Homotopy Intertwining Solution

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In 1406.3021 and 1909.11675 C. Maccaferri and I developed the so-called intertwining solution which purports to demonstrate the nonperturbative background independence of Witten's open bosonic SFT.

What this means:

1. Witten's theory on a given D-brane supports classical solutions representing all D-brane systems which share the same closed string background.
2. Witten theories formulated on distinct D-branes which share the same closed string background are related by a field transformation which is invertible up to gauge transformation.

Here we describe a formal extension which may apply to more general SFTs whose gauge invariance is based on a homotopy algebras.

The idea is to get a deeper and more general understanding of the structure, and suggest new realizations of the intertwining solution which may be useful.

However, in most situations it will not be realistic to expect that the solution can be made fully explicit. In a few circumstances it can, and the ideas can hopefully be put to the test.

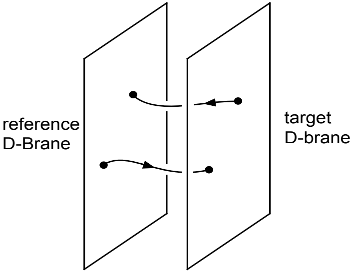
But what is perhaps most interesting is the possibility that such a structure may apply to classical solutions in closed string field theory.

To entertain this idea, it will be necessary to postulate that closed SFT has a tachyon vacuum—a classical solution whose kinetic operator does not support cohomology.

Otherwise, I have tried to be mindful of assumptions in the construction that may distinguish between open and closed strings.

This being said, I will present formulas in the language of A_∞ algebras and the tensor algebra, rather than L_∞ algebras and the symmetrized tensor algebra, since I find this more comfortable. All equations have analogous expression on the symmetrized tensor algebra.

Intertwining solution



We have Witten's SFT on a reference D-brane system, and Witten's SFT on a target D-brane system in the same closed string background. The SFTs are related by the intertwining solution:

$$\Psi^{\text{int}} = \Psi^{\text{tv}} - \Sigma\Psi^{\text{tv}}\overline{\Sigma}$$

- Σ, $\overline{\Sigma}$: intertwining fields
- Ψ^{tv} (in $\Sigma\Psi^{\text{tv}}\overline{\Sigma}$): target tachyon vacuum
- Ψ^{tv}: reference tachyon vacuum
- Ψ^{int}: solution on reference D-brane describing target D-brane

The equations of motion are satisfied with two conditions:

Invariance constraint: $Q^{\text{tv}}\Sigma = 0, \qquad Q^{\text{tv}}\overline{\Sigma} = 0$

Nondegeneracy constraint: $\overline{\Sigma}\Sigma = 1 =$ identity string field

Setup

- Reference background: dynamical field:= $\Psi_{\text{ref}} \in V_{\text{ref}}$
  Target background: dynamical field:= $\Psi_{\text{targ}} \in V_{\text{targ}}$
- String products

$$\begin{array}{rl} \text{on } V_{\text{ref}}: & Q^{\text{pv}} = M_1^{\text{pv}}, \quad M_2^{\text{pv}}, \quad M_3^{\text{pv}}, \quad \ldots \\ \text{on } V_{\text{targ}}: & Q^{\text{pv}} = M_1^{\text{pv}}, \quad M_2^{\text{pv}}, \quad M_3^{\text{pv}}, \quad \ldots \end{array}$$

  defining cyclic A_∞ algebras. The products form coderivations $\mathbf{M}^{\text{pv}}$ satisfying

$$(\mathbf{M}^{\text{pv}})^2 = 0$$
$$\langle\omega|\pi_2\mathbf{M}^{\text{pv}} = 0$$

  The superscript "pv" indicates that these are the A_∞ structures which appear when expanding the theories around the perturbative vacuum $\Psi_{\text{ref}}, \Psi_{\text{targ}} = 0$.
- Tachyon vacuum solutions $\Psi^{\text{tv}} \in V_{\text{ref}}$ and $\Psi^{\text{tv}} \in V_{\text{targ}}$

If we expand the theories around the tachyon vacuum,

$$\Psi_{\text{ref}} = \Psi^{\text{tv}} + \text{fluctuation}$$
$$\Psi_{\text{targ}} = \Psi^{\text{tv}} + \text{fluctuation}$$

we obtain another pair of cyclic A_∞ algebras given by string products

$$Q^{\text{tv}} = M_1^{\text{tv}}, \quad M_2^{\text{tv}}, \quad M_3^{\text{tv}}, \quad \ldots$$

The relation to the products of the perturbative vacuum is given by a cohomomorphism $\mathbf{S}$ called the shift:

$$\mathbf{M}^{\text{tv}} = \mathbf{S}^{-1}\mathbf{M}^{\text{pv}}\mathbf{S}$$

A key assumption in the following, and a defining feature of the tachyon vacuum, is that its kinetic operator Q^{tv} does not support cohomology.

Homotopy Intertwining Solution

We look for a solution connecting reference and target backgrounds of the form

$$\Psi^{\text{int}} = \Psi^{\text{tv}} - \mathcal{I}_1^{\text{tv}}(\Psi^{\text{tv}}) + \mathcal{I}_2^{\text{tv}}(\Psi^{\text{tv}}, \Psi^{\text{tv}}) - \mathcal{I}_3^{\text{tv}}(\Psi^{\text{tv}}, \Psi^{\text{tv}}, \Psi^{\text{tv}}) - \dots$$

where $\mathcal{I}_1^{\text{tv}}(\Psi^{\text{tv}})$ is analogous to $\Sigma\Psi^{\text{tv}}\overline{\Sigma}$. In this general context, it seems unrealistic to assume that the series terminates at first order in the tachyon vacuum.

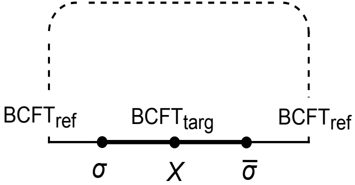
The string products $\mathcal{I}_n^{\text{tv}}$ map the target vector space into the reference vector space:

$$\begin{aligned}
\mathcal{I}_1^{\text{tv}} &: V_{\text{targ}} \to V_{\text{ref}} \\
\mathcal{I}_2^{\text{tv}} &: V_{\text{targ}} \otimes V_{\text{targ}} \to V_{\text{ref}} \\
\mathcal{I}_3^{\text{tv}} &: V_{\text{targ}} \otimes V_{\text{targ}} \otimes V_{\text{targ}} \to V_{\text{ref}} \\
&\vdots
\end{aligned}$$

and are called intertwining products (around the tachyon vacuum).

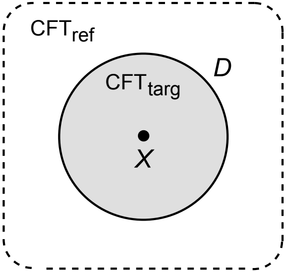
Open string case: boundary condition changing operators

$$I_1^{\text{tv}}(X) \sim$$



Closed string case: conformal defects

$$I_1^{\text{tv}}(X) \sim$$



The intertwining products naturally define the products of a cohomomorphism $\mathbf{I}^{\mathrm{tv}}$:

$$I_n^{\mathrm{tv}} = \pi_1 \mathbf{I}^{\mathrm{tv}} \pi_n$$

In terms of this the solution may be written

$$\Psi^{\mathrm{int}} = \Psi^{\mathrm{tv}} + \pi_1 \mathbf{I}^{\mathrm{tv}} \frac{1}{1 + \Psi^{\mathrm{tv}}}$$

We may define intertwining products around the perturbative vacuum using the shift

$$\mathbf{I}^{\mathrm{pv}} = \mathbf{S}\mathbf{I}^{\mathrm{tv}}\mathbf{S}^{-1}$$

and the solution may be written even more simply as

$$\Psi^{\mathrm{int}} = \mathbf{I}^{\mathrm{pv}} \cdot 1_{TV_{\mathrm{targ}}}$$

where $1_{TV_{\mathrm{targ}}}$ is the identity of the tensor algebra of the target background.

Solving the equations of motion

The homotopy intertwining solution must solve the EOM on the reference background:

$$\pi_1 \mathbf{M}^{\mathrm{pv}} \frac{1}{1-\Psi^{\mathrm{int}}} = 0$$

This follows if the intertwining products relate the A_∞ structures of the reference and target backgrounds:

$$\mathbf{M}^{\mathrm{tv}} \mathbf{I}^{\mathrm{tv}} = \mathbf{I}^{\mathrm{tv}} \mathbf{M}^{\mathrm{tv}}$$

i.e. they defiine a morphsm of $A\infty$ structures. In this way, the EOM of the intertwining solution follows from the EOM for the tachyon vacuum in the target background.

This generalizes the invariance constraint of the intertwining fields $\Sigma, \overline{\Sigma}$.

Expanding leads to a hierarchy of relations between the intertwining products:

(1) $[Q^{\text{tv}}, \mathcal{I}_1^{\text{tv}}] = 0$

(2) $[Q^{\text{tv}}, \mathcal{I}_2^{\text{tv}}] = \mathcal{I}_1^{\text{tv}} M_2^{\text{tv}} - M_2^{\text{tv}}(\mathcal{I}_1^{\text{tv}} \otimes \mathcal{I}_1^{\text{tv}})$

(3) $$[Q^{\text{tv}}, \mathcal{I}_3^{\text{tv}}] = \mathcal{I}_2^{\text{tv}}(M_2^{\text{tv}} \otimes \mathbb{I} + \mathbb{I} \otimes M_2^{\text{tv}}) - M_2^{\text{tv}}(\mathcal{I}_2^{\text{tv}} \otimes \mathcal{I}_1^{\text{tv}} + \mathcal{I}_1^{\text{tv}} \otimes \mathcal{I}_2^{\text{tv}}) + \mathcal{I}_1^{\text{tv}} M_3^{\text{tv}} - M_3^{\text{tv}}(\mathcal{I}_1^{\text{tv}} \otimes \mathcal{I}_1^{\text{tv}} \otimes \mathcal{I}_1^{\text{tv}})$$

$\vdots$

The important observation is that the right hand side of these equations is automatically Q^{tv}-closed. But since Q^{tv} has no cohomology, a solution for the intertwining products is guaranteed to exist.

To solve the first relation (1), we can postulate

$$I_1^{\text{tv}} = [Q^{\text{tv}}, bI_1^{\text{bare}}]$$

where b is an operator of ghost number -1 and I_1^{bare} represents a conjugate pair of boundary condition changing operators or a defect operator.

We can choose I_1^{bare} based on our interests and convenience.

To solve the second relation (2) we write the 2-product around the tachyon vacuum in Q^{tv}-exact form:

$$M_2^{\text{tv}} = [Q^{\text{tv}}, u_2]$$

u_2 may be called a gauge 2-product. Then we can take

$$I_2^{\text{tv}} = I_1^{\text{tv}} u_2 - u_2(I_1^{\text{tv}} \otimes I_1^{\text{tv}})$$

To solve the third relation (3) we introduce a gauge 3-product u_3 so that

$$M_3^{\text{tv}} = \frac{1}{2}\Big([Q^{\text{tv}}, u_3] + [M_2^{\text{tv}}, u_2]\Big)$$

We then find

$$\begin{aligned} \mathcal{I}_3^{\text{tv}} = \frac{1}{2}\Big(& \mathcal{I}_1^{\text{tv}} u_3 - u_3(\mathcal{I}_1^{\text{tv}} \otimes \mathcal{I}_1^{\text{tv}} \otimes \mathcal{I}_1^{\text{tv}}) + \mathcal{I}_1^{\text{tv}} u_2(u_2 \otimes \mathbb{I} + \mathbb{I} \otimes u_2) \\ & -2u_2(\mathcal{I}_1^{\text{tv}} \otimes \mathcal{I}_1^{\text{tv}})(u_1 \otimes \mathbb{I} + \mathbb{I} \otimes u_2) \\ & +u_2(u_2 \otimes \mathbb{I} + \mathbb{I} \otimes u_2)(\mathcal{I}_1^{\text{tv}} \otimes \mathcal{I}_1^{\text{tv}} \otimes \mathcal{I}_1^{\text{tv}})\Big) \end{aligned}$$

The general story has close parallel with homotopy algebraic constructions of superstring field theories.

We have a hierarchy of gauge products $u_n, n \geq 2$ which can be assembled into a coderivation which depends on a parameter t:

$$\mathbf{u}(t) = \mathbf{u}_2 + t\mathbf{u}_3 + t^2\mathbf{u}_3 + \ldots$$

With this we form a cohomomorphism defined as a path ordered exponential:

$$\mathbf{U} = \mathcal{P} \exp \left[\int_0^1 dt\, \mathbf{u}(t) \right]$$

The gauge products are determined to satisfy

$$\mathbf{M}^{\text{tv}} = \mathbf{U}^{-1}\mathbf{Q}^{\text{tv}}\mathbf{U}$$

where $\mathbf{Q}^{\text{tv}}$ is the coderivation corresponding to Q^{tv}.

$\mathbf{U}$ relates the A_∞ structure around the tachyon vacuum to the trivial A_∞ structure defined by the tachyon vacuum kinetic operator.

The intertwining products are given by

$$\mathbf{I}^{\mathrm{tv}} = \mathbf{U}^{-1}\mathbf{I}_1^{\mathrm{tv}}\mathbf{U}$$

where $\mathbf{I}_1^{\mathrm{tv}}$ is the cohomomorphism corresponding to $\mathcal{I}_1^{\mathrm{tv}}$. Around the perturbative vacuum we have

$$\mathbf{I}^{\mathrm{pv}} = \mathbf{S}\mathbf{U}^{-1}\mathbf{I}_1^{\mathrm{tv}}\mathbf{U}\mathbf{S}^{-1}$$

Comments

Comment 1: A_∞ notation aside, there is nothing in what we have said so far which essentially distinguishes between open and closed strings, except perhaps concerning the existence of a tachyon vacuum.

Comment 2: The cohomomorphism $\mathbf{U}$ must be singular when acting on any physically interesting field configuration.

To see why, consider the field redefinition

$$\Psi' = \pi_1 \mathbf{US}^{-1} \frac{1}{1-\Psi}$$

This maps the EOM around the perturbative vacuum as

$$\pi_1 \mathbf{M}^{\mathrm{pv}} \frac{1}{1-\Psi} \quad \longrightarrow \quad Q^{\mathrm{tv}} \Psi' = 0$$

The later EOM has no physically nontrivial solution. So unless the theory is physically vacuous, the field redefinition cannot be allowed.

The homotopy intertwining solution may nevertheless be well defined in a similar way as the A^{∞} structure around the tachyon vacuum, which may be expressed as $\mathbf{M}^{\text{tv}} = \mathbf{U}^{-1}\mathbf{Q}^{\text{tv}}\mathbf{U}$.

But this cannot be taken for granted.

Comment 3: The solution does not manifestly require anything analogous the nondegeneracy constraint $\overline{\Sigma}\Sigma = 1$.

This is good news, since this contraint is the main source of difficulty in attempting to realize the solution.

- For the solution of 1406.3021 it requires restricting to time independent backgrounds, tensoring boundary condition changing operators with an unphysical timelike Wilson line deformation, and associativity anomalies.
- For the solution of 1909.11675, it requires working with complicated and singular Riemann surfaces. Also, the perturbative vacuum is represented by a nonperturbative string field.

But without the nondegeneracy constraint, we need to deal with a solution containing an unbounded number of boundary condition changing or defect operators, and we have to confront the question of convergence.

At a practical level, explicit computations become more difficult.

Examples

Example 1: The original intertwining solution

$$\Psi^{\text{int}} = \Psi^{\text{tv}} - \overline{\Sigma}\Psi^{\text{tv}}\Sigma$$

- Intertwining products:

$$\mathcal{I}_1^{\text{tv}}(X) = \Sigma X \overline{\Sigma}$$
$$\mathcal{I}_n^{\text{tv}} = 0, \quad (n \geq 2)$$

- Gauge products:

$$u_2(X,Y) = -XAY$$
$$u_n = 0, \quad (n \geq 3)$$

where A is the homotopy operator (string field) of the tachyon vacuum satisfying

$$Q^{\text{tv}} A = 1 = \text{identity string field}$$

The fact that the higher intertwining products can be taken to vanish is a result of

$$u_2(\mathcal{I}_1^{\mathrm{tv}}(X), \mathcal{I}_1^{\mathrm{tv}}(Y)) = \mathcal{I}_1^{\mathrm{tv}}(u_2(X, Y))$$

which follows from the nondegeneracy constraint $\overline{\Sigma}\Sigma = 1$.

Since the solution has only two terms, there is clearly not an issue with convergence, even though it can be expressed in terms of **U**.

In this example, the problematic field redefinition is

$$\pi_1 \mathbf{U}\mathbf{S}^{-1}\frac{1}{1-\Psi} = (\Psi - \Psi^{\mathrm{tv}})\frac{1}{1+A(\Psi-\Psi^{\mathrm{tv}})}$$

Taking for example the perturbative vacuum $\Psi = 0$, this is singular in a well-understood way related to Okawa's expression of Schnabl's solution in pure gauge form.

Example 2: Schnabl gauge intertwining solution.

Can be obtained from an ordinary intertwining solution using the Zeze map:

$$(\Psi^{\text{int}})' = \sqrt{F}(\Psi^{\text{tv}} - \Sigma\Psi^{\text{tv}}\overline{\Sigma})\frac{1}{1 + B\frac{1-F}{K}(\Psi^{\text{tv}} - \Sigma\Psi^{\text{tv}}\overline{\Sigma})}\sqrt{F}$$

where $F \in$ wedge algebra.

This can also be written as an ordinary intertwining solution:

$$(\Psi^{\text{int}})' = (\Psi^{\text{tv}})' - \Sigma'\Psi^{\text{tv}}\overline{\Sigma}'$$

where

$$\begin{aligned}
(\Psi^{\text{tv}})' &= (\Psi^{\text{int}})' + \Sigma'\Psi^{\text{tv}}\overline{\Sigma}' \\
\Sigma' &= \frac{1}{\sqrt{F}}\left(1 + B\frac{1-F}{K}(\Psi^{\text{tv}} - \Sigma\Psi^{\text{tv}}\overline{\Sigma})\right)\Sigma \\
\overline{\Sigma}' &= \overline{\Sigma}\frac{1}{1 + B\frac{1-F}{K}(\Psi^{\text{tv}} - \Sigma\Psi^{\text{tv}}\overline{\Sigma})}\sqrt{F}
\end{aligned}$$

But this expression is rather artificial; the tachyon vacuum is not universal, and $\overline{\Sigma}'$ contains an infinite number of boundary condition changing operators.

The Schnabl gauge solution can more naturally be written as a homotopy intertwining solution by taking reference and target tachyon vacuum solutions

$$\widetilde{\Psi}^{\text{tv}} = \sqrt{F}\Psi^{\text{tv}}\frac{1}{1 - B\frac{1-F}{K}\Psi^{\text{tv}}}\sqrt{F} \in V_{\text{ref}}$$

$$\widetilde{\Psi}^{\text{tv}} = \Psi^{\text{tv}} \in V_{\text{targ}}$$

We introduce

$$\widetilde{\Sigma} = \sqrt{F}\frac{1}{1 + \Psi^{\text{tv}}B\frac{1-F}{K}}\Sigma$$

$$\widetilde{\overline{\Sigma}} = \overline{\Sigma}\frac{1}{1 + B\frac{1-F}{K}\Psi^{\text{tv}}}\sqrt{F}$$

and write A for the homotopy operator of the original tachyon vacuum Ψ^{tv} in V_{targ} and $\widetilde{A}$ for the tachyon vacuum $\widetilde{\Psi}^{\text{tv}}$ of V_{ref}.

With this the gauge products are

$$
\begin{aligned}
u_2(X,Y) &= -X\widetilde{A}Y, \quad X,Y\in V_{\text{ref}}\\
u_2(X,Y) &= -XAY, \quad X,Y\in V_{\text{targ}}\\
u_n &= 0, \quad (n\geq 3)
\end{aligned}
$$

and the intertwining products are

$$\mathcal{I}_n^{\text{tv}}(X_1,X_2,...,X_n) = (-1)^{n+1}\widetilde{\Sigma}X_1\left(A-\widetilde{\overline{\overline{\Sigma}}}\widetilde{A}\widetilde{\Sigma}\right)X_2...\left(A-\widetilde{\overline{\overline{\Sigma}}}\widetilde{A}\widetilde{\Sigma}\right)X_n\widetilde{\overline{\overline{\Sigma}}}$$

Now the tachyon vacuum solutions can be universal, and the intertwining product $\mathcal{I}_n^{\text{tv}}$ contains no more than n conjugate pairs of boundary condition changing operators.

The convergence of this solution is generally a nontrivial matter. But this structure applies, for example, to solutions for marginal deformations in Schnabl gauge, which are well-known and regular.

Background Independence

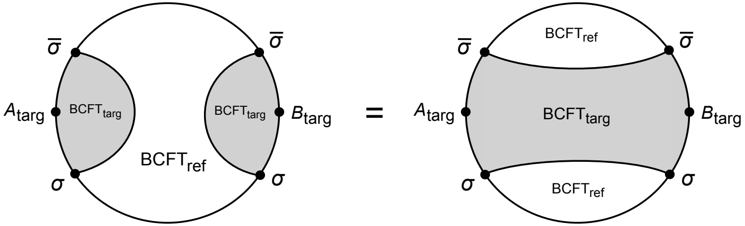
To establish background independence we need to see that the actions of the reference and target background can be related by a field transformation.

The natural transformation provided by the homotopy intertwining solution is

$$\Psi_{\text{ref}} = \pi_1 \mathbf{I}^{\text{pv}} \frac{1}{1 - \Psi_{\text{targ}}}$$

We just plug this into the action of the reference background and see if it works.

Here I encountered some problems, since the analogous computation in Witten's theory requires the following:



For the open string, this is obvious since the bulk CFTs are the same and it makes no difference how to draw the lines.

For the closed string, we cannot expect this to be true.

Nevertheless I was able to make progress by assuming that the gauge products are cyclic: $\langle\omega|\pi_2\mathbf{u}_n = 0$.

This seems like a natural condition, but is not realized in known intertwining-type solutions in Witten's theory.

If the gauge products are cyclic, the infinite number of defect/boundary condition changing operators in the action cancel down to two, which can be further expressed:

$$(I_1^{\text{tv}})^\star I_1^{\text{tv}} = \mathbb{I} - [Q^{\text{tv}}, \mathcal{B}]$$

gives terms which vanish on-shell

gives the action of the target background

Convergence and nondegeneracy

Other than the existence of a tachyon vacuum, the big unanswered question behind the construction is whether the solution converges.

We also have not assumed anything analogous to the nondegeneracy constraint.

I believe that these issues are related.

Indeed, in the ordinary intertwinining solution the nondegeneracy constraint ensures convergence by allowing a solution to be written with a finite number of terms.

The nondegeneracy constraint essentially says that the intertwining products transfer all information about the target background into the reference background.

If they do not, it is hard to believe that the action of the reference background can reproduce the action of the target background.

This leads to a conjecture:

Conjecture: The homotopy intertwining solution faithfully represents the target background only if the bilinear form

$$\langle A_{\text{ref}}, \mathcal{I}_1^{\text{bare}} B_{\text{targ}} \rangle$$

is nondegenerate.

If all of this applies to closed strings, the implication is that two bulk CFTs can be dynamically related only if there is a defect operator D such that the bilinear form

$$\langle A_{\mathsf{CFT}_1}, D\, B_{\mathsf{CFT}_2} \rangle$$

is nondegenerate.

It would be interesting to see if this leads to a reasonable picture of the string landscape.

Thank you!