Equivariant BV formalism for string worldsheet

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Introduction: “What is string theory?”

What properties should a 2D sigma model satisfy to serve as a string worldsheet theory?

Let us think of it as a BV theory.

Integration over Lagrangian submanifolds

there is canonical integration measure (more precisely, a PDF) on any families of Lagrangan submanifolds, not only on families of conormal bundles. If gauge fermions $\Psi_1, \ldots, \Psi_n$ define n tangent vectors to LAG at the point $L \in LAG$ then the measure is:

$$\int_L \Psi_1 \cdots \Psi_n \, \exp(S_{BV}) \tag{1}$$

Kalkman element

It seems that there is a general theme which plays a role here. It is basically Kalkman map between Cartan and Weyl models of equivariant cohomology.

Restriction of form to orbit

Let us illustrate it by the following simple example. Consider a manifold M with the action of a Lie group G. Consider differential forms of mixed degree on M. Pick a point $x_0 \in M$ and consider the orbit Gx_0. Then, for every ω we can consider the differential form on G:

$$x \mapsto g.x,\ \ dx \mapsto (dg).x$$

This can be represented as follows:

$$\omega \mapsto \left(\exp\left((dgg^{-1})^a \iota_a\right)\omega\right)(x = gx_0, dx = 0)$$

(We think of forms on M as functions on ΠTM, and forms on G as functions on ΠTG.) The secret of the formula is:

$$\left(d_M + \frac{1}{2}c^a c^b f^c_{ab}\frac{\partial}{\partial c} + \mathcal{L}\langle c\rangle\right)\exp \iota\langle c\rangle \ = \ \exp \iota\langle c\rangle\left(d_M + \frac{1}{2}c^a c^b f^c_{ab}\frac{\partial}{\partial c}\right) \tag{2}$$

Slight generalization

After the first step (which was $\omega \mapsto \left(\exp\left((dgg^{-1})^a \iota_a\right)\omega\right)$), we could have (instead of putting $dx = 0$) integrated over a submanifold $gN_0 \subset M$ where N_0 is some compact submanifold of M. (Choice of a point x_0 is a particular case, when N_0 is one point.) If we take G the whole group of diffeomorphisms of M, then this prescription cooks out of ω an inhomogeneous differential form on the space of submanifolds $N \subset M$.

Generalization

We have some representation V of $C\mathfrak{g}$. Then Eq. (2) still holds, where d_M should be compatible with the differential of $C\mathfrak{g}$.

PDF on LAG

The group of odd canonical transformations, and its Lie superalgebra $\mathfrak{g}$, acts on **half-densities** on the BV phase space. Moreover, there is an action of $C\mathfrak{g}$, where $\iota\langle v\rangle$ is multiplication of a half-density by the BV Hamiltonian of v (which we denote $\underline{v}$):

$$\iota\langle v\rangle\rho_{1/2} = \underline{v}\rho_{1/2}$$

The canonical BV Laplacian Δ_{can} is compatible with $\mathsf{d}_{C\mathfrak{g}}$.

The Eq. (2) produces the PDF on G (the string measure):

$$\Omega(\mathsf{g}, \mathsf{dg}) = \int_{\mathsf{g}L_0} \exp\left(\underline{\mathsf{dgg}^{-1}} \right) \rho_{1/2} \tag{3}$$

Worldsheet diffeomorphisms

We believe that **diffeomorphisms of the string worldsheet** are crucial ingredient in string worldsheet theory.

How do they act in the BV phase space?

They should preserve S_{BV}. As they are gauge symmetries, it is natural to conjecture that their BV hamiltonians should be exact. In other words, for every vector field ξ on the worldsheet we should get some function $\phi\langle\xi\rangle$ on the BV phase space such that $\{S_{BV}, \phi\langle\xi\rangle\}$ generates the action of diffeomorphisms. We should definitely require:

$$\{\{S_{BV}, \phi\langle\xi\rangle\}, \{S_{BV}, \phi\langle\eta\rangle\}\} = \{S_{BV}, \phi\langle[\xi,\eta]\rangle\}$$

where $[\xi,\eta]$ is the commutator of two vector fields on Σ.

What else should we require? We need to turn Ω into a base form. It is already invariant, but it is not horizonthal.

But it has some `special property` (which we explain on next slide: ⟦ Special properties of Ω ⟧) which helps to turn it into a base form.

Case of bosonic string

The BV phase space is:

$$M = \Pi T^* \left(\frac{\Pi TH \times X}{H}\right)$$

Here we quotient by the action of H where H acts on ΠTH **from the right**. The Q_{BRST} comes from the canonical nilpotent vector field on ΠTH. We want Q_{BRST} to preserve the volume on $\frac{\Pi TH \times X}{H}$. This is some condition on the trace of the structure constant and the div's of generators.

But the **left** action of H on ΠTH remains. It is generated by an exact Hamiltonian:

$$H\langle\xi\rangle = \Delta(\xi^\alpha c^\star_\alpha)$$

This means that:

Gauge symmetries act on the BV phase space

An equivariant form is given by:

$$\int_L \exp\left(S_{BV} + \Psi + \mathbf{t}^\alpha c^\star_\alpha\right)$$

Special properties of Ω

The string form Ω defined by Eq. (3) is very special. Besides $d\Omega = 0$ it satisfies, for any $v \in \mathfrak{h} \subset \mathrm{Vect}(\mathrm{LAG})$, the following special properties:

$$v \mapsto \iota\langle v\rangle\Omega \quad \texttt{is injective map} \tag{4}$$

$$\text{exists } d_\Omega : d\, \iota\langle v\rangle\Omega = \iota\langle d_\Omega v\rangle\Omega \tag{5}$$

where d_Ω corresponds to the BV Laplacian w.r.to the half-density $\rho_{1/2}$. Eqs (4) and (5) are very special. They imply:

$$de^{\iota\langle v\rangle}\Omega = e^{\iota\langle v\rangle}\iota\left\langle d_\Omega v + \frac{1}{2}[v, v]\right\rangle\Omega$$

Therefore, the **equivariant** analogue of Ω in the Cartan model can be constructed as follows:

$$\Omega^C(\mathbf{t}) = \exp\left(\iota\langle i(\mathbf{t})\rangle\right)\Omega$$

$$\text{where:} \quad d_\Omega i(\mathbf{t}) + \frac{1}{2}[i(\mathbf{t}), i(\mathbf{t})] = l(\mathbf{t})$$

The construction of the Cartan equivariant form is recast as a construction of a representation of $D\mathfrak{h}$ in functions on BV phase space. In other words, we need a morphism

$$D\mathfrak{h} \longrightarrow \mathfrak{g}$$

(where $\mathfrak{g}$ is the Lie superalgebra of infinitesimal BV-canonical transformations) such that $d_{D\mathfrak{h}}$ agrees with d_Ω (remember d_Ω is the same as $\{S_{BV}, \_\}$). We denote it like this:

$$(D\mathfrak{h}, d_{D\mathfrak{h}}) \longrightarrow (\mathfrak{g}, \{S_{BV}, \_\})$$

Deformation complex

Equivariant Master Equation

Let us reformulate. The equivariant Master Equation is:

$$\Delta_{can}\rho_{1/2}(\mathbf{t}) = \underline{l(\mathbf{t})}\rho_{1/2}(\mathbf{t})$$

where $l(\mathbf{t})$ is the odd moment map of $\mathfrak{h}$. We should also require $\mathfrak{h}$-covariant dependence on $\mathbf{t}$. We write:

$$\rho_{1/2}(\mathbf{t}) = \exp\left(S_{BV} + \underline{i(\mathbf{t})}\right)$$

Deformations of solutions (integrated vertex operators)

Such solutions admit deformations, corresponding to the deformations of string background. Infinitesimal deformations can be represented in the form:

$$\delta\rho_{1/2}(\mathbf{t}) = f(\mathbf{t})\rho_{1/2}(\mathbf{t})$$

where $f(\mathbf{t})$ satisfies the equation:

$$d_{D\mathfrak{h}}f + \{\underline{i(\mathbf{t})}, f\} = 0$$

This is Cartan model. If we relax the $\mathfrak{h}$-covarance of $f(\mathbf{t})$, and allow it to also depend on $\boldsymbol{\theta}$ then it is:

$$(d_{D\mathfrak{h}} + d_W)f(\mathbf{t},\boldsymbol{\theta}) + \{\underline{l\langle\boldsymbol{\theta}\rangle}, f(\mathbf{t},\boldsymbol{\theta})\} + \{\underline{i(\mathbf{t})}, f(\mathbf{t},\boldsymbol{\theta})\} = 0$$

where d_W is the differential of the Weyl algebra of $\mathfrak{h}$ formed by letters $\boldsymbol{\theta}$ and $\mathbf{t}$ (sometimes θ is denoted A and $\mathbf{t}$ is denoted F). The differential:

$$d_{D\mathfrak{h}} + d_W + \{\underline{l\langle\boldsymbol{\theta}\rangle}, \_\} + \{\underline{i(\mathbf{t})}, \_\} \qquad (6)$$

is nilpotent.

> Weyl algebra $dA = F + AA$, $dF = AF$ is not minimal, because linearized diffrential is nonzero: $d_0A = F$. We can turn it into a minimal algebra by adding a central element Z and saying $dA = ZF + AA$, $dF = AF$. This is Koszul dual to the Lie superalgebra $D\mathfrak{h}' = D\mathfrak{h} \oplus \mathbf{C}d_{D\mathfrak{h}}$.

Another point of view on the deformation complex

Unintegrated vertices

Instead of $\mathfrak{h}$-invariant f, let us consider arbitrary v satisfying $\{S_{BV}, v\} = 0$. This is "unintegrated vertex operator". We want to construct from v a closed PDF on H. The insertion procedure would require integration of this PDF over an appropriate cycle.

Following the ⟦Kalkman element⟧ we would write something like:

$\omega \overset{?}{=} \int_{hL} \exp\left\{\underline{\iota\langle dhh^{-1}\rangle}, \_\right\} v$ But there is no $\iota\langle\ldots\rangle$ ... We only have $i(\ldots)$. In bosonic string $i\langle\xi\rangle = \int_\Sigma \xi^\alpha c^\star_\alpha$, and it works:

Bosonic string:

$$\Omega_C = \int_{gL} \exp\left(S_{BV} + \underline{dgg^{-1}} + c^\star_\alpha \mathbf{t}^\alpha\right)$$

$$\left\{\underline{\iota\langle dhh^{-1}\rangle}, \_\right\} \quad \texttt{gives} \quad \left\{\int_\Sigma \xi^\alpha c^\star_\alpha\,,\ \_\right\} \quad \texttt{(Just ``strips'' c and } \bar{c}\texttt{)}$$

Remember the ⟦Kalkman element⟧ required a representation of $C\mathfrak{h}$; any representation of $C\mathfrak{h}$ gives a representation of $D\mathfrak{h}$, but not vice-versa.

But will it work in general case, when $i(\xi)$ is nonlinear in ξ and $[i(\xi), i(\xi)] \neq 0$? It turns out that the answer is "yes", although the naive $\exp\left\{\underline{"i(dhh^{-1})"}, \_\right\}$ (whatever it would mean) will not work. But actually exists a generalization of the Kalkman formula which only requires a representation of $D\mathfrak{h}$:

$$\omega\langle V\rangle = \int_{hL} \left[\left(P\exp\int_0^1 \mathcal{A}_\tau d\tau\right) V\right] \rho_{1/2} \qquad (7)$$

$$\text{where} \quad \mathcal{A}_\tau d\tau = \left\{ \left.\frac{d}{du}\right|_{u=0} \underline{i(u\, d\tau\, dgg^{-1} + (\tau - \tau^2)(dgg^{-1})^2)} \quad , \quad \_ \right\}$$

Derivation

We know that $\{l(A), \_\} + \{i(F), \_\}$ is Maurer-Cartan (because the differential given by Eq. (6) is the differential in the deformation complex of the equivariant half-density!).

Then we consider the Kalkman embedding:

$$W\mathfrak{g} \to W\mathfrak{g} \otimes \Omega^1(\mathbf{R}_s)$$

$$\boldsymbol{\theta}_a \mapsto \widetilde{\boldsymbol{\theta}}_a = \tau \boldsymbol{\theta}_\mathbf{a}$$

$$\mathbf{t}_a \mapsto \widetilde{\mathbf{t}}_a = d\tau \boldsymbol{\theta}_a + \tau \mathbf{t}_a + (\tau^2 - \tau) f_a^{bc} \boldsymbol{\theta}_b \boldsymbol{\theta}_c$$

We have:

$$\left(d_{D\mathfrak{h}} + d_W + d\tau \frac{\partial}{\partial \tau} + \{\underline{l\langle\widetilde{\boldsymbol{\theta}}\rangle}, \_\} + \{\underline{i(\widetilde{\mathbf{t}})}, \_\}\right)^2 = 0 \qquad (8)$$

This implies:

$$\left(P\exp\int_0^1 \mathcal{A}_\tau d\tau\right)(d_{D\mathfrak{h}} + d_W) = \left(d_{D\mathfrak{h}} + d_W + \{\underline{l\langle\boldsymbol{\theta}\rangle}, \_\} + \{\underline{i(\mathbf{t})}, \_\}\right)\left(P\exp\int_0^1 \mathcal{A}_\tau d\tau\right)$$

Remember that $\mathsf{d}_{\mathsf{D}\mathfrak{h}} = \Delta_{\mathsf{S}_{\mathsf{BV}}} = \Delta_0 + \{\mathsf{S}_{\mathsf{BV}}, \_\}$.

Let us put $\mathbf{t} = 0$ and $\boldsymbol{\theta} = \mathsf{dhh}^{-1}$.

Let us put $\mathsf{F} = 0$ and $\mathsf{A} = \mathsf{dhh}^{-1}$. But, consider some vertex operator V which is **not** H-invariant. Our V will not contain any A and F, and will satisfy the Master Equation for vertex operators:

$$\mathsf{d}_{\mathsf{D}\mathfrak{g}}\mathsf{V} = 0$$

Then:

$$\left(\mathsf{d}_{\mathsf{Weyl}} + \{\mathsf{I}(\mathsf{dhh}^{-1}), \_\}\right) \int_{\mathsf{hL}} \left[\mathsf{P}\exp\left(\int_0^1 \mathcal{A}_\tau \mathsf{d}\tau\right)\mathsf{V}\right]\rho_{1/2} \;=\; 0$$

This is the same as to say:

$$\mathsf{d}_{(\mathsf{h})} \int_{\mathsf{hL}} \left[\mathsf{P}\exp\left(\int_0^1 \mathcal{A}_\tau \mathsf{d}\tau\right)\mathsf{V}\right]\rho_{1/2} \;=\; 0$$

In other words, we have constructed a closed differential form on H. When we are inserting an unintegrated vertex V, we have to integrate this closed form over an appropriate cycle in H.

Applications

At this time we do not have a single example of string worldsheet theory where $\mathrm{i}(\mathbf{t})$ would not be linear. We suspect that pure spinor formalism is an example. We constructed $\mathrm{i}(\mathbf{t})$ for pure spinor string in $AdS_5 \times S^5$, but only to the first order in $\mathbf{t}$.