Gauge Invariant Perturbation Theory via Homotopy Transfer

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[JHEP 05(2021)236 , arXiv: 2012.12249]

SFT Cloud, September 24, 2021

The Big Picture

Are there signatures of massive string modes in the CMB?

Double Field Theory $\longrightarrow$ Cosmology

(weakly constrained)

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Need systematic approach to gauge invariant perturbation theory.

Outline

1. Warm Up: what gauge invariant variables?
2. L_∞-algebras
3. Procedure
4. Gravity on Flat Space
5. Gravity on FLRW

Electrodynamics

Maxwell's Theory:

$$S = -\frac{1}{4}\int \mathrm{d}^4x\, F^{\mu\nu}F_{\mu\nu}\,, \quad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}\,, \text{ invariant under } \delta A_\mu = \partial_\mu \Lambda\,.$$

Goal: Write Maxwell's theory in manifestly gauge invariant form.

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Decomposition of vector field:

$$A_i = \widehat{A}_i + \partial_i\psi\,, \qquad \partial_i \widehat{A}^i = 0\,,$$
$$\psi = \Delta^{-1}(\partial_i A^i)\,, \quad \Delta \equiv \partial^i\partial_i \text{ invertible.}$$

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Maxwell's eqs $\partial_\mu F^{\mu\nu} = 0$ in terms of $(\Phi, \widehat{A}_i)$, where $\Phi \equiv A_0 - \dot{\psi}$:

$$\Delta\Phi = 0 \quad \to \quad \Phi = 0\,, \quad \text{(using } \Delta \text{ invertible)}$$
$$\Box\, \widehat{A}_i + \partial_i \dot{\Phi} = 0 \quad \to \quad \Box \widehat{A}_i = 0\,.$$

Manifestly gauge-invariant action: $\quad S = \frac{1}{2}\int \mathrm{d}^4x \big(\widehat{A}^i\, \Box\, \widehat{A}_i - \Phi\Delta\Phi\big)\,.$

L_∞-algebras

An L_∞-algebra is a graded vector space,

$$X = \bigoplus_{i \in \mathbb{Z}} X_i \,,$$

with multilinear maps,

$$b_k : X^{\otimes k} \to X \,,$$

which are graded symmetric,

$$b_k(x_1, \cdots, x_i, x_{i+1}, \cdots, x_k) = (-1)^{x_i x_{i+1}} b_k(x_1, \cdots, x_{i+1}, x_i, \cdots, x_k) \,,$$

and satisfy generalized Jacobi identites:

$$\sum_{n=1}^{\infty} \sum_{i+j=n+1} b_i b_j = 0 \,, \quad i, j \geq 1 \,.$$

Free Theory

Free theory data are encoded in a chain complex.
Example: Maxwell's theory,

$$\begin{array}{ccccccc} X_1 & \xrightarrow{\partial_1} & X_0 & \xrightarrow{\partial_0} & X_{-1} & \xrightarrow{\partial_{-1}} & X_{-2} \\ \{\Lambda\} & & \{A_\mu\} & & \{E^\mu\} & & \{f\} \\ \textit{GPs} & & \textit{Fields} & & \textit{EOMs} & & \textit{Bianchi IDs} \end{array}$$

where the differentials (which satisfy $\partial_{i-1} \circ \partial_i = 0$) act as:

$$\begin{aligned} \partial_1(\Lambda)_\mu &= \partial_\mu \Lambda\,, \\ \partial_0(A)_\mu &= \Box A_\mu - \partial_\mu(\partial^\nu A_\nu)\,, \\ \partial_{-1}(E) &= \partial_\mu E^\mu\,. \end{aligned}$$

To describe interactions: equip chain complex with an L_∞-algebra.

L_∞-algebras of Gauge Theories: Interactions

Gauge transformations on $\phi \in X_0$ generated by $\Lambda \in X_1$ written as:

$$\delta_\Lambda \phi = \sum_{n \geq 1} \frac{1}{(n-1)!} b_n(\phi, \cdots \phi, \Lambda)$$

The equations of motion are:

$$0 = \sum_{n \geq 1} \frac{1}{n!} b_n(\phi, \cdots, \phi)$$

Homotopy Transfer Theorem

How to obtain theory in terms of gauge invariant $\bar{A}_\mu \equiv (\Phi, \widehat{A}_i)$?

$$
\begin{array}{ccccccc}
X_1 & \underset{s_0}{\overset{\partial_1}{\rightleftarrows}} & X_0 & \underset{s_{-1}}{\overset{\partial_0}{\rightleftarrows}} & X_{-1} & \underset{s_{-2}}{\overset{\partial_{-1}}{\rightleftarrows}} & X_{-2} \\
\iota_0 \Big\uparrow \Big\downarrow p_0 & & \iota_0 \Big\uparrow \Big\downarrow p_0 & & \iota_0 \Big\uparrow \Big\downarrow p_0 & & \iota_0 \Big\uparrow \Big\downarrow p_0 \\
\{0\} & \xrightarrow{\bar{\partial}=0} & \bar{X}_0 & \xrightarrow{\bar{\partial}} & \bar{X}_{-1} & \xrightarrow{\bar{\partial}=0} & \{0\}
\end{array}
$$

HTT states:
Given two quasi-isomorphic chain complexes $(X_\bullet, \partial)$ and $(\bar{X}_\bullet, \bar{\partial})$, L_∞ structure on $X_\bullet$ can be transferred to $\bar{X}_\bullet$ if $\exists s_i : X_i \to X_{i+1}$ s.t:

$$(\iota_0 \circ p_0)_i = \mathrm{id}_{X_i} - \partial_{i+1} \circ s_i - s_{i-1} \circ \partial_i \,,$$

Homological Perturbation Lemma

Homotopy Equivalence Data

$$(X_\bullet, b_1)\,,\ s_0$$

$$\iota_0 \Big\uparrow \Big\downarrow p_0$$

$$(Y_\bullet, c_1)$$

$\iota_0\,, p_0$ quasi-isomorphisms

$$(\iota_0 \circ p_0) = 1 - b_1 \circ s_0 - s_0 \circ b_1$$

Perturbed Data

$$(X_\bullet, b_1 + \delta)\,,\ s$$

$$\iota \Big\uparrow \Big\downarrow p$$

$$(Y_\bullet, c')$$

$$\iota(\iota_0, s_0, \delta)\,,\ p(p_0, s_0, \delta)$$

$$s(s_0, \delta)$$

For small δ, perturbed data is a homotopy equivalence data, i.e.

$\iota\,, p$ are quasi-isomorphisms and $\iota \circ p$ is identity up to homotopy.

The perturbed maps are given by HPL:

$$\iota = (1 + s \circ \delta)^{-1} \circ \iota_0\,, \quad p = p_0 \circ (1 + \delta \circ s)^{-1}\,.$$

In our case, we take $\delta := \sum_{k \geq 2} b_k$.

Application of HTT and HPL

- Applying HPL to obtain gauge invariant variable to all orders:

$$\widehat{A}_\mu = p(A_\mu) = p_0(1 + \delta s)^{-1} A_\mu$$

- We can also define the map (non-linear extension of ι_0):

$$A_\mu = e^{\Delta_{\phi(A)}} \widehat{A}_\mu \,,$$

  where $\phi(A)$ generates infinitesimal gauge transformations.

  For Yang-Mills,

$$A_\mu = \widehat{A}_\mu + \partial_\mu \phi + [\widehat{A}_\mu, \phi] + \frac{1}{2}[\partial_\mu \phi, \phi] + \dots .$$

- Manifestly gauge invariant action can be obtained by replacing A_μ with $\widehat{A}_\mu$.

Gravity on Flat Space

$$
\begin{array}{cccc}
X_1 \xrightarrow{\partial_1} & X_0 \xrightarrow{\partial_0} & X_{-1} \xrightarrow{\partial_{-1}} & X_{-2} \\
\{\xi_\mu\} & \{h_{\mu\nu}\} & \{E_{\mu\nu}\} & \{F_\mu\} \\
\textit{GPs} & \textit{Fields} & \textit{EOMs} & \textit{Bianchi IDs}
\end{array}
$$

where

$$
\begin{aligned}
\partial_1(\xi)_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \,, \\
\partial_0(h)_{\mu\nu} &= G_{\mu\nu} \,, \\
\partial_{-1}(E)_\mu &= \partial^\nu E_{\nu\mu} \,.
\end{aligned}
$$

Decompose $h_{\mu\nu}$:

$$
\begin{aligned}
h_{00} &= -2\phi \,, \\
h_{0i} &= B_i + \partial_i B \,, \\
h_{ij} &= \widehat{h}_{ij} + \partial_i E_j + \partial_j E_i + 2\partial_i \partial_j E + 2\delta_{ij}\big(C - \tfrac{1}{3}\Delta E\big) \,,
\end{aligned}
$$

where $\quad \partial^i B_i = \partial^i E_i = 0 \,, \quad \partial^i \widehat{h}_{ij} = \delta^{ij} \widehat{h}_{ij} = 0 \,.$

Gravity on Flat Space

Gauge invariant combinations (Bardeen variables):

$$\widehat{h}_{ij}\,, \quad \Sigma_i \equiv \dot{E}_i - B_i\,, \quad \Psi \equiv -C + \tfrac{1}{3}\Delta E\,, \quad \Phi \equiv \phi + \dot{B} - \tfrac{1}{2}\ddot{E}$$

We project from the space of fields to the space of gauge invariant ones:

$$p(h_{00}) = -2\Phi\,, \quad p(h_{0i}) = -\Sigma_i\,, \quad p(h_{ij}) = \widehat{h}_{ij} - 2\Psi\delta_{ij}\,.$$

We can then write the original field $h_{\mu\nu}$ in terms of the projection plus pure gauge term:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu\,, \quad \text{where } A_\mu = (A_0\,, A_i) = (B - \dot{E}\,, E_i + \partial_i E)\,.$$

We define a homotopy map $s_0(h)_\mu = A_\mu$ such that

$$(\iota p - \mathrm{id})(h_{\mu\nu}) = \bar{h}_{\mu\nu} - h_{\mu\nu} = -\partial_\mu A_\nu - \partial_\nu A_\mu = -\partial(A)_{\mu\nu}\,.$$

Linearized Einstein-Hilbert action in manifestly gauge invariant form:

$$S = \int d^4x \left\{ \tfrac{1}{4}\widehat{h}^{ij}\,\Box\,\widehat{h}_{ij} - \tfrac{1}{2}\Sigma^i\Delta\Sigma_i + (4\Phi - 2\Psi)\Delta\Psi + 6\Psi\ddot{\Psi} \right\}$$

Gravity on FLRW Backgrounds

Einstein-Hilbert Action with Minimally Coupled Scalar Field:

$$S = \int d^4x \sqrt{-g} \left\{ R - \tfrac{1}{2} g^{\mu\nu} \partial_\mu \mathcal{X} \partial_\nu \mathcal{X} - V(\mathcal{X}) \right\},$$

Expand around FLRW:

$$\begin{aligned} g_{\mu\nu}(\eta, x) \; &= \; a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu}(\eta, x)), \\ \mathcal{X}(\eta, x) \; &= \mathcal{X}^{(0)}(\eta) + \varphi(\eta, x). \end{aligned}$$

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To quadratic order,

$$\begin{aligned} S = \int d^4x\; a^2\Big\{ & \tfrac{1}{4}\dot{h}^{ij}\dot{h}_{ij} + \tfrac{1}{4}h_{ij}\Delta h^{ij} + \tfrac{1}{2}\partial_j h^{ij}\partial^k h_{ik} + \tfrac{1}{2}(h_i{}^i - h_{00})\partial^j\partial^k h_{jk} \\ & + \tfrac{1}{2}h_{0i}\Delta h^{0i} - \tfrac{1}{2}(\partial_i h^{0i})^2 + \partial^i h^{0j}\dot{h}_{ij} + \partial^i h_{0i}\dot{h}_j{}^j - \tfrac{1}{4}(\dot{h}_i{}^i)^2 - \tfrac{1}{4}h_i{}^i\Delta h_j{}^j \\ & + \tfrac{1}{2}h_i{}^i\Delta h_{00} \; - \tfrac{1}{2}(\dot{H} + 2H^2)h_{00}^2 - Hh_{00}\dot{h}_i{}^i + 2Hh_{00}\partial^i h_{0i} \\ & + \tfrac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\varphi\Delta\varphi - \tfrac{1}{2}a^2V''(\mathcal{X}^{(0)})\varphi^2 - \tfrac{1}{2}\dot{\mathcal{X}}^{(0)}\varphi\big(\dot{h}_{00} + \dot{h}_i{}^i + 2\partial_i h^{0i}\big) \\ & + a^2V'(\mathcal{X}^{(0)})\varphi h_{00}\Big\}. \end{aligned}$$

Gravity on FLRW Backgrounds

Chain complex:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{\partial_1} & X_0 & \xrightarrow{\partial_0} & X_{-1} & \xrightarrow{\partial_{-1}} & X_{-2} \\ \{\xi_\mu\} & & \{h_{\mu\nu}\,,\varphi\} & & \{E_{\mu\nu}\} & & \{F_\mu\} \\ \textit{GPs} & & \textit{Fields} & & \textit{EOMs} & & \textit{Bianchi IDs} \end{array}$$

where

$$\partial_1(\xi)_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - 2H\xi_0\eta_{\mu\nu}\ ,$$
$$\partial_1(\xi)_\bullet = -\dot{\mathcal{X}}^{(0)}\xi_0\ .$$

Gauge invariant variables:

$$\widehat{h}_{ij}\,, \qquad \Sigma_i = \dot{E}_i - B_i\,, \qquad \Psi = -C + \frac{1}{3}\Delta E - H(B - \dot{E})\,,$$
$$\Phi = \phi + H(B - \dot{E}) + \dot{B} - \ddot{E}\,, \quad \Theta = \varphi + \dot{\mathcal{X}}^{(0)}(B - \dot{E})\,.$$

Projected fields

$$\bar{h}_{ij} = \widehat{h}_{ij} - 2\Psi\delta_{ij}\ , \qquad \bar{h}_{0i} = -\Sigma_i\ , \qquad \bar{h}_{00} = -2\Phi\ , \qquad \bar{\varphi} = \Theta\ .$$

Gravity on FLRW Backgrounds

Write fields as projections plus pure gauge terms:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu - 2HA_0\eta_{\mu\nu} \ ,$$
$$\varphi = \bar{\varphi} - \dot{\mathcal{X}}^{(0)} A_0 \ ,$$

where

$$A_\mu = (A_0, A_i) = (B - \dot{E}, E_i + \partial_i E) \ .$$

We can find the homotopy maps by computing

$$(\iota p - \mathrm{id})(h_{\mu\nu}) = \bar{h}_{\mu\nu} - h_{\mu\nu} = -\partial_\mu A_\nu - \partial_\nu A_\mu + 2HA_0\eta_{\mu\nu} = -\partial(A)_{\mu\nu} \ ,$$
$$(\iota p - \mathrm{id})(\varphi) = \bar{\varphi} - \varphi = \dot{\mathcal{X}}^{(0)} A_0 = -\partial(A)_\bullet \ ,$$

from which we can infer

$$s(h) = s(\varphi) = A_\mu \in X_1 \ .$$

Gravity on FLRW Backgrounds

Gauge invariant E-H action expanded to quadratic order around FLRW:

$$
\begin{aligned}
S \;=\; \int d^4x\; a^2 \Big\{ & \frac{1}{4}\dot{\widehat{h}}^{ij}\dot{\widehat{h}}_{ij} + \frac{1}{4}\widehat{h}^{ij}\Delta\widehat{h}_{ij} - \frac{1}{2}\Sigma_i\Delta\Sigma^i \\
& + 4\Psi\Delta\Phi - 2\Psi\Delta\Psi - 6(\dot{\Psi} + H\Phi)^2 + \frac{1}{2}\dot{\mathcal{X}}^{(0)2}\Phi^2 \\
& + \frac{1}{2}\dot{\Theta}^2 + \frac{1}{2}\Theta\Delta\Theta - \frac{1}{2}a^2V''(\mathcal{X}^{(0)})\Theta^2 \\
& + \dot{\mathcal{X}}^{(0)}\Theta(\dot{\Phi} + 3\dot{\Psi}) - 2a^2V'(\mathcal{X}^{(0)})\Theta\Phi \Big\}\,.
\end{aligned}
$$

How can we simplify/diagonalize the action containing the scalar modes? $\rightarrow$ Need to deal with constraints!

Gravity on FLRW Backgrounds

After solving constraints, we find the Mukhanov-Sasaki action,

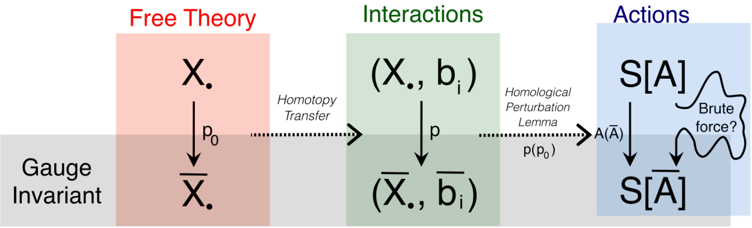
$$S = \int d^4x \left\{ \frac{1}{2} v \Delta v + \frac{1}{2} \dot{v}^2 + \frac{1}{2} \frac{\ddot{z}}{z} v^2 \right\},$$

where $v \equiv a\Theta + z\Psi$ and $z \equiv a\dot{\chi}^{(0)}/H$. To write gauge invariant action to cubic order, we use the perturbation lemma to write:

$$\hat{h}_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2} p_0[s(\varphi\,,h)\,,(1+\iota_0 p_0)h_{\mu\nu}]$$

$$\hat{\varphi} = \bar{\varphi} + \frac{1}{2} p_0[s(\varphi\,,h)\,,(1+\iota_0 p_0)\varphi]$$

Summary and Outlook



- How to apply algebraic framework to compute n-pt functions in manifestly gauge invariant way?
- Can one use homotopy transfer to solve constraints?
- Application to cosmological double field theory.