

Lecture 2:

L_∞ & Field Theory

Dictionary:

chain complex:

$$\dots \rightarrow X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\partial_0} X_{-1} \rightarrow \dots$$

gauge
parameters
 $\{\lambda\}$

fields
 $\{A\}$

EoM

E.o.M:

$$0 = \partial A + \frac{1}{2} [A, A] + \frac{1}{3!} [A, A, A] + \dots$$

Yang-Mills transformans.

$$S_{\Lambda, \mathcal{A}} = \partial \Lambda + [\Lambda, \mathcal{A}] + \underbrace{\frac{1}{2} [\Lambda, \mathcal{A}, \mathcal{A}]}_{\dots} + \dots$$

Action:

$$I = \frac{1}{2} \langle \mathcal{A}, \partial \mathcal{A} \rangle + \frac{1}{3!} \langle \mathcal{A}, [\mathcal{A}, \mathcal{A}] \rangle + \dots$$

Example 1: ϕ^3 - theory

$$I = \int d^4x \left(-\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{3!} \phi^3 \right)$$

$$0 \xrightarrow{\partial_1=0} X_0 \xrightarrow{\partial_0} X_{-1} \xrightarrow{\partial_{-1}=0} 0$$

$\{\phi\}$

E.o.M

$$\partial(\phi) = (\square - m^2)\phi \in X_{-1}$$

2-bracket: $[\phi_1, \phi_2] = -2\lambda \phi_1 \phi_2 \in X_{-1}$

\Rightarrow e.o.m.

$$\boxed{\partial\phi + \frac{1}{2} [\phi, \phi] = 0}$$

L_∞ relations:

$$\partial^2 = 0 \quad \checkmark$$

Leibniz: $\partial([\phi_1, \phi_2]) \stackrel{!}{=} -[\partial\phi_1, \phi_2] - [\phi_1, \partial\phi_2]$

$\underbrace{X_{-1}}_{0} \qquad \underbrace{-1 \quad 0}_{=0} \qquad \underbrace{0 \quad -1}_{=0}$

Solvability [Integrability]

$$\partial A + \frac{1}{2} g [A, A] + \frac{1}{3!} g^2 [A, A, A] + \dots = 0$$

ansatz: $A = A^{(0)} + g A^{(1)} + g^2 A^{(2)} + \dots$

$$\theta(g^0) : \underline{\partial(A^{(0)}) = 0}$$

$$\theta(g^1) : \boxed{\partial(A^{(1)}) + \frac{1}{2} [A^{(0)}, A^{(1)}] = 0} //$$

$$\theta(g^2) : \boxed{\partial(A^{(2)}) + [A^{(0)}, A^{(1)}] + \frac{1}{3!} [A^{(0)}, A^{(0)}, A^{(1)}] = 0}$$

$$\theta(g^1) \quad \underbrace{\partial^2(A^{(1)})}_{\equiv 0} + \frac{1}{2} \underbrace{\partial([A^{(0)}, A^{(0)}])}_{\begin{array}{c} \\ \parallel \\ = 0 \end{array}} + \frac{1}{1!} \underbrace{-2[\partial A^{(0)}, A^{(0)}]}_{\begin{array}{c} \\ \parallel \\ = 0 \end{array}}$$

$$\theta(g^2) : 0 = \underbrace{\partial^2 A^{(2)}}_{\equiv 0} + \underbrace{\partial[A^{(0)}, A^{(1)}]}_{\begin{array}{c} \\ \parallel \\ = 0 \end{array}} + \frac{1}{2!} \underbrace{\partial[A^{(0)}, A^{(0)}, A^{(0)}]}_{\begin{array}{c} \\ \parallel \\ = 0 \end{array}}$$

$$- [\underbrace{\partial A^{(0)}, A^{(1)}}_{\begin{array}{c} \\ \parallel \\ = 0 \end{array}}]$$

$$- [A^{(0)}, \underbrace{\partial A^{(1)}}_{\begin{array}{c} \\ \parallel \\ = 0 \end{array}}]$$

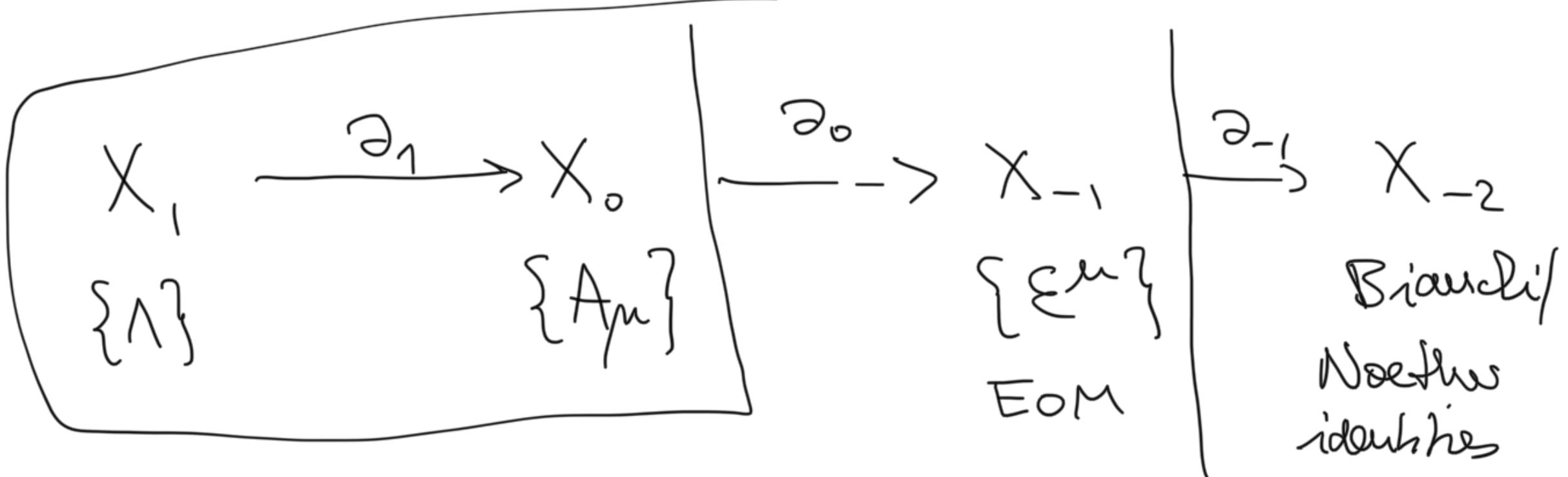
Jacobi
↓

$$= -\frac{1}{2} \left[\partial_{\mu} A^{(0)}, A^{(0)}, A^{(0)} \right]$$

$$= 0 \quad . \quad /$$

$$= \frac{1}{2} [A^{(0)}, [A^{(0)}, A^{(0)}]]$$

Example: 2: Yang-Mills theory



$$\times [\partial_1(\Lambda)]_\mu := \partial_\mu \Lambda$$

$$\times [\partial_0(A)]^\mu := \square A^\mu - \partial^\mu (\partial_\nu A^\nu)$$

$$\times [\partial_{-1}(\varepsilon)] := \partial_\mu \varepsilon^\mu$$

$\partial_0 \circ \partial_1 = 0 \Leftrightarrow$ (lin.) gauge invariance

$\partial_{-1} \circ \partial_0 = 0 \Leftrightarrow$ (lin.) Bianchi identity

2-brackets:

$$b_2(\Lambda_1, \Lambda_2) = [\Lambda_1, \Lambda_2] \in X_1$$

$\begin{matrix} 1 & 1 \end{matrix} \Rightarrow$ antisymmetric

$$b_2(A_\mu, \Lambda) = b_2(\Lambda, A_\mu) \equiv [A_\mu, \Lambda] \in X_0$$

0 1

$$b_2(\varepsilon, \Lambda) = -[\varepsilon, \Lambda] \in X_{-1}$$

-1 1

$$b_2(A, A)_\mu = 2 \partial^\nu [A_\nu, A_\mu] + 2 \underbrace{[\partial_\mu A_\nu - \partial_\nu A_\mu, A^\nu]}$$

Checks: $S A_\mu = b_1(\Lambda)_\mu + b_2(\Lambda, A_\mu)$

$$= \partial_\mu \Lambda + [A_\mu, \Lambda] \quad \checkmark$$

E.O.M.: $\circlearrowleft = \partial(A) + \frac{1}{2} b_2(A, A) + \frac{1}{3!} b_3(A, A, A)$

$$\partial_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}]$$

$$\rightarrow \Gamma_A^\mu \Delta^\nu \gamma + \Gamma_A [\partial^\mu A^\nu - \partial^\nu A^\mu]$$

$$= \dots + \partial^\mu [F^\nu, F^\rho] - L'(\mu) \dots$$

$$+ [A_\mu, [A^\mu, A^\nu]]$$

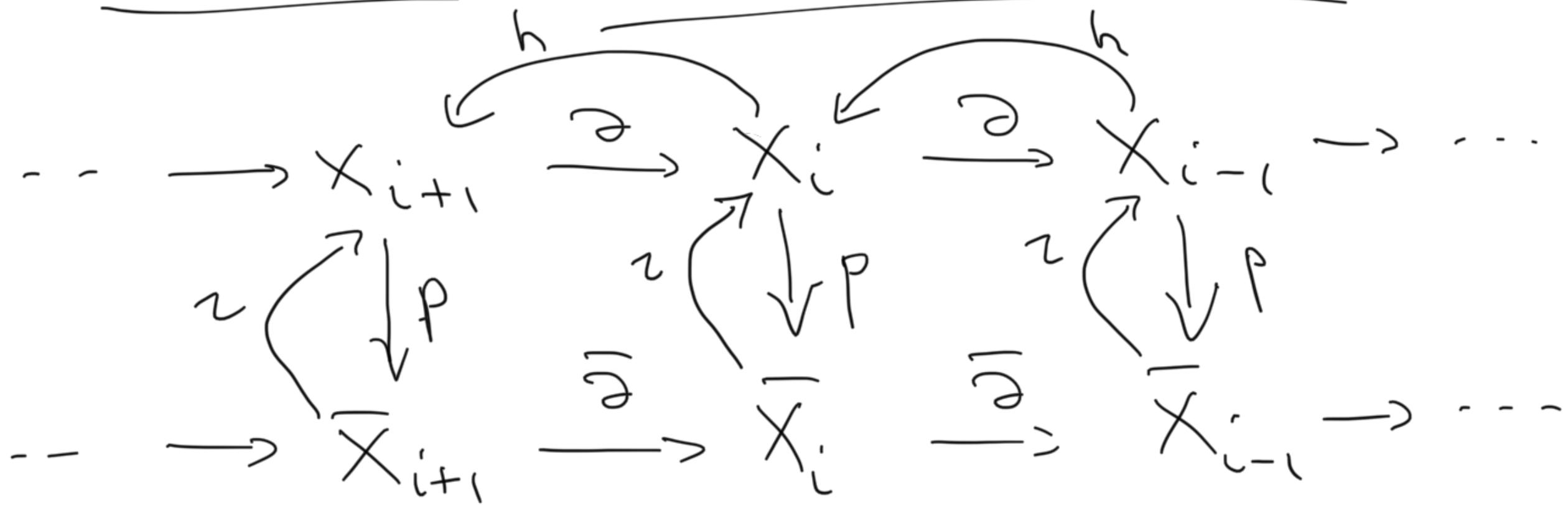
$$\rightarrow \text{3-bracket} : b_3(A_1, A_2, A_3)_\mu$$

$$= [A_1^\nu, [A_{2\nu}, A_{3\mu}]] + 2 \text{ terms}$$

$$F_{\mu\nu}$$

$$\partial_\mu F_{\nu\rho} = 0, \quad \partial^\mu F_{\mu\nu} = 0$$

Lecture 3: Homotopy Transfer



$$P : \boxed{\bar{\partial} P = P \partial}$$

$$P \circ L = id_{\bar{X}}$$

$$L \circ P \neq id_X$$

homotopy equivalence:

$$\boxed{z \circ p = id_X + \partial \circ h + h \circ \bar{\partial}}$$

for degree + 1 maps h

\Rightarrow Homologies equivalent:

$$H_i := \frac{\text{Ver}(\partial_i)}{\text{Um}(\partial_{i+1})} = \left\{ [x] \mid x \in X, \partial x = 0 \right\}$$

$$[x] = [x + \partial \alpha]$$

$$\cong \overline{H_i}$$

Transported L_∞ -brackets:

$$c \quad \vee : \text{dol}_n (\partial, [\cdot, \cdot])$$

say \wedge is "y" - "z"

thus L $_\infty$ algebra on \bar{X}

$\bar{x}, \bar{y} \in \bar{X}$: $x := \gamma(\bar{x}), y := \gamma(\bar{y})$

$$b_2(\bar{x}, \bar{y}) = P([x, y])$$

Jacobians:

$$b_2(b_2(\bar{x}_1, \bar{x}_2), \bar{x}_3) + \text{cycl.}$$

$$= P([\gamma(b_2(\bar{x}_1, \bar{x}_2)), \gamma(\bar{x}_3)]) + \dots$$

$$= P([\gamma P([x_1, x_2]), x_3]) + \dots$$

\nwarrow
 ~~$X + \partial h + h\partial$~~

'by Jaashi

$$= p \left(\underbrace{[\partial(h([x_1, x_2])), x_3]}_{\text{fixed by}} + \underbrace{[h\partial([x_1, x_2]), x_3]}_{\dots} \right)$$

Claim: fixed by

$$b_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) = -p \left(\underbrace{[h([x_1, x_2]), x_3]}_{\text{fixed by}} + \dots \right)$$

$$\begin{aligned} \bar{\partial}(b_3(\bar{x}_1, \bar{x}_2, \bar{x}_3)) &= -p \partial \left[h([x_1, x_2]), x_3 \right] + \dots \\ &= p \left(\underbrace{[\partial h([x_1, x_2]), x_3]}_{\text{fixed by}} + \dots \right) \end{aligned}$$

$$b_3(\bar{\partial}\bar{x}_1, \bar{x}_2, \bar{x}_3) = -p \left(\underbrace{[h([\partial x_1, x_2]), x_3]}_{h\partial[x_1, x_2]} + \dots \right)$$

works to all orders,
proof in terms of contributions

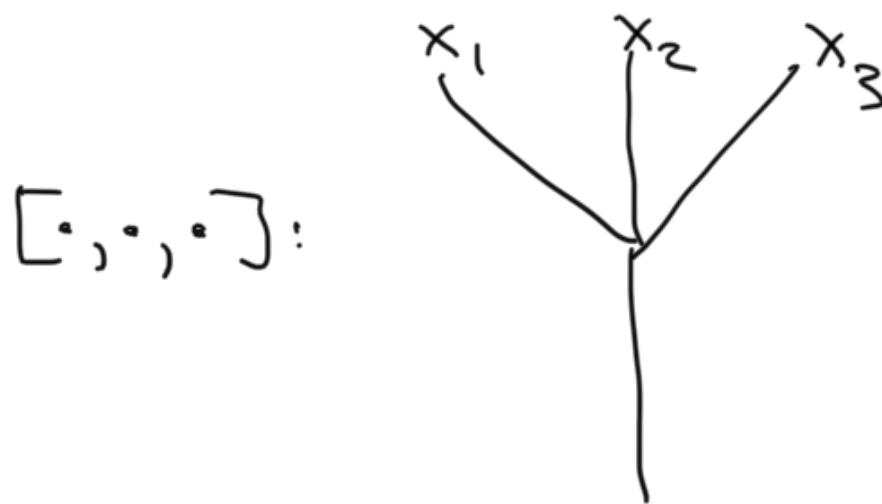
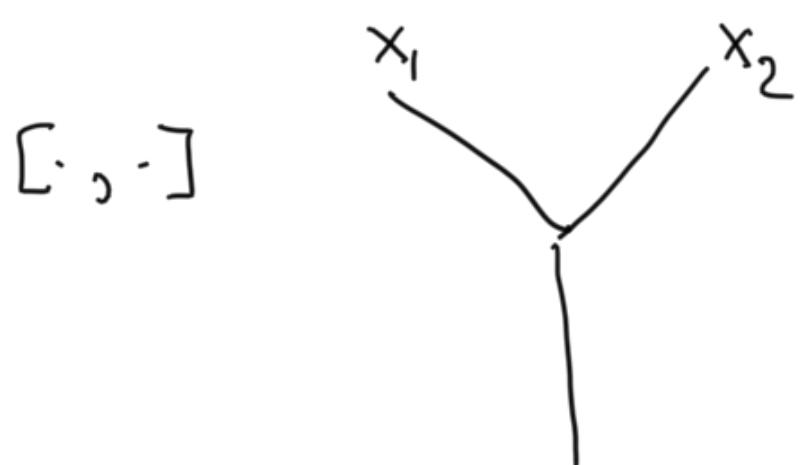
$$\mathcal{D} = \partial + B, \quad B = b_2 + b_3 + \dots$$

$$\mathcal{D}^2 = 0$$

$$\begin{aligned} \rightsquigarrow \bar{\mathcal{D}} &= \bar{\partial} + \bar{B}, \quad \bar{B} = p(B + BhB + BhBhB \\ &\quad + \dots) \\ \bar{\mathcal{D}}^2 &= 0 \end{aligned}$$
$$= p B (-h B)^{-1} L$$

uplift of h ?

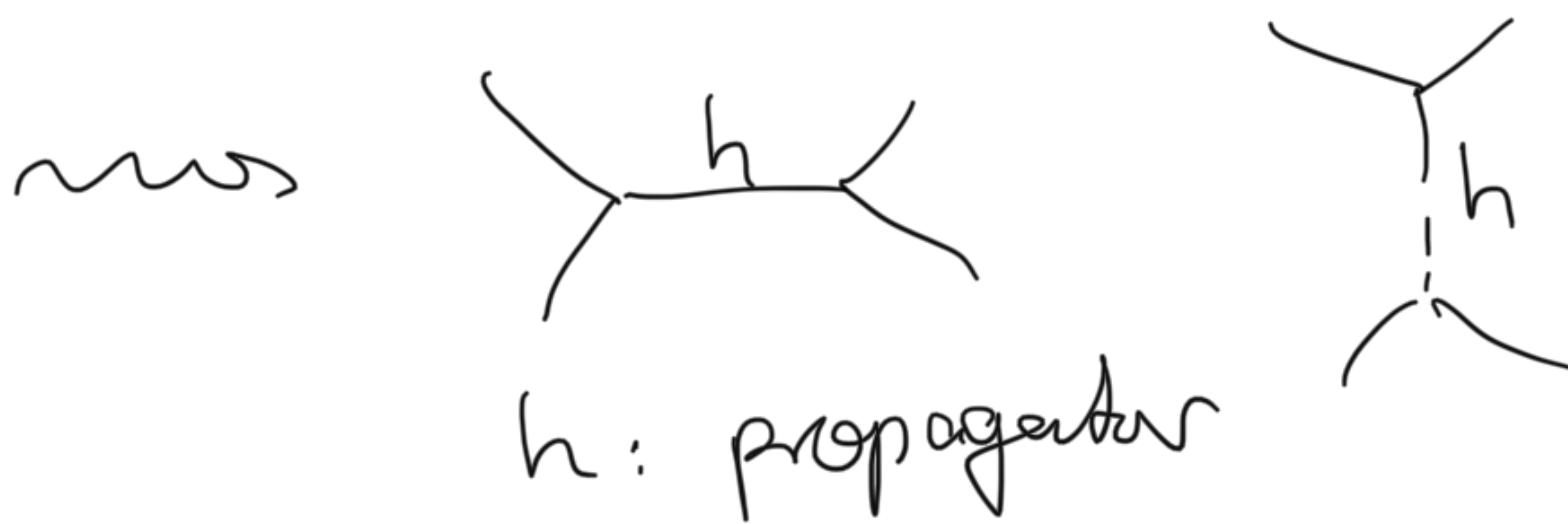
Graphical representations:


$$[x_1, x_2]$$

$$\partial \left(\begin{array}{c} x_1 \\ \diagdown \\ x_3 \\ \diagup \\ x_2 \end{array} \right) = \partial x_1 \begin{array}{c} x_1 \\ \diagdown \\ x_3 \\ \diagup \\ x_2 \end{array} + \begin{array}{c} x_1 \\ \diagdown \\ x_3 \\ \diagup \\ x_2 \end{array}^2 + \begin{array}{c} x_1 \\ \diagdown \\ x_3 \\ \diagup \\ x_2 \end{array}^2$$
$$\equiv 0$$

3-bracket from homotopy transfer:

 $x \quad x_1 \quad x_2$ $x \quad x_1 \quad v$



0D scalar toy model:

$$\phi^i \in \mathbb{R}^n, \quad i = 1, \dots, n$$

$$I(\phi) = \frac{1}{2} A_{ij} \phi^i \phi^j + \frac{\lambda}{3!} A_{ijk} \phi^i \phi^j \phi^k + \dots$$

$$h_0 = 0$$

$$h_{-1}$$

$$c_{n+k}$$

$$0 \rightarrow X_0 \cong \mathbb{R}^n \xrightarrow{\quad} X_{-1} \cong (\mathbb{R}^n)^\perp \rightarrow 0$$

$$(\partial\phi)_i = A_{ij}\phi^j$$

$$[\phi_1, \phi_2]_i = \lambda A_{ijk} \phi^j \phi^k$$

⋮
⋮

split fields: $\phi^i = (\phi^{\bar{i}}, x^u)$

homotopy transfr to $\{\phi^{\bar{i}}\}$ \equiv integrating out x

assume $A_{ij} = \begin{pmatrix} A_{\bar{i}\bar{j}} & 0 \\ 0 & \underline{\underline{A_{uv}}} \end{pmatrix}$

$$P\begin{pmatrix} \phi \\ x \end{pmatrix} = \phi, \quad \iota(\phi) = \begin{pmatrix} \phi \\ 0 \end{pmatrix}$$

$$\begin{aligned}
 (\omega_p - 1)(\begin{pmatrix} \phi \\ x \end{pmatrix}) &= \begin{pmatrix} 0 \\ \cancel{-x} \end{pmatrix} \\
 &\stackrel{!}{=} (\cancel{\partial h} + h \cancel{\partial})(\begin{pmatrix} \phi \\ x \end{pmatrix}) \\
 &= h \begin{pmatrix} A_{\bar{i}\bar{j}} \phi^j \\ A_{uv} x^v \end{pmatrix}
 \end{aligned}$$

$$\Rightarrow h := \boxed{\begin{pmatrix} 0 & 0 \\ 0 & -(A^{-1})^{uv} \end{pmatrix}}$$

2-bracket: $\bar{b}_2(\phi_1, \phi_2)_{\bar{i}} = A_{\bar{i}\bar{j}\bar{k}} \phi_1^j \phi_2^k$ ✓

3-bracket: $[\phi_1, \phi_2] = \lambda A_{ijk} \phi_1^j \phi_2^k$

$$\bar{B}_3(\phi_1, \phi_2, \phi_3)_{\bar{i}} = P\left([h([\phi_1, \phi_2]), \phi_3]\right) + 2 \text{ terms}$$

||

$$\lambda A_{\bar{i}\bar{j}\bar{k}} h([\phi_1, \phi_2])^j \phi_3^{\bar{k}} + \dots$$

||

$$\lambda A_{\bar{i}\bar{j}\bar{k}} h^{jl} \lambda A_{l\bar{m}\bar{n}} \phi_1^{\bar{m}} \phi_2^{\bar{n}} \phi_3^{\bar{k}} + \dots$$

||

$$-3\lambda^2 A_{\bar{i}\bar{k}(u)} (A^{-1})^{uv} A_{\bar{m}\bar{n})v} \phi_1^{\bar{m}} \phi_2^{\bar{n}} \phi_3^{\bar{k}} + \dots$$

integrating out π classically (tree-level):

$$A_{...} \pi^v + \lambda A_{u\bar{i}\bar{k}} \phi^{\bar{j}} \phi^{\bar{k}} = 0$$

$$\Rightarrow \boxed{x^u = -\frac{\lambda}{2} (A^{-1})^{uv} A_{v\bar{i}\bar{j}} \phi^{\bar{i}} \phi^{\bar{j}}}$$

$$\rightsquigarrow S_{\text{quartic}} = -\frac{\lambda^2}{8} A_{\bar{i}\bar{j}u} (A^{-1})^{uv} A_{v\bar{k}\bar{l}} \phi^{\bar{i}} \phi^{\bar{j}} \phi^{\bar{k}} \phi^{\bar{l}}$$

$$\rightsquigarrow \bar{b}_3 = \dots$$