}_\lambda$

Although each $\mathcal{T}_\lambda^{\vec{\theta}}$ is convex, we prove: $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ is non-convex when $n > 2$

Furthermore, we prove: $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ is path-connected

LES tube can be thickened in order to make it open. To holomorphically extend $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$, we apply Bochner's tube theorem

**Bochner (1937)**

Any open connected tube $\mathbb{R}^m + iA$ has a holomorphic extension to the domain $\mathbb{R}^m + i\text{Ch}(A)$

$\text{Ch}(A)$: smallest convex set containing $A$, called the convex hull of $A$

# Extension of LES tube

Extension of $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ to $\mathbb{R}^{(n-1)D} + i\mathrm{Ch}(\bigcup_{\vec{\theta}} \mathcal{C}_\lambda^{\vec{\theta}})$ is non-trivial when $n > 2$

$\mathcal{C}_\lambda \supset \mathrm{Ch}(\bigcup_{\vec{\theta}} \mathcal{C}_\lambda^{\vec{\theta}})$, since $\mathcal{C}_\lambda$ is a convex cone containing $\bigcup_{\vec{\theta}} \mathcal{C}_\lambda^{\vec{\theta}}$

For 3-, 4-point functions, for each $\mathcal{C}_\lambda$ above extension yields the full of $\mathcal{C}_\lambda$, i.e. $\mathrm{Ch}(\bigcup_{\vec{\theta}} \mathcal{C}_\lambda^{\vec{\theta}}) = \mathcal{C}_\lambda$, and for 5-point functions we obtained subcases in which we are able to prove this equality

For 2-point functions, $\mathcal{T}_\lambda = \bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ . That is, whenever $p \in \mathcal{T}_\lambda$, $\mathrm{Im}\ p_1$ $(= -\mathrm{Im}\ p_2)$ lies on some 2d Lorentzian plane

# Four-point functions

## Primitive tubes, 32 in numbers

$$\mathcal{T}_a^{\pm} = \left\{ p \in \mathbb{C}^{3D} : \ \mathrm{Im}\ p \in \mathcal{C}_a^{\pm} \right\}, \ \mathcal{T}_{ab}^{\pm} = \left\{ p \in \mathbb{C}^{3D} : \ \mathrm{Im}\ p \in \mathcal{C}_{ab}^{\pm} \right\}, \quad (3)$$

$p = (p_1, \ldots, p_4)$ linked by $p_1 + \cdots + p_4 = 0, \ \ 1 \leq a, b \leq 4, \ a \neq b$

## Conical bases

$$\mathcal{C}_a^{+} = -\mathcal{C}_a^{-} = \left\{ \mathrm{Im}\ p : \ \mathrm{Im}\ p_b, \ \mathrm{Im}\ p_c, \ \mathrm{Im}\ p_d \in V^{+} \right\}$$
$$\mathcal{C}_{ab}^{+} = -\mathcal{C}_{ab}^{-} = \left\{ \mathrm{Im}\ p : \ -\mathrm{Im}\ p_b, \ \mathrm{Im}\ (p_b + p_c), \ \mathrm{Im}\ (p_b + p_d) \in V^{+} \right\} \quad (4)$$

$(abcd)$ = a permutation of $(1234)$

For each of 32 bases, the no. of $\mathrm{Im}\ P_I$ that are required to be in specific lightcones $= 3$. This in turn, determines all other $\mathrm{Im}\ P_I$ be in specific lightcones for a base

# Four-point functions

$$(\mathbf{p_b},\ \mathbf{p_c},\ \mathbf{p_d}) \longleftrightarrow (-\mathbf{p_b},\ \mathbf{p_b}+\mathbf{p_c},\ \mathbf{p_b}+\mathbf{p_d})$$

$\nwarrow$ INVERTIBLE CHANGE OF BASIS

$\mathrm{Im}\ p_1 + \cdots + \mathrm{Im}\ p_4 = 0 \implies$ each of 32 bases $\subset \mathbb{R}^{3D}$. They can be brought to a common form: $\mathcal{C} = \{\vec{Q} :\ P_\alpha, P_\beta, P_\gamma \in V^+\}$

$\mathcal{C} \ni \vec{Q} = (P_\alpha, P_\beta, P_\gamma)$ can be written as a $D \times 3$ matrix:

$$\vec{Q} \ = \begin{pmatrix} P_\alpha^0 & P_\beta^0 & P_\gamma^0 \\ P_\alpha^1 & P_\beta^1 & P_\gamma^1 \\ \vdots & \vdots & \vdots \\ P_\alpha^{D-1} & P_\beta^{D-1} & P_\gamma^{D-1} \end{pmatrix}, \qquad (5)$$

where $P_r^0 > +\sqrt{\sum_{i=1}^{D-1}(P_r^i)^2} \quad \forall r = \alpha, \beta, \gamma$

# $\mathrm{Ch}(\cup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}) = \mathcal{T}_\lambda$ holds

Consider $\vec{Q}_1 \in \mathcal{C}^{\vec{\theta}_1}$, $\vec{Q}_2 \in \mathcal{C}^{\vec{\theta}_2}$, $\vec{Q}_3 \in \mathcal{C}^{\vec{\theta}_3}$ as follows

$$\begin{aligned} \vec{Q}_1 &= 3 \begin{pmatrix} P_\alpha^0 - \epsilon & \epsilon/2 & \epsilon/2 \\ P_\alpha^1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ P_\alpha^{D-1} & 0 & 0 \end{pmatrix}, \quad \vec{Q}_2 = 3 \begin{pmatrix} \epsilon/2 & P_\beta^0 - \epsilon & \epsilon/2 \\ 0 & P_\beta^1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & P_\beta^{D-1} & 0 \end{pmatrix}, \\ \vec{Q}_3 &= 3 \begin{pmatrix} \epsilon/2 & \epsilon/2 & P_\gamma^0 - \epsilon \\ 0 & 0 & P_\gamma^1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & P_\gamma^{D-1} \end{pmatrix} \end{aligned} \tag{6}$$

Choose: $0 < \epsilon < \min\left\{P_r^0 - \sqrt{\sum_i (P_r^i)^2},\ r = \alpha, \beta, \gamma\right\}$. Implies for each $r$ $P_r^0 - \epsilon > +\sqrt{\sum_{i=1}^{D-1} (P_r^i)^2}$. And, we have

$$\frac{\vec{Q}_1}{3} + \frac{\vec{Q}_2}{3} + \frac{\vec{Q}_3}{3} = \vec{Q} \tag{7}$$

# Five-point functions, no. of primitive tubes = 370

Bases for 20 of them are: $\mathcal{C}_{ab}^{(5)+}$, $\mathcal{C}_{ab}^{(5)-} = -\mathcal{C}_{ab}^{(5)+}$

$$\mathcal{C}_{ab}^{(5)+} = \Big\{ \mathrm{Im}\; p : \; \mathrm{Im}\,(p_a + p_c),\; \mathrm{Im}\,(p_a + p_d),\; \mathrm{Im}\,(p_a + p_e), \\ \mathrm{Im}\,(p_b + p_c),\; \mathrm{Im}\,(p_b + p_d),\; \mathrm{Im}\,(p_b + p_e) \in V^+ \Big\} \quad (8)$$

$p = (p_1, \ldots, p_5)$ linked by $p_1 + \cdots + p_5 = 0$

$(abcde)$= a permutation of (12345)

$\mathcal{C}_{ab}^{(5)+} \ni \vec{Q}$ will have 4 independent $D$-momenta. The no. of $\mathrm{Im}\; P_I$ that are required to be in specific lightcones = 6

Formula to decompose a general $\vec{Q}$ into terms, each from $\bigcup_{\vec{\theta}} \mathcal{C}_{ab}^{(5)+,\vec{\theta}}$ is still missing *ongoing work in progress*

# Five-point functions: remaining tubes *work in progress*

– can be brought to the form: $\mathcal{C}^{(5)+} =$

$$\left\{ \vec{Q} = (P_1, P_2, P_3, P_4) :\ P_1,\ P_2,\ P_3,\ P_4,\ \underbrace{P_2 + P_4 - P_1}_{P_5},\ \underbrace{P_3 + P_4 - P_1}_{P_6} \in V^+ \right\}$$

Here, $P_i - P_j,\ i \neq j$ are timelike/null/spacelike

$\vec{Q}:\ P_1 - P_4$ is timelike or null. Then $\vec{Q}$ can be decomposed into terms, each of which belongs to the LES tube

$\vec{Q}:\ P_1 - P_4$ is spacelike. Then go to a Lorentz frame s.t. $P_1^0 - P_4^0 = 0$

Now, $s_r = P_r^0 - \|\mathbf{P}_r\| > 0,\ r = 1, \ldots, 6;\ l_1 = \|\mathbf{P}_1 - \mathbf{P}_4\| \geq 0$; and various inequalities among them – allowed primitive space in $(s_1, \ldots, s_6, l_1)$

Various decompositions of $\vec{Q}$ into points from LES tube are found covering various parts of allowed region. Searching left-over points, if any ...

# Summary and outlook

Extended the LES domain to a larger subset of the primitive domain, holomorphically. For 2-, 3- and 4-point functions in closed SFT, the full primitive domain is recovered. For 5-point functions, analysed limited subcases. More complete analysis – *work in progress*

Rather dealing case by case for all $n$, we need to automate the analysis in a computer to carry out the proof for arbitrary $n$-point functions

Superstring amplitudes might have potential singularities on primitive domain arising from non-perturbative effects. Local QFTs are free from those. Will be interesting to find any differences in context of analyticity

Holomorphic envelope, i.e. largest analytic extension of PD – still remains a challenging open problem since Bros, Messiah, Stora (1961)

# THANK YOU!