Magnetically Deformed Neutron Stars: An Analytic Approach

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Overview

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   - The supposed origin of the magnetic field
   - Magnetic Field Configuration
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   - Magnetic field configurations
   - Magnetic Deformations
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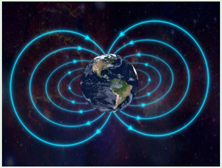
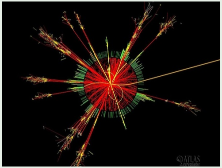
Motivation: Magnetic Fields

Earth $B \sim 0.5G$

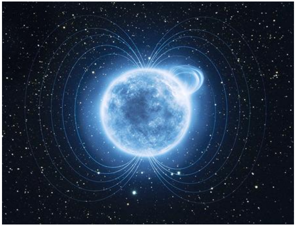


MRI $\sim 10^4 G$

LHC $B \sim 10^{20} G$



- Typical NS magnetic field $B \sim 10^{12} G$
- Magnetars [1] $B \sim (10^{14} - 10^{15}) G$
- powered by mag. energy



AXps and SGRs: Two Calsses of Magnetars

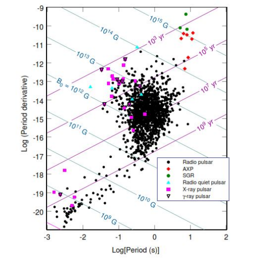
Anomalous X-ray Pulsars

- $L_x \sim (10^{34} - 10^{36})\frac{erg}{s}$
- Soft X-ray spectrum
- Rotational period of a few seconds
- Spin down $(0.05 - 4) \times 10^{-11}\frac{s}{s}$

Soft Gamma-Ray Repeaters

- $L_x \sim (10^{35} - 10^{36})\frac{erg}{s}$
- Emit repeating bursts of hard X / soft gamma-rays
- Rotational period of a few seconds
- Spin down $(10^{-11} - 10^{-10})\frac{s}{s}$

$p - \dot{p}$ diagram



Main properties

1. No evidence of binary companion.
2. Young objects (2-220 kyrs).
3. About 10% of all neutron

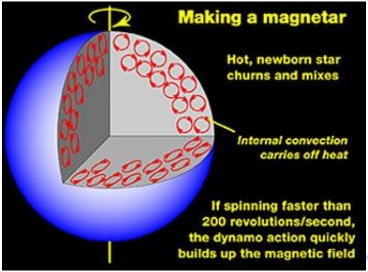
The supposed origin of the magnetic field

It is as it is, because it was as it was!

Small magnetic field of a progenitor star can be amplified during the star's collapse due to magnetic flux conservation

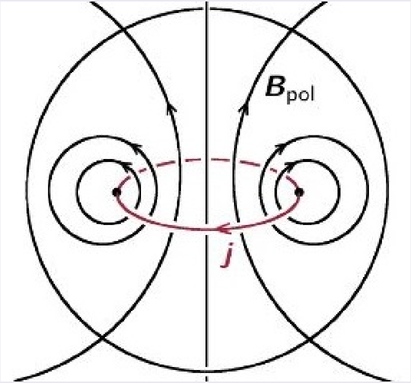
Magnetohydrodynamic dynamo mechanism!

The amplification of the stellar magnetic fields through the combination of rapid rotation and convective processes in the plasma during the proto–neutron star phase.

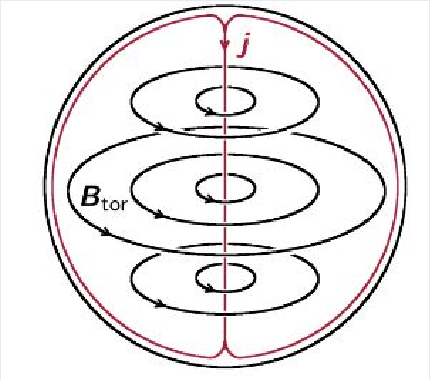


Magnetic Field Configuration

Purely Poloidal Magnetic Field



Purely Toroidal Magnetic Field



$$\mathbf{B} = \mathbf{B_{pol}} + \mathbf{B_{tor}}$$

An Analytic Treatment of Magnetically Deformed Neutron Stars: Overview

Density profiles are obtained from various analytic approaches

1. n=1 solution of Lane-Emden equation
2. parabolic profile
3. a fourth-order polynomial

Barotropic equation of state

$$\rho = \rho(p)$$

Euler's MHD equations are solved

1. purely poloidal
2. purely toroidal

We regard magnetic fields as perturbation

Magnetic field produces small changes from a spherically symmetric star.

magnetic deformation

1. prolate shape for positive distortion
2. oblate shape for negative distortion

LOCV method

In the LOCV method, we consider a trial many-body wave function of the form $\psi = \mathcal{F}\phi$, in which ϕ is Slater determinant of the plane waves and $\mathcal{F} = \mathcal{F}(1 \cdots N)$ is a proper N-body correlation operator which can be replaced by a Jastrow form i.e.,

$$\mathcal{F} = \mathcal{S} \prod_{i>j} f(ij),$$

E is the total energy per particle of spin polarized neutron matter:

$$E = \frac{1}{N} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = E_1 + E_2.$$

We consider the cluster expansion of the energy functional up to the two-body term.

LOCV method

$$E_1 = \sum_{\sigma=\uparrow,\downarrow} \sum_{k \le k_F{}^\sigma} \frac{\hbar^2 k^2}{2m_n}$$

Fermi momentum: $k_F{}^\sigma = (6\pi^2 \rho^\sigma)^{1/3}$

$$E_2 = \frac{1}{2N} \sum_{ij} \langle ij | \nu(12) | ij - ji \rangle$$

$$\nu(12) = -\frac{\hbar^2}{2m_n(B)} [f(12), [\nabla_{12}^2, f(12)]] + f(12) V(12) f(12)$$

$f(12)$: two-body correlation function, $V(12)$: two-body potential

Energy Density

$$\varepsilon_m = \rho(E + m_n)$$

Density configurations

Important to know

Our main reason for focusing on these models is that they are sufficiently accurate and they allow us to do the calculations analytically.

model 1: n=1 solution of Lane-Emden equation [3]

$$\rho_0(y) = \rho_c \frac{\sin y}{y}$$

model 2: parabolic profile

$$\rho_0(x) = \rho_c(1 - x^2)$$

model 3: fourth-order polynomial with a free parameter [4]

$$\rho_0(x) = \rho_c(1 - \alpha x^2 + (\alpha - 1)x^4)$$

$$y = \frac{\pi r}{R_*}, \qquad \rho_c = 3.95 \times 10^{15}\ gcm^{-3}, \qquad x = \frac{r}{R_*}.$$

Describing free parameter α in model 3

$\alpha = 1.31$ (by calculating M_* in Newtonian gravity)

$$\alpha = \frac{80\pi\rho_c R_*^3 - 105 M_*}{24\pi\rho_c R_*^3}$$

$\alpha = 1.38$ (by fitting density profile of model 3 to the realistic profile gained numerically from microscopic EoS obtained from the LOCV method [5])

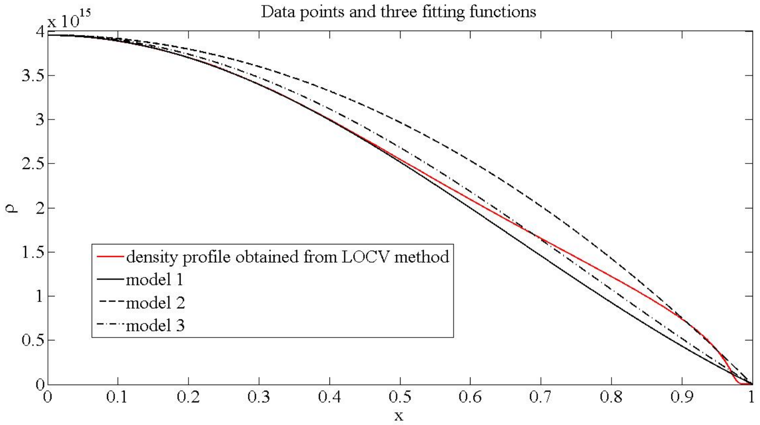
$$\alpha = a_0 + a_1\left(\frac{C^n}{\rho_c R_*^2}\right) + a_2\left(\frac{C^n}{\rho_c R_*^2}\right)^2$$

Table: parameters of Eq. 1

a_0	a_1	a_2	n	α
3.8066	-1.4627	0.0378	1.0028	1.38

$C = \frac{M_*}{R_*}$, $M_* = 3.41 \times 10^{33}$ g, $R_* = 8.4$ km.

Denstity profiles



The density profile of non-magnetized neutron matter obtained from LOCV method and three considered fitting functions.

Magnetic field configurations

General relations of magnetic field configurations

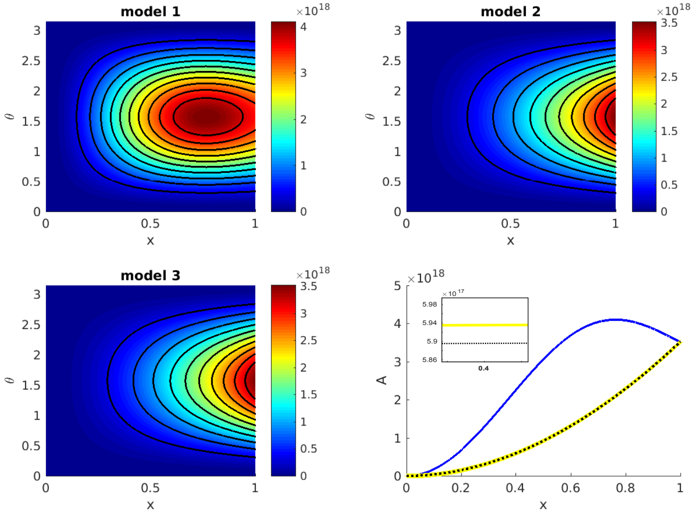
$$\mathbf{B} = \mathbf{B}_p + \mathbf{B}_t$$
$$\mathbf{B} = \nabla S(r,\theta) \times \nabla\phi + \beta(r,\theta)\nabla\phi$$
$$S(r,\theta) = A(r)\sin^2\theta$$
$$\mathbf{B}_p = (\frac{2A(r)\cos\theta}{r^2}, \frac{-A'(r)\sin\theta}{r}, 0)$$
$$\mathbf{B}_t = (0, 0, \frac{\beta(S)}{r\sin\theta})$$

Euler equations in Newtonian framework

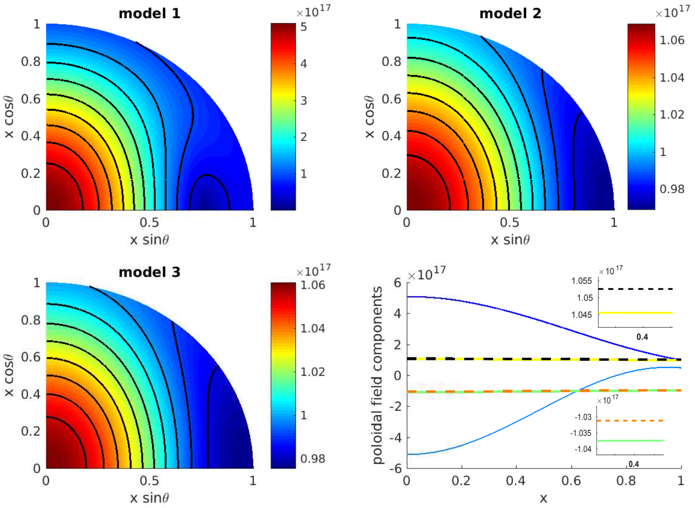
$$\frac{\nabla p}{\rho} + \nabla\phi = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi\rho}$$
$$\nabla^2\phi = 4\pi G\rho$$
$$\nabla.\mathbf{B} = 0$$
$$\nabla \times \left[\frac{\mathbf{B} \times (\nabla \times \mathbf{B})}{\rho}\right] = 0$$

$$\frac{\partial}{\partial r}\left[\frac{2A\sin\theta\cos\theta}{\rho r^2}(A'' - \frac{2A}{r^2}) + \frac{\beta}{\rho r^2\sin^2\theta}\frac{\partial\beta}{\partial\theta}\right]$$
$$-\frac{\partial}{\partial\theta}\left[\frac{A'\sin^2\theta}{\rho r^2}(A'' - \frac{2A}{r^2}) + \frac{\beta}{\rho r^2\sin^2\theta}\frac{\partial\beta}{\partial r}\right] = 0$$

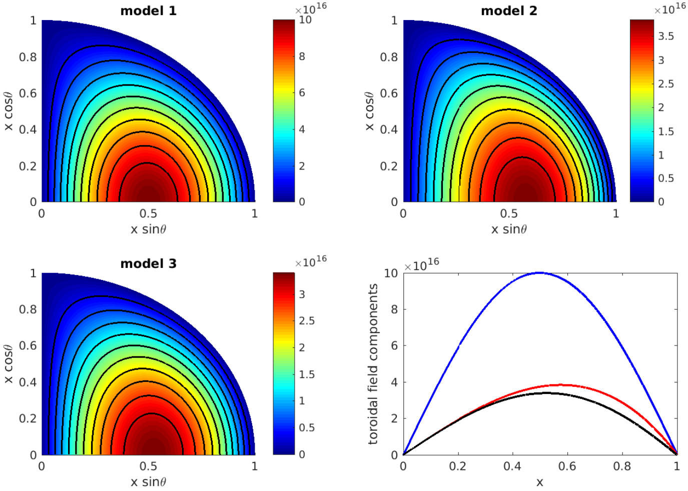
Contour plots of stream function



Contour plots of poloidal magnetic field



Contour plots of toroidal magnetic field



Magnetic Deformations

We assume that the magnetic field only produces small changes from a spherically symmetric star. This allows us to expand all the equilibrium quantities in the form,

$$\begin{aligned}
\rho(r,\theta) &= \rho_0(r) + \delta\rho(r)P_l(\cos\theta) \\
p(r,\theta) &= p_0(r) + \delta p(r)P_l(\cos\theta) \\
\phi(r,\theta) &= \phi_0(r) + \delta\phi(r)P_l(\cos\theta)
\end{aligned}$$

For the perturbed equilibrium condition, the following equations can be derived,

$$\left(\frac{d\delta p}{dr} + \rho_0\frac{d\delta\phi}{dr} + \delta\rho\frac{d\phi_0}{dr}\right)P_2 = \frac{[(\nabla\times\mathbf{B})\times\mathbf{B}]_r}{4\pi} = \frac{L_r}{4\pi}$$

$$(\delta p + \rho_0\delta\phi)\frac{dP_2}{d\theta} = r\frac{[(\nabla\times\mathbf{B})\times\mathbf{B}]_\theta}{4\pi} = \frac{L_\theta}{4\pi}$$

which must be solved together with the perturbed Poisson equation,

$$\frac{d^2\delta\phi}{dr^2} + \frac{2}{r}\frac{d\delta\phi}{dr} - \frac{6}{r^2}\delta\phi = 4\pi G\delta\rho$$

Values of distortion at the surface

Table: The values of distortion at the surface (ε) and the radius deviation from background radii (δR) for all density distributions and for purely poloidal and purely toroidal magnetic fields [7].

model	magnetic field	$\varepsilon = -\frac{\delta\rho(R)}{R}\left[\frac{d\rho_0}{dr}\right]^{-1}_{r=R}$	$\delta R(km)$
1	poloidal	-0.0025	0.0206
2	poloidal	-1.1442×10^{-4}	9.6066×10^{-4}
3	poloidal	-4.5221×10^{-5}	3.7965×10^{-4}
1	toroidal	0.0061	0.0512
2	toroidal	0.0126	0.1061
3	toroidal	0.0409	0.3434

We begin by the hydromagnetic equilibrium equation,

$$\frac{\nabla p}{\rho} + \nabla\phi = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi\rho} = \frac{\mathbf{L}}{4\pi\rho}$$

In barotropic stars ∇p can be described in the form of a function gradient [6].

$$\nabla p = -\rho\nabla\left(\phi + \omega\Omega_A^2 a(1 - P_2)\right) = -\rho\nabla\Psi$$

where $\Omega_A^2 = \frac{B_s^2}{4\pi R_*^2 \rho_c}$ is the Alfv̀en frequency, a is a stream function without the normalization factor and ω is a constant.
The function Ψ and ρ may be expanded in terms of the parameter Ω_A:

$$\begin{aligned}
\Psi(r,\theta) &= \Psi_0(r) - 2R_*^2\Omega_A^2\left[\psi_0(r) + \psi_2(r)P_2(cos\theta)\right] \\
\rho(r,\theta) &= \rho_0(r) - 2R_*^2\Omega_A^2\frac{d\rho_0}{d\Psi_0}\left[\psi_0(r) + \psi_2(r)P_2(cos\theta)\right]
\end{aligned}$$

Change in mass of the star due to the magnetic field

toroidal magnetic field

$$\nabla p = -\rho \nabla \left(\phi + \frac{R_*^2}{3} \Omega_A^2 a (1 - P_2) \right) = -\rho \nabla \Psi$$

$$M = 2\pi \int \rho r^2 dr \sin\theta d\theta = M_* + \Delta m$$

$$\Delta m = \frac{R_*^2}{G} \frac{d\delta\phi}{dr}$$

Change in mass of the star due to the magnetic field

Table: The constant ω, the stream function without the normalization factor (a) and the change in the mass of the star (Δm) for all density distributions and for purely poloidal and purely toroidal magnetic fields.

model	field	ω	a	$\Delta m \times 10^{31}(g)$
1	p	$\frac{-\pi^4 R_*^2}{2(\pi^2-6)^2}$	$y^2 + \sin y(3y - \frac{6}{y}) + 6\cos y$	-3.8422
2	p	$\frac{-2R_*^2}{3}(\frac{70}{131})^2$	$x^2 - \frac{1}{10}x^4 + \frac{1}{28}x^6$	-4.0084
3	p	$\frac{-2R_*^2}{3\gamma^2}$	$x^2 - \frac{1}{10}x^4 + \frac{\alpha}{28}x^6 - \frac{\alpha-1}{54}x^8$	-1.1331
1	t	-	$y \sin y$	5.4454
2	t	-	$x^2 - x^4$	0.70760
3	t	-	$x^2 - \alpha x^4 + (\alpha - 1)x^6$	2.4301

Summary

1. The magnetic field profiles are influenced by the choice of the density distributions but increase relatively slowly from the surface to the center of magnetars for all the density profiles;
2. The poloidal magnetic fields pervade most of the interior of the stars, which their maximum occur in the center of the star;
3. The toroidal magnetic fields are confined inside the stars, which their maximum occur in the center of torus bounding the toroidal field;
4. The poloidal magnetic fields induces an oblate distortions, while the toroidal magnetic field cause the stellar deformation be prolate;
5. The magnetic field configuration acquired from the model 3 can be generalized to various realistic EoSs, regarding to its free parameter α;
6. We assume that the magnetic field distortions are small which enables us a perturbative treatment;

Next step

Constructing realistic models for strongly magnetized relativistic rotating neutron stars (employing numerical relativity and GRMHD codes).

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Thank you