

L_∞ -Algebras, Homotopy Transfer

and Field Theory

Plan:

lecture 1 : L_∞ -algebras

lecture 2 : Homotopy Transfer

lecture 3 : Field theory



References: • with Barton, 1701.08824

• with Arvanitakis, Hull, Lekan,

2007. 07924, 2106. 08343

Motivation | Big Picture:

Yang-Mills gauge field

1-form: $A = A_\mu dx^\mu = \underbrace{A_\mu{}^a t_a}_{A_\mu \in \mathfrak{g}} dx^\mu$

$A_\mu \in \mathfrak{g}$ (lie algebra)

0-form: $\Lambda = \Lambda^a t_a$

$\delta A = d\Lambda + [A, \Lambda]$

2-form:

Curvature

$$F = dA + \frac{1}{2} \underbrace{[A, A]}_{[A_\mu, A_\nu] dx^\mu \wedge dx^\nu}$$

→ differential graded lie algebra : (dgLa)

• chain complex (X, d) ; X : graded vector space

$$\dots \rightarrow X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} X_2 \xrightarrow{d_2} \dots$$

$$\underline{d^2 = 0}$$

$$(d_1 \circ d_0 = 0, d_2 \circ d_1 = 0, \dots)$$

- graded antisymmetric bracket $\underline{[\cdot , \cdot]}$

$$\underline{\deg([x_1, x_2]) = \deg(x_1) + \deg(x_2)}$$

$$\underline{[x_1, x_2] = (-1)^{1+x_1 x_2} [x_2, x_1]}$$

(e.g. $A \in X_1$: $[A, A] \in X_2$ symmetric)

so that

i) $d^2 = 0$

ii) $d([x, y]) = [dx, y] + (-1)^x [x, dy]$
(Leibniz)

iii) $\Gamma[x, y] z] + (-1)^{(x+y)z} \Gamma[z, x] y]$

$$+ (-1)^{(y+z)x} [[y, z], x] = 0$$

(Jacobi)

→ Chern - Simons - theory in 2+1 dimensions:

$$I_{CS} = \int_{M_3} \langle A, dA + \frac{1}{3} [A, A] \rangle$$

$$\rightarrow E.O.M.: F(A) := dA + \frac{1}{2} [A, A] = 0$$

(Maurer - Cartan)

uses only dgLa (plus inner product)

Main Message:

Any (classical) field theory encoded
in L_∞ -algebra (generalization of dgLa)

E.o.M., gauge transformations, etc. encoded in
higher brackets, e.g.

$$0 = \partial A + \frac{1}{2} [A, A] + \frac{1}{3!} [A, A, A] + \dots$$

L_∞ -algebras:

Comment: L_∞ -algebras closely related to
classical limit of Batalin-Vilkovisky (BV) formalism,
but with less structure, no extra fields, ...
[co-derivation vs. derivation/vector field
co-algebra vs. algebra]

Def.: L_∞ -algebra is graded vector space

$$X = \bigoplus_{i \in \mathbb{Z}} X_i, \quad \begin{array}{l} x \in X_i \\ \deg(x) = i \end{array}$$

equipped with multi-linear,
graded symmetric brackets

$$[x_1, \dots, x_n] = b_n(x_1, \dots, x_n) \in X_{\overbrace{\sum_i |x_i| - 1}}$$

$$n = 1, 2, 3, \dots$$

satisfying generalized Jacobi identities:

("b-picture convention")

$$\underbrace{b}_1, \underbrace{[\cdot, \cdot]}_2 = b_2, \dots$$

$$\mathcal{D} := b_1 + b_2 + b_3 + \dots$$

$$\boxed{D^2 = 0}$$

(as coderivation on graded symmetric coalgebra)

Def.: Graded symmetric tensor algebra

$$\boxed{S = S(X) \equiv \bigoplus_{n=1}^{\infty} S^n X}$$

$$\boxed{S^1 X \equiv X}$$

$$S^2 X : \quad \overbrace{x_1 \wedge x_2}^{x_1 x_2} = (-1)^{x_1 x_2} \overbrace{x_2 \wedge x_1}^{x_2 x_1}$$

⋮

$$S^n X : \quad x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma; x) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}$$

$\underbrace{\quad \quad \quad}_{\text{Koszul sign for permutation } \sigma}$
 $\underbrace{\quad \quad \quad}_{\text{of } n \text{ objects } x_1, \dots, x_n}$

Walgebra: ω product $\Delta : S \xrightarrow{\quad} S \otimes S$
 $\Delta(x) = 0$ $(-i)^x \equiv (-1)^{\deg(x)}$

$* \Delta(x_1 \wedge x_2) = x_1 \otimes x_2 + (-1)^{x_1 x_2} x_2 \otimes x_1$

$\Delta(x_1 \wedge x_2 \wedge x_3) = \underbrace{x_1 \otimes (x_2 \wedge x_3)}_{+ (-1)^{x_1 x_2} x_2 \otimes (x_1 \wedge x_3)} + (-1)^{x_3 (x_1 + x_2)} x_3 \otimes (x_1 \wedge x_2)$

$+ (x_1 \wedge x_2) \otimes x_3 \pm (2 \text{ terms})$

On general monomial:

$$\Delta(x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in \{i, n-i\}} \underline{e(\sigma; x)}.$$

$$\cdot (x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(i)}) \otimes (x_{\sigma(i+1)} \wedge \cdots \wedge x_{\sigma(n)})$$

where $\sigma(1) \leq \cdots \leq \sigma(i)$, $\sigma(i+1) \leq \cdots \leq \sigma(n)$

"unshuffles"

Prop.: Δ is cassociative:

$$S \xrightarrow{\Delta} S \otimes S$$

$$S \otimes S \xrightarrow{1 \otimes \Delta} S \otimes S \otimes S$$

$$(\Delta \otimes 1) \Delta = (1 \otimes \Delta) \Delta$$

commutes:

"dual" to associativity of product

$$m: S \otimes S \rightarrow S :$$

$$\underline{m(1 \otimes m)} = m(m \otimes 1)$$

Def.: A coderivation is a map

$\textcircled{1}: S \rightarrow S$ satisfying

co-Leibniz (w.r.t. coproduct)

$$\Delta \mathcal{D} = (1 \otimes \mathcal{D} + \mathcal{D} \otimes 1) \Delta$$

dual to derivation d w.r.t. product m :

$$dm = m(1 \otimes d + d \otimes 1)$$

Def.: An L_∞ -algebra is a \mathbb{Z} -graded vector space X with derivation

$$\mathcal{D} = \sum_{i=1}^{\infty} b_i \text{ of degree } -1$$

so that

$$D^2 = 0.$$

Examples for action of b_i :

$$\underline{b_1(x)}$$

$$b_2(x) = 0, \quad b_3(x) = 0, \quad \dots$$

$$\rightarrow \boxed{b_1(x \wedge y) = b_1(x) \wedge y + (-1)^x \times \wedge b_1(y)}$$

(in order to obey $\Delta b_1 = (1 \otimes b_1 + b_1 \otimes 1) \Delta$)

$$\rightarrow b_2(x \wedge y) \equiv b_2(x, y) \equiv [x, y] \quad \checkmark$$

$$\rightarrow b_2(x_1 \wedge x_2 \wedge x_3) = \underbrace{b_2(x_1, x_2) \wedge x_3}_{+} + \underbrace{(-1)^{(x_2+x_3)x_1} b_2(x_2, x_3) \wedge x_1}_{+}$$

$$b_2^2(x_1 \wedge x_2 \wedge x_3) = b_2(b_2(x_1, x_2), x_3) + (-1)^{(x_1+x_2)x_3} b_2(x_3, x_1) \wedge x_2$$

In general:

$$b_i : S^j X \longrightarrow \underbrace{S^{j-i+1} X}_{+ \dots}$$

trivial for $j < i$

$$b_i(x_1 \wedge \dots \wedge x_j) = \sum_{\sigma \in (i, j-1)} \epsilon(\sigma, x) b_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}) \wedge x_{\sigma(i+1)} \wedge \dots \wedge x_{\sigma(j)}$$

Explicit L_∞ -relations:

$$\mathcal{D}^2 = 0 \quad \text{for} \quad \mathcal{D} = b_1 + b_2 + b_3 + \dots$$

$$(i) \quad b_1^2 = 0$$

$$(ii) \quad b_1 b_2 + b_2 b_1 = 0$$

$$(iii) \quad \underline{b_2^2 + b_1 b_3 + b_3 b_1 = 0}$$

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$$\underline{(i) \Rightarrow \mathcal{D} = b_1 \text{ nilpotent } \mathcal{D}^2 = 0}$$

$$(ii) \Rightarrow 0 = (\underline{b_1 b_2} + \underline{b_2 b_1}) (x_1 \wedge x_2)$$

$$= \underline{b_1 (b_2 (x_1, x_2))} + \underline{b_2 (b_1 (x_1) \wedge x_2 + (-1)^{x_1} x_1 \wedge b_1 (x_2))}$$

$$\begin{aligned}
 (\text{iii}) \Rightarrow 0 &= \underbrace{b_2(b_2(x_1, x_2), x_3)}_{+ b_1(b_3(x_1, x_2, x_3))} \pm (\text{2 terms}) \\
 &\quad + b_3(b_1(x_1), x_2, x_3) \pm (\text{2 terms})
 \end{aligned}$$

$\Rightarrow b_2 = [\cdot, \cdot]$ need not to obey Jacobi

$\Rightarrow L_\infty$ is generalization of dgLa.

L_∞ morphisms:

Given vector spaces X and X' with L_∞ structures

an L_∞ morphism is a collection of
graded symmetric maps

* $F = (f_1, f_2, f_3, \dots) : S(X) \rightarrow S(X')$

of intrinsic degree zero so that

1) $\boxed{\Delta' F = (F \otimes F) \Delta}$

(morphism of coalgebras)

2) $D' F = F D$

(respects comultiplication)

1) determines how f_1, f_2, f_3 extend to higher monomials:

$$F(x) = f_1(x)$$

$$\rightarrow F(x_1 \wedge x_2) = f_2(x_1, x_2) + f_1(x_1) \wedge f_1(x_2)$$

⋮

check:

$$\begin{aligned}(F \otimes F) \Delta(x_1 \wedge x_2) &= (F \otimes F)(x_1 \otimes x_2 + (-1)^{x_1 x_2} x_2 \otimes x_1) \\&= f_1(x_1) \otimes f_1(x_2) + (-1)^{x_1 x_2} f_1(x_2) \otimes f_1(x_1) \\&= \Delta' F(x_1 \wedge x_2)\end{aligned}$$

since $\Delta'(f_2(x_1, x_2)) = 0$

Def.: A cyclic L_∞ algebra is one equipped with inner product

$$\kappa: X \times X \rightarrow \mathbb{R} \quad \text{so that}$$

$$\kappa(x_1, b_n(x_2, x_3, \dots, x_{n+1})) = (-1)^{x_1 x_2} \kappa(x_2, b_n(x_1, x_3, \dots, x_{n+1}))$$

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i.e. is graded symmetric.