Doubled Spacetime, Homotopy Algebras, and Puzzles of String Fields on Tori

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Based on

- Homotopy Transfer and Effective Field Theory II: Strings and Double Field Theory, ASA, Chris Hull, Olaf Hohm, Victor Lekeu, [2106.08343]
- Homotopy Transfer and Effective Field Theory I: Tree-level, — *idem* —, [2007.07942]

Consider the low-energy effective theory of closed bosonic strings on Minkowski:

$$S[g,B,\phi] = \int \sqrt{-g}e^{-2\phi}(R_{\text{icci}} + (\partial\phi)^2 - (\partial B)^2)\,.$$

This tells us string theory is a theory of gravity (coupled to $\phi, B \ldots$).

Can we find *fundamentally stringy* features of this quantum gravity theory?

For this talk: 'effective' theory of strings on $\mathbb{R}^n \times T^d$: *double field theory (DFT)* [Hull Zwiebach 2009], known to **cubic order**.

It exhibits a **doubled spacetime**, courtesy of the torus factor T^d:

$$\text{field } \Psi = \Psi(y,x,\tilde{x}) \qquad y \in \mathbb{R}^n\,, \quad x \in T^d\,, \quad \tilde{x} \in \widetilde{T}^d \text{ ("dual" torus)}\,.$$

Question:

Does doubled spacetime survive beyond cubic order?

We employ **homotopy algebra** (L_∞-algebras to be precise) to provide an algorithm for the quartic and higher DFT vertices:

[arXiv:2106.08343]

$$\text{Closed bosonic string field theory} \xrightarrow{\text{homotopy transfer}} \text{Double field theory}$$

Manifestly doubled from beginning to end; consistent with gauge invariance.
(Contrast here *strongly-constrained DFT* [Hohm Hull Zwiebach 2010] — known to all orders; $O(d,d)$-covariant; but not *doubled*.)

Implementation of [Sen 2016] proposal for general, (non-)Wilsonian effective actions from SFT. [See also: Masuda Matsunaga 2020; Erbin Maccaferri Schnabl Vošmera 2020 both (largely) for A_∞/open strings]

Outstanding technical issues involving tachyons and signature (Euclidean is best...).

Closed strings...

- on $\mathbb{R}^d$: Oscillators $\alpha_n, \bar{\alpha}_n$; **one** zero-mode $\alpha_0 = \bar{\alpha}_0 = p$ (centre-of-mass momentum)
- on T^d: Oscillators $\alpha_n, \bar{\alpha}_n$; **two** zero-modes

$$\begin{aligned} p &= \bar{\alpha}_0 + \alpha_0 && \text{momentum,} \\ w &= \bar{\alpha}_0 - \alpha_0 && \text{winding.} \end{aligned}$$

String states, e.g. the level $N = \bar{N} = 1$ state (a g, B, or ϕ perturbation)

$$\Psi(x, \tilde{x}) = \int dpdw\; e^{i(px + w\tilde{x})} \Psi_{ij}(p, w) \alpha^i_{-1} \bar{\alpha}^j_{-1} |p, w\rangle \,,$$

see the double torus $T^{\mathbf{2}d}$ because we simultaneously have momenta *and* windings.

T-duality swaps $p \leftrightarrow w$. So we must take dual positions $\tilde{x}$ as seriously as positions x!

[Hull Zwiebach 2009]: truncation of **bosonic closed SFT** to levels $N = \bar{N} = 1$ (graviton, dilaton, B-field) is consistent to **cubic** order — **double field theory** (DFT).

Who ordered that? Wilson didn't...

...because (for a spacetime $\mathbb{R}^n \times T^1$, where T^1 of radius R)

$$\text{mass}^2 = (p/R)^2 + (wR/\alpha')^2 \,.$$

Lagrangian resembles Fierz-Pauli (+ cubic) for $\Psi = (e_{ij}, d)$ — perturbations of a *generalised metric* and *dilaton*. (e_{ij} non-symmetric.)

String field gauge symmetry gives **double diffeos** (gauge params $\lambda_i, \bar{\lambda}_i$). Action is **local** in doubled space; Fields *and* gauge params **obey level-matching** ("weak section")

$$S_{\text{DFT}} = \int dx d\tilde{x} \left[e_{ij}(\partial^2 + \tilde{\partial}^2)e^{ij} + (\partial^i e_{ij})^2 + \dots \right], \qquad \frac{\partial^2}{\partial x^i \partial \tilde{x}_i}(\text{any}) = 0 \,.$$

(This is also $O(d,d)$-covariant with respect to $\eta(\mathbb{P}, \mathbb{P}) = 2p_i w^i$ for $\mathbb{P} = (p, w)$.)

Brute-force construction to e.g. quartic order?

Combinatorial explosion!

- Projectors $[\cdots]$ to $\frac{\partial}{\partial x}\frac{\partial}{\partial \tilde{x}} = 0$!

$$[(\partial\Psi)^2]\Psi^2\,, [\partial\Psi]\Psi^2\partial\Psi\,, [\partial\Psi\Psi]\partial\Psi\Psi\,, \ldots$$

  Only relevant at $\mathcal{O}(\Psi^4)$ due to momentum conservation.

- "Cocycle signs"! factors that appear already in torus string field theory of the form

$$\exp(i\pi p w')\,,$$

  (relative to $\mathbb{R}^d$ SFT) relevant for gauge invariance at $\mathcal{O}(\Psi^4)$ [HIKKO 87]

Homotopy transfer does not suffer from these difficulties.
(It suffers *differently*.)

Definition (L_∞-algebra aka *homotopy* Lie algebra)

Vector spaces X_n in chain complex	$\cdots \xrightarrow{\partial} X_1 \xrightarrow{\partial} X_0 \xrightarrow{\partial} X_{-1} \xrightarrow{\partial} \cdots, \qquad \partial^2 = 0,$
Jacobi identities	$\begin{cases} [[x,y],z] + [[y,z],x] + [[z,x],y] = \partial[x,y,z], \\ [[x_1, x_2 \ldots x_k], x_{k+1}, \ldots x_\ell] + \text{perms} = 0, \ldots \end{cases}$

1 bracket $b_n : X^n \to X$ of n arguments for each $n = 1, 2, \ldots \infty$.

Lie algebra with Jacobi "up to homotopy" — **i.e. up to ∂-exact.**

L_∞-algebra $\leftrightarrow$ field theory dictionary: [Hohm Zwiebach 17] [Also Hohm's talk on Thursday]

$$\cdots \xrightarrow{\partial} \text{gauge params} \xrightarrow{\partial} \text{fields} \xrightarrow{\partial} \text{Eq. of Motion} \xrightarrow{\partial} \cdots$$

∂ :	linearised EoM & gauge transformations,
$b_{n\geq 2}$:	**interaction** $(n+1)$-**vertex**,
$\kappa(-,-)$:	inner product/antibracket (if present).

Example (Covariant bosonic tree-level closed-string field theory [Zwiebach 92])

$$\Psi \in X \iff \Psi \in \mathcal{F}_{\text{ock}}\,, \text{ and } \quad (L_0 - \bar{L}_0)\Psi = (b_0 - \bar{b}_0)\Psi = 0.$$

$$\begin{aligned} \partial : &\qquad \text{worldsheet } Q_{\text{BRST}}, \\ b_{n\geq 2} : &\qquad \text{Sirs Not-Appearing-In-This-Seminar}, \\ \kappa(-,-) : &\qquad \text{Essentially BPZ.} \end{aligned}$$

For any* "matter" CFT with $c = +26$, $\exists b_n$ so $(X; \kappa; \partial, b_1, b_2 \dots)$ is L_∞-algebra.

Unfortunately b_3 and above are nonzero: we need an L_∞-algebra [Sonoda Zwiebach 89].

However: equivalent lightcone type theories for some backgrounds (e.g. [HIKKO 87]) **can** be cubic. Cubic theories $\leftrightarrow$ (differential graded) Lie algebras.

We need an *equivalence* of L_∞-algebras that connects e.g. covariant to lightcone SFT. Fortunately this is known: *homotopy equivalence*. [Stasheff 92, Kontsevich 95]

This is a tower of linear maps $\phi_n : X^n \to Y$ of which ϕ_1 preserves ∂-cohomologies:

$$\phi_1(H(X)) = H(Y) \qquad (\star)$$

plus conditions involving brackets (∂^X, b_n^X) on X, (∂^Y, b_n^Y) on Y.

Homotopy transfer

Say you have an L_∞-algebra on X, some chain complex Y, and a ϕ_1 with $(\star)$.
Give Y brackets $b_2^Y, b_3^Y \ldots$, to turn it into an L_∞-algebra (equivalent to X).

This is an algorithm. If Y is a subspace PX, the input includes a **homotopy map**

$$\boxed{h : X \to X\,, \qquad \text{with} \qquad P = 1 + \partial h + h\partial\,.}$$

The conditions for homotopy transfer *only* involve ∂, h, P — the *free theory*!

One can view [Sen 2016] as an ansatz for h:

$$h = G(1 - P)\,,$$

depending on the physical data of a *propagator* G and projector P to effective d.o.f's.

For string field theory $\partial = Q_{\mathrm{BRST}}$: ∂-cohomology is worldsheet BRST cohomology.

The 2×10^6-dollar question: take P projector on DFT states $N = \bar{N} = 1$.
Can we find an h?

. . . No.

However: *can* take

$$P = P_{N=\bar{N}=1} + \underbrace{P_{N=\bar{N}=0}}_{\text{tachyon}} \qquad \text{(generic torus moduli, Euclidean signature)}\,.$$

Homotopy transfer *construction* of vertices & gauge transfs of "{DFT + tachyon}".

Agrees with cubic result of [Hull Zwiebach 09] *modulo* elimination of auxiliaries.

Given

- *cyclic* (with inner product κ) L_∞-algebra of closed bosonic SFT
- (Siegel gauge) propagator G
- projector P & homotopy $h = G(1-P)$ satisfying $P = 1 + Q_{\text{BRST}}h + hQ_{\text{BRST}}$

we obtain another *cyclic* L_∞-algebra $\leftrightarrow$ theory of "effective" $\mathit{im}\,P$ modes given by BV master action.

Do we have a starting cyclic L_∞-algebra given by CSFT?

Subtlety involving κ/BPZ inner product for torus CSFT, actually.

A torus CFT paradox. . . ?

Old news: tachyon vertex ops $\mathcal{V}_{k,\bar{k}}(z,\bar{z}) =: \exp\left(ik\hat{X}(z) + i\bar{k}\hat{\bar{X}}(\bar{z})\right):$ **aren't bosons**

$$\mathcal{V}_{k,\bar{k}}(z,\bar{z})\mathcal{V}_{\ell,\bar{\ell}}(w,\bar{w}) = (-1)^{(\bar{k}\bar{\ell}-k\ell)}\mathcal{V}_{\ell,\bar{\ell}}(w,\bar{w})\mathcal{V}_{k,\bar{k}}(z,\bar{z})\,, \qquad |w| = |z| \pm \varepsilon\,.$$

Fix by introducing "cocycle operator" $C_{k,\bar{k}}$ depending on momenta/windings (via $k,\bar{k}$)

$$\mathcal{V}_{k,\bar{k}} \to \mathcal{V}_{k,\bar{k}}\underbrace{\exp(i\pi/2(k-\bar{k})(\hat{\alpha}_0 + \hat{\bar{\alpha}}_0))}_{C_{k,\bar{k}}} = \mathcal{V}^{\text{correct}}_{k,\bar{k}}\,.$$

(Can be absorbed into position zeromodes $x,\bar{x}$ to make them non-commutative.) [Sakamoto 89; Erler et al 92; . . . Freidel Leigh Minic 17, Zwiebach 17(? unpublished)]

$\mathcal{V}_{k,\bar{k}}$ and $\mathcal{V}^{\text{correct}}_{k,\bar{k}}$ create the same ket (state-operator map). . .

$$C_{k,\bar{k}}|\mathbf{1}\rangle = |\mathbf{1}\rangle\,, \quad \text{so} \quad \mathcal{V}^{\text{correct}}_{k,\bar{k}}(0)|\mathbf{1}\rangle = \mathcal{V}_{k,\bar{k}}(0)|\mathbf{1}\rangle$$

A torus CFT paradox. . . ?

. . . but they create *different* bras:

$$\langle \mathbf{1}|\mathcal{V}^{\text{correct}}_{k,\bar{k}}(\infty) = \langle \mathbf{1}|\mathcal{V}_{k,\bar{k}}(\infty) C_{k,\bar{k}} = (\textbf{sign})\langle \mathbf{1}|C_{k,\bar{k}}\mathcal{V}_{k,\bar{k}}(\infty)\,, \\ = (\textbf{sign})\langle \mathbf{1}|\mathcal{V}_{k,\bar{k}}(\infty)\,.$$

This (cocycle) **sign** is -1 for states with

$$\mathrm{pw} = 1 \mod 2$$

It enters $\kappa(x,y) = R(x,(c_0-\bar{c}_0)y)$, through the **reflector** R:

$$R(x,y) = \lim_{z\to 0, w\to 0} \langle \mathbf{1}(|I\circ \mathcal{V}_x)(z)\mathcal{V}_y(w)|\mathbf{1}\rangle \qquad I(z) = -z^{-1}\,,$$

with associated **transpose** ^{T} (BPZ conjugation)

$$R(\mathcal{O}x,y) = (-1)^{x\mathcal{O}} R(x,\mathcal{O}^T y)\,.$$

Inversion $I(z) = -z^{-1}$ realised (?) on operators by transpose:

$$I \circ \mathcal{O} = \mathcal{O}^T .$$

Oscillator $\alpha_n, \bar{\alpha}_n$ transpose known because $i\partial X(z)$ is a conformal primary ($\alpha_n^T \propto \alpha_{-n}$).

Position zeromode $x, \bar{x}$ transpose not fixed. Demand $I \circ \mathcal{V}^{\text{correct}}_{k,\bar{k}} = (\mathcal{V}^{\text{correct}}_{k,\bar{k}})^T$ to find

$$\left(\exp(ik\hat{x})\exp(i\bar{k}\hat{\bar{x}})\right)^T = C_{k,\bar{k}} \exp(ik\hat{x})\exp(i\bar{k}\hat{\bar{x}}) C_{k,\bar{k}}(\cdots)$$

$\mathcal{V}_{k,\bar{k}}$ & $\mathcal{V}^{\text{correct}}_{k,\bar{k}}$ are not simultaneously primary under global conformal transformations. Neither are $\mathcal{V}^{\text{correct}}_{k,\bar{k}}$, $\mathcal{V}^{\text{correct}\prime}_{k,\bar{k}}$ for different choices of cocycle operator.

- Choice of $C_{k,\bar{k}}$ fixes conformal primary tachyon operator & a compatible R
- Bose commutations select $\mathcal{V}^{\text{correct}}_{k,\bar{k}}$ over $\mathcal{V}_{k,\bar{k}}$ (also advocated by [Hellerman Walcher 06] for T-duality reasons.)
- With above $\mathcal{V}^{\text{correct}}_{k,\bar{k}}$, R agrees with [HIKKO 87; Kugo Zwiebach 92]

Remarks & open questions

Manifestly doubled & gauge-invariant "$\{$DFT $+$ tachyon$\}$" from bosonic closed Euclidean SFT, as an application of very general homotopy-algebraic technology.

- Tachyon? At least two resolutions:
  1. **Consistent L_∞-truncation** to remove the tachyon. Possible if $\exists$ injective L_∞-morphism. This is *a fortiori* a consistent truncation à la SUGRA [see our paper II].
  2. **Superstring field theory.** Recent development! (Type II: [Kunitomo Sugimoto 2019])
- Locality?
  - Need (?) to allow 'cocycle' signs $\exp(i\pi p w')$ — related **ambiguity** in torus SFT.
  - We resolved it for the free theory via $\mathcal{V}^{\text{correct}}_{k,\bar{k}}$ with its *associated* reflector $R \implies$ **cocycle sign prescription for higher brackets/vertices?**
- Background independence?
  - The *strongly-constrained DFT* of [Hohm Hull Zwiebach 2010] is non-polynomial and background-independent. . .

. . . does this remain true when we go double?

Thank you!