The Gravitational Energy-Momentum Pseudo-Tensor in Higher Order Theories of Gravity

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Summary

We discuss the generalization of gravitational energy-momentum pseudo-tensor to Extended Theories of Gravity, in particular to higher-order theories in curvature invariants. This result is achieved by imposing that the local variation of gravitational action of any order $n$ vanishes under rigid translations. We also prove that this tensor, in general, is not covariant but only affine, that is, it is a pseudo-tensor. The pseudo-tensor $\tau^\mu_{\alpha}$ is calculated in the weak-field limit up to a first non-vanishing term of order $h^2$, where $h$ is the metric perturbation. The average value of the pseudo-tensor, over a suitable spacetime domain, is obtained. Finally, we calculate the emitted power, per unit solid angle $\Omega$, carried by a gravitational wave in the direction $\hat{x}$ for a fixed wave number $k$ under a suitable gauge.
1. The Energy-Momentum Pseudo-Tensor in General Relativity

2. The Energy-Momentum Pseudo-Tensor for Lagrangians of order $n$
   - Non-covariance of gravitational energy-momentum tensor

3. The Energy-Momentum Pseudo-Tensor of $L_{\Box^k R}$ Lagrangians
   - Weak field limit of Energy-Momentum Pseudo-Tensor for $L_{\Box^k R}$ Lagrangian at the order $h^2$ in harmonic gauge
   - Particular cases
   - Average of the energy-momentum pseudo-tensor
   - Power emitted by a gravitational radiating source

In GR, there is no unanimously accepted definition of energy-momentum of the gravitational field. Some prescriptions have been given by Einstein, Landau-Lifshitz, Papapetrou, Weinberg, and Möller.

The "non-tensoriality" and the "affine" character of the gravitational energy-momentum "tensor" make the energy and momentum of the gravitational field non-localizable.

However, it is possible to define the energy-momentum of total gravitational field in an asymptotically flat spacetime almost independently of the coordinates.

The continuity equation of Special Relativity $\partial_{\mu} T^{\mu\nu} = 0$, in GR, becomes

$$\nabla_{\mu} T^{\mu\nu} = 0 \quad \rightarrow \quad \frac{1}{\sqrt{-g}} \partial_{\nu} \left( \sqrt{-g} T^{\mu\nu} \right) + \Gamma^{\mu}_{\nu\lambda} T^{\lambda\nu} = 0.$$ 

It is not, in principle, a conservation law.
Einstein postulated a local conservation law by introducing a pseudo-tensor $\tau^{\mu\nu}$ related to the energy-momentum of the gravitational field

$$\partial_\mu \left( \sqrt{-g} \left( T^{\mu\nu} + \tau^{\mu\nu} \right) \right) = 0$$

$$\sqrt{-g} \tau^\nu_\mu = \frac{1}{16\pi} \left( \delta^\nu_\mu L - \frac{\partial L}{\partial g^{\rho\sigma},_\nu} g^{\rho\sigma},_\mu \right)$$

depending on the metric $g_{\mu\nu}$ and its derivatives, being

$$L = \sqrt{-g} g^{\mu\nu} \left( \Gamma^\sigma_\mu_\nu \Gamma^\rho_\sigma_\rho - \Gamma^\sigma_\mu_\rho \Gamma^\rho_\nu_\sigma \right).$$

The pseudo-tensor $\tau^{\mu\nu}$ does not transform as a tensor under generic coordinate transformations but under affine transformations.
From the pseudo-tensor character of $\tau^{\mu\nu}$, we have:

- Energy-momentum of the gravitational field, in a given region of the universe, depends on the coordinate system, i.e. it is not localizable.

If we choose a space-time domain $\Omega$, verifying the

- spatial asymptotic flatness condition where the metric asymptotically joins with continuity with the Minkowski one and fields and derivatives rapidly go to zero, by the Gauss theorem, we can

- define the energy-momentum of the gravitational field plus that of non-gravitational fields, contained in $V$ independently of the coordinate choice as

$$ P^\nu = \int_V \sqrt{-g} \left( T^{0\nu} + \tau^{0\nu} \right) d^3x $$

here $V$ is an infinite spatial hypersurface defined at $t$ constant.
In order to calculate the gravitational pseudo-tensor for fourth-order Lagrangians, let us consider the Noether Theorem for rigid translations. Let us define

$$L = L \left( g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\lambda}, g_{\mu\nu,\rho\lambda\xi}, g_{\mu\nu,\rho\lambda\xi\sigma} \right)$$

The variation $\tilde{\delta}$ with respect to the metric $g_{\mu\nu}$ and coordinates $x^\mu$ is

$$S = \int_\Omega d^4x L \rightarrow \tilde{\delta}S = \int_{\Omega'} d^4x' L' - \int_\Omega d^4x L = \int_\Omega d^4x \left[ \delta L + \partial_\mu (L \delta x^\mu) \right] .$$

Here $\delta$ represents the variation with fixed coordinates $x^\mu$. An infinitesimal translation is:

$$x'^\mu = x^\mu + \epsilon^\mu \left( x \right)$$

and the variation of metric tensor is

$$\delta g_{\mu\nu} = g'_{\mu\nu} \left( x \right) - g_{\mu\nu} \left( x \right) = -\epsilon^\alpha \partial_\alpha g_{\mu\nu} - g_{\mu\alpha} \partial_\nu \epsilon^\alpha - g_{\nu\alpha} \partial_\mu \epsilon^\alpha$$
The Energy-Momentum Pseudo-Tensor for Lagrangians of order $n$

The metric variation, under global transformations $\partial_\lambda \epsilon^\mu = 0$, is $\delta g_{\mu\nu} = -\epsilon^\alpha \partial_\alpha g_{\mu\nu}$ and, if we require the action invariant under these transformations, i.e. $\tilde{\delta} S = 0$, for an arbitrary integration domain $\Omega$, we get:

$$0 = \delta L + \partial_\mu (L \delta x^\mu) = \left( \frac{\partial L}{\partial g_{\mu\nu}} - \partial_\rho \frac{\partial L}{\partial g_{\mu\nu,\rho}} + \partial_\rho \partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}} - \partial_\rho \partial_\lambda \partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} + \partial_\rho \partial_\lambda \partial_\xi \partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} \right) \delta g_{\mu\nu} - \partial_\eta \left( 2\chi \sqrt{-g} \tau^\eta_\alpha \right) \epsilon^\alpha.$$

From the Euler-Lagrange equations, we have:

$$\frac{\partial L}{\partial g_{\mu\nu}} - \partial_\rho \frac{\partial L}{\partial g_{\mu\nu,\rho}} + \partial_\rho \partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}} - \partial_\rho \partial_\lambda \partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} + \partial_\rho \partial_\lambda \partial_\xi \partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} = 0.$$

We obtain the continuity equation:

$$\partial_\eta \left( \sqrt{-g} \tau^\eta_\alpha \right) = 0$$

for any $\epsilon^\alpha$ where $\tau^\eta_\alpha$ is the gravitational energy-momentum pseudo-tensor.
The case of Energy-Momentum Pseudo-Tensor for Lagrangians of order 4

The energy-momentum pseudo-tensor for Lagrangians depending on fourth-order derivatives in the metric $g_{\mu\nu}$ is

$$\tau^{\eta}_{\alpha} = \frac{1}{2\chi \sqrt{-g}} \left[ \left( \frac{\partial L}{\partial g_{\mu\nu,\eta}} - \partial_{\lambda} \frac{\partial L}{\partial g_{\mu\nu,\eta\lambda}} + \partial_{\lambda} \partial_{\xi} \frac{\partial L}{\partial g_{\mu\nu,\eta\lambda\xi}} \right) g_{\mu\nu,\alpha} + \left( \frac{\partial L}{\partial g_{\mu\nu,\rho\eta}} - \partial_{\xi} \frac{\partial L}{\partial g_{\mu\nu,\rho\eta\xi}} + \partial_{\xi} \partial_{\sigma} \frac{\partial L}{\partial g_{\mu\nu,\rho\eta\xi\sigma}} \right) g_{\mu\nu,\alpha\rho} \right.$$  
$$\left. + \left( \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\eta}} - \partial_{\sigma} \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\eta\sigma}} \right) g_{\mu\nu,\rho\lambda\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} g_{\mu\nu,\rho\lambda\xi\alpha} - \delta^{\eta}_{\alpha} L \right]$$

where $\chi = \frac{8\pi G}{c^4}$ is the gravitational coupling and the metric derivatives are up to 7-th order.
Let us consider now a general Lagrangian density depending on the $n$-th derivatives of $g_{\mu\nu}$

$$ L = L (g_{\mu\nu}, g_{\mu\nu, i_1}, g_{\mu\nu, i_1 i_2}, g_{\mu\nu, i_1 i_2 i_3}, \cdots, g_{\mu\nu, i_1 i_2 i_3 \cdots i_n}) $$

The gravitational pseudo-tensor for Lagrangians of order $n$ is

$$ \tau_{\eta\alpha} = \frac{1}{2 \chi \sqrt{-g}} \left[ \sum_{m=0}^{n-1} (-1)^m \left( \frac{\partial L}{\partial g_{\mu\nu, \eta i_0 \cdots i_m}} \right)_{,i_0 \cdots i_m} g_{\mu\nu, \alpha} ight. $$

$$ + \Theta[2, +\infty] \left[ (n) \sum_{j=0}^{n-2} \sum_{m=j+1}^{n-1} (-1)^j \left( \frac{\partial L}{\partial g_{\mu\nu, \eta i_0 \cdots i_m}} \right)_{,i_0 \cdots i_j} g_{\mu\nu, i_{j+1} \cdots i_m \alpha} - \delta_{\eta\alpha} L \right] $$

depending up to $2n - 1$ derivatives in the metric $g_{\mu\nu}$.

$()_{,i_0} = 1$

$()_{,i_0 \cdots i_m} = \begin{cases} 
(),_1 & \text{if } m = 1 \\
(),_{1i_2} & \text{if } m = 2 \\
(),_{1i_2i_3} & \text{if } m = 3 \\
\text{and so on} & 
\end{cases}$

S. Capozziello, M. Capriolo
Continuity Equation

The field equations associated to a generic Lagrangian, in presence of matter, are now $P^{\eta \alpha} = \chi T^{\eta \alpha}$ where

$$P^{\eta \alpha} = \frac{1}{\sqrt{-g}} \frac{\delta L_g}{\delta g_{\eta \alpha}}, \quad \text{with the coupling} \quad \chi = \frac{8\pi G}{c^4}$$

From the Lagrangian invariance for rigid translations and from the symmetry of $T^{\eta \alpha}$, we have

$$\partial_{\eta} \left[ \sqrt{-g} \left( \tau^{\eta \alpha} + T^{\eta \alpha} \right) \right] = \sqrt{-g} T^{\eta \alpha}_{\alpha;\eta}$$

Continuity Equation

$$P^{\eta \alpha}_{;\eta} = 0 \iff T^{\eta \alpha}_{;\eta} = 0 \iff \partial_{\eta} \left[ \sqrt{-g} \left( \tau^{\eta \alpha} + T^{\eta \alpha} \right) \right] = 0$$

The conserved quantities are not only the energy and momentum associated to the matter and non-gravitational fields but the overall contribution of these fields plus the energy-momentum of the gravitational field.
Let us now integrate the continuity equation on a spatial domain $\Sigma$, which is the foliation of the 4D space-time at a fixed $t$, where fields and their derivatives go to zero in a sufficiently rapid way on the boundary $\partial \Sigma$. Using the Gauss theorem, the surface integrals go to zero

$$\partial_0 \int_{\Sigma} d^3 x \sqrt{-g} \left( T^{\mu 0} + \tau^{\mu 0} \right) = - \int_{\partial \Sigma} d\sigma_i \sqrt{-g} \left( T^{\mu i} + \tau^{\mu i} \right) = 0$$

The overall contribution of energy-momentum in the volume $\Sigma$ is defined as

$$P^\mu = \int_{\Sigma} d^3 x \sqrt{-g} \left( T^{\mu 0} + \tau^{\mu 0} \right)$$

depending on the coordinate choice. These conditions are often realized for isolated systems where it is possible to derive the spatial asymptotic flatness so that $P^\mu$ is independent of the coordinates and transforms as a 4-vector.
Non-covariance of gravitational energy-momentum tensor

It is possible to demonstrate that $\tau^\eta_\alpha$ is not, in general, a covariant tensor but it behaves as a tensor only under affine transformations. This means it is a pseudo-tensor. Let us consider first the particular case with a Lagrangian density of order 2

$$\tau^\eta_\alpha = \frac{1}{2\chi \sqrt{-g}} \left[ \left( \frac{\partial L}{\partial g_{\mu\nu,\eta}} - \partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\eta\lambda}} \right) g_{\mu\nu,\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\eta\xi}} g_{\mu\nu,\xi\alpha} - \delta_\alpha^\eta L \right]$$

In general, under diffeomorphisms $x' = x' (x)$, it is

$$\tau'^\eta_\alpha (x') \neq J^\eta_\sigma J^{-1\tau}_\alpha \tau^\sigma_\tau (x)$$

where the Jacobian matrix and the determinant are defined as

$$J^\eta_\sigma = \frac{\partial x'^\eta}{\partial x^\sigma} \quad J^{-1\tau}_\alpha = \frac{\partial x^\tau}{\partial x'^\alpha} \quad \text{det} \left( J^\alpha_\beta \right) = J$$

On the other hand, under linear affine transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \quad J^\mu_\nu = \Lambda^\mu_\nu \quad |\Lambda| \neq 0$$

the tensor transforms as

$$\tau'^\eta_\alpha (x') = \Lambda^\eta_\sigma \Lambda^{-1\tau}_\alpha \tau^\sigma_\tau (x)$$
Non-covariance of gravitational energy-momentum tensor

\[ g'_{\mu\nu,\alpha}(x') = J^{-1\alpha}_{\mu} J^{-1\beta}_{\nu} J^{-1\gamma}_{\tau} g_{\alpha\beta,\gamma}(x) + \partial'_{\tau} \left[ J^{-1\alpha}_{\mu} J^{-1\beta}_{\nu} \right] g_{ab}(x) \]

\[ \tau'_{\eta\alpha}(x') = J_{\sigma}^{\eta} J^{-1\tau}_{\sigma} \tau^{\tau}(x) + \left\{ \text{containing terms} \frac{\partial^2 x}{\partial x'^2}, \frac{\partial^3 x}{\partial x'^3} \right\} \]

This result derives from the non-covariance of metric tensor \( g_{\mu\nu} \) derivatives. These derivatives give rise to the affine tensor. In general,

\[ g'_{\mu\nu,i_1\ldots i_m\alpha}(x') = J^{-1\alpha}_{\mu} J^{-1\beta}_{\nu} J^{-1\gamma}_{i_1} \ldots J^{-1\tau}_{i_m} g_{\alpha\beta,\gamma,i_1\ldots i_m}(x) \]

\[ + \left\{ \text{containing terms} \frac{\partial^2 x}{\partial x'^2}, \ldots, \frac{\partial^{m+2} x}{\partial x'^{m+2}} \right\} \]

and

\[ \frac{\partial L'}{\partial g'_{\mu\nu,\eta i_0\ldots i_m}} = J^{-1\gamma}_{\mu} J^{\nu}_{\rho} J^{\eta}_{\tau} J^{i_1}_{j_1} \ldots J^{i_m}_{j_m} \frac{\partial L}{\partial g_{\gamma\rho,\tau,j_1\ldots j_m}} \]

tensor densities \((m+3,0)\)

do weight \( w = -1 \)

from which the non-covariance of tensor \( \tau_{\eta\alpha} \).
Let us calculate now the energy-momentum pseudo-tensor $\tau^\eta_\alpha$ for the gravitational Lagrangian

$$L^{\square k R} = (\bar{R} + a_0 R^2 + \sum_{k=1}^{p} a_k R^{\square k R})\sqrt{-g}$$

where $\bar{R}$ is the part of curvature Ricci scalar $R$ depending only on $g_{\mu\nu}$ and its first derivatives. If the variation of action $S^{\square k R}$ is zero for rigid translations $\tilde{\delta}_{g, x} S^{\square k R} = 0$ with $g_{\mu\nu}$ satisfying the Euler-Lagrange equations, we have

$$\tau^\eta_\alpha = \tau^\eta_\alpha|_{GR} + \tilde{\tau}^\eta_\alpha$$

with

$$\tau^\eta_\alpha|_{GR} = \frac{1}{2\chi} \left( \frac{\partial \bar{R}}{\partial g_{\mu\nu, \eta}} g_{\mu\nu, \alpha} - \delta^\eta_\alpha \bar{R} \right)$$

which, in the weak field limit and harmonic gauge, becomes, up to the order $h^2$,

$$\tau^\eta_\alpha|_{GR} \approx \frac{h^2}{2\chi} \left[ \frac{1}{2} h^\mu\nu, \eta h_{\mu\nu, \alpha} - h^{\eta\mu, \nu} h_{\mu\nu, \alpha} - \frac{1}{4} \delta^\eta_\alpha \left( h^{\sigma\lambda, \rho} h_{\lambda\sigma, \rho} - 2 h^{\sigma\lambda, \rho} h^{\rho} h_{\lambda, \sigma} \right) \right]$$

depending quadratically on first derivatives of metric perturbations $h_{\mu\nu}$. If a source is far, it is $h_{\mu\nu} \sim 1/r$, $h_{\mu\nu, \alpha} \sim 1/r^2$, and $\tau^\eta_\alpha \sim 1/r^4$. 

S. Capozziello, M. Capriolo
The Energy-Momentum Pseudo-Tensor for $L^k R$ Lagrangians

\[
\tau_{\eta}^{\alpha} = \tau_{\alpha|GR} + \frac{1}{2 \chi \sqrt{-g}} \left\{ \sqrt{-g} \left( 2a_0 R + \sum_{k=1}^{p} a_k R^k \right) \left[ \frac{\partial R}{\partial g_{\mu\nu, \eta}} g_{\mu\nu, \alpha} \right. \right.
\]
\[
+ \left. \frac{\partial R}{\partial g_{\mu\nu, \eta \lambda}} g_{\mu\nu, \lambda \alpha} \right] - \partial_\lambda \left[ \sqrt{-g} \left( 2a_0 R + \sum_{k=1}^{p} a_k R^k \right) \frac{\partial R}{\partial g_{\mu\nu, \eta \lambda}} \right] g_{\mu\nu, \alpha}
\]
\[
+ \Theta_{[1, +\infty]} \left( p \right) \sum_{h=1}^{p} \left\{ \sum_{q=0}^{2h+1} \left( -1 \right)^q \partial_{i_0 \cdots i_q} \left[ \sqrt{-g} a_h R \frac{\partial R^h}{\partial g_{\mu\nu, \eta i_0 \cdots i_q}} \right] g_{\mu\nu, \alpha} \right. \\
\]
\[
\left. + \sum_{j=0}^{2h} \sum_{m=j+1}^{2h+1} \left( -1 \right)^j \partial_{i_0 \cdots i_j} \left[ \sqrt{-g} a_h R \frac{\partial R^h}{\partial g_{\mu\nu, \eta i_0 \cdots i_m}} \right] g_{\mu\nu, i_{j+1} \cdots i_m \alpha} \right\}
\]
\[
- \delta_{\alpha}^{\eta} \left( a_0 R^2 + \sum_{k=1}^{p} a_k R^k \right) \sqrt{-g} \right\}
\]
Continuity Equation for $\Box^k R$ gravity

\begin{align*}
G^{\eta\alpha}_{\ ;\eta} = 0 & \iff P^{\eta\alpha}_{\ ;\eta} = 0 & \iff T^{\eta\alpha}_{\ ;\eta} = 0 & \iff \partial_\eta \left[ \sqrt{-g} \left( \tau^{\eta}_{\alpha} + T^{\eta}_{\alpha} \right) \right] = 0
\end{align*}

In other words, the Bianchi identities imply the conservation of gravitational fields + matter. For a spatial domain where fields and derivatives go to zero at boundaries and, in the asymptotic flatness hypothesis, we have

**Energy and momentum for $\Box^k R$ gravity contained in the volume $\Sigma$**

\[ P^\mu = \int_\Sigma d^3 x \sqrt{-g} \left( T^{\mu 0} + \tau^{\mu 0} \right) \]

where $P^\mu$ is a 4-vector independent of the chosen coordinates.
Weak field limit of Energy-Momentum Pseudo-Tensor for $L^{\Box k} R$ Lagrangian at the order $h^2$ in harmonic gauge

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1 \quad \text{in harmonic gauge} \]

\[ \frac{h^2}{2\chi} \left\{ \frac{1}{4} \left( \sum_{k=0}^{p} a_k \Box^{k+1} h \right) h^\eta \alpha + \frac{1}{2} \sum_{t=0}^{p} a_t \Box^{t+1} h,\lambda \left( h^{\eta \lambda} - \eta^{\eta \lambda} h \right),\alpha \right. \]

\[ + \left. \frac{1}{2} \sum_{h=0}^{1} \sum_{j=h}^{p} \sum_{m=j}^{p} (-1)^h a_m \Box^{m-j} \left( h^{\eta \lambda} - \eta^{\eta \lambda} h \right),\alpha_{i_h} \Box^{j+1-h} h,\lambda_{i_h} \right. \]

\[ + \frac{1}{4} \sum_{l=0}^{p} a_l \Box^l \left( h^\eta \alpha - \Box h^\delta \alpha \right) \Box h + \Theta[1,+\infty] (p) \left[ (D_p)^\eta _\alpha + (F_p)^\eta _\alpha \right] \}

where we used the conventions:

\[ (),_{\alpha i_0} = () ,_{\alpha} \quad h,\lambda_{i_0} = h,\lambda \]

depending up to the $2p + 3$ derivatives of metric perturbations $h_{\mu\nu}$ in the hypothesis $h_{\mu\nu} \sim 1/r$. We have $\Box^{p+1} h h^\eta \alpha \sim 1/r^{2p+6}$.
Here $\Theta$ is the Haeviside function and $(D_\rho)^{\eta}_\alpha$ and $(F_\rho)^{\eta}_\alpha$ are two terms containing the partial derivatives of $\Box^h R$ with respect to $g_{\mu\nu}$ derivatives, when the permutations of the $(\mu\nu)$ and $(\eta i_1 \ldots i_{2h+1})$ indices are considered, namely

$$
\frac{\partial \Box^h R}{\partial g_{\mu\nu, \eta i_1 \ldots i_{2h+1}}} = g^{j_2 j_3} \ldots g^{j_{2h} j_{2h+1}} g^{ab} g^{cd} \left\{ \delta_a^\mu \delta_d^\nu \delta_c^\eta \delta_b^{i_1} \delta_j^{i_2} \ldots \delta_j^{i_{2h}} \delta_j^{i_{2h+1}} - \delta_a^\mu \delta_b^\nu \delta_c^\eta \delta_d^{i_1} \delta_j^{i_2} \ldots \delta_j^{i_{2h}} \delta_j^{i_{2h+1}} \right\}
$$
Particular cases $p = 0$ and $p = 1$

Let us consider the corrections to $\tilde{\tau}_\alpha^n$ where $p$ is 0 and 1.

For $p = 0$ and $L_g = (R + a_0 R^2) \sqrt{-g}$, we get fourth-order gravity where $\tilde{\tau}_\alpha^n$ depends up to the third derivatives of $h_{\mu\nu}$ and $\tilde{\tau}_\alpha^n = \mathcal{O} \left( 1/r^6 \right)$

$$
\tilde{\tau}_\alpha^n \overset{h.g.}{=} \frac{a_0}{2\chi} \left( \frac{1}{2} h^{\eta}_{\alpha} \Box h + h^{\eta}_{\lambda,\alpha} \Box h^{\lambda} - h_{\alpha} \Box h^{\eta} - \frac{1}{4} (\Box h)^2 \delta^{\eta}_{\alpha} \right)
$$

For $p = 1$, that is $L_g = (R + a_0 R^2 + a_1 R \Box R) \sqrt{-g}$, sixth-order gravity where $\tilde{\tau}_\alpha^n$ depends up to the fifth derivatives of $h_{\mu\nu}$ and $\tilde{\tau}_\alpha^n = \mathcal{O} \left( 1/r^6 \right) + \mathcal{O} \left( 1/r^8 \right)$

$$
\tilde{\tau}_\alpha^n \overset{h.g.}{=} \frac{1}{2\chi} \left\{ \frac{1}{4} (2a_0 \Box h + a_1 \Box^2 h) h^{\eta}_{\alpha} + \frac{1}{2} (2a_0 \Box h,_{\lambda} + a_1 \Box^2 h,_{\lambda}) (h^{\eta\lambda} - \eta^{\eta\lambda} h),_{\alpha} 
+ \frac{1}{2} a_1 \Box (h^{\eta\lambda} - \eta^{\eta\lambda} h),_{\alpha} \Box h,_{\lambda} + \frac{1}{2} a_1 (h^{\eta\lambda} - \eta^{\eta\lambda} h),_{\alpha} \Box^2 h,_{\lambda} 
- \frac{1}{2} a_1 (h^{\eta\lambda} - \eta^{\eta\lambda} h),_{\sigma\alpha} \Box h,_{\lambda}^{\sigma} + \frac{1}{4} a_1 \Box h^{\eta}_{\alpha} \Box h 
- \frac{1}{4} \delta^{\eta}_{\alpha} \left[ a_0 (\Box h) + a_1 (\Box^2 h) \right] \Box h + (D_1)^{\eta}_{\alpha} + (F_1)^{\eta}_{\alpha} \right\}
$$
In order to derive the emitted power from a radiating gravitational source, we have to average on a space-time region $\Omega$ so that $|\Omega| \gg \frac{1}{|k|}$, in short wavelength approximation, to remove integrals containing $e^{i(k_i - k_j \cdot x \cdot \alpha)}$, in the harmonic gauge $g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0$. We can use the modified gravitational waves derived in S. Capozziello, M. Capriolo and L. Caso, Int. J. Geom. Methods Mod. Phys. 16, 1950047 (2019), namely

$$h_{\mu\nu} (x) = \sum_{m=1}^{p+2} \frac{d^3k}{(2\pi)^3} \int_{\Omega} (B_m)_{\mu\nu} (k) e^{i(k_m \cdot x \cdot \alpha)} + c.c. \quad (1)$$

where

$$(B_m)_{\mu\nu} (k) = \begin{cases} C_{\mu\nu} (k) & \text{for } m = 1 \\ \frac{1}{3} \left[ \frac{\eta_{\mu\nu}}{2} + \frac{(k_m \cdot k_m \cdot \nu)}{k_m^2} \right] A_m (k) & \text{for } m \geq 2 \end{cases} \quad (2)$$
Here "c.c." stands for the complex conjugate, $A_m(k)$ is the amplitude of $m-th$ modified gravitational waves and $C_{\mu\nu}(k)$ is the transverse polarization tensor of the massless gravitational waves predicted by Einstein.

The trace is

$$\langle B_m \rangle^\lambda_\lambda(k) = \begin{cases} C^\lambda_\lambda(k) & \text{for } m = 1 \\ A_m(k) & \text{for } m \geq 2 \end{cases}$$  \hspace{1cm} (3)$$

and $k^\mu_m = (\omega_m, k)$ with $k^2_m = \omega^2_m - |k|^2 = M^2$ where $k^2_1 = 0$ and $k^2_m \neq 0$ for $m \geq 2$. If we average on $\Omega$ spacetime region, the following terms vanish

$$\langle (D_p)^{\eta}_{\alpha} \rangle = \langle (F_p)^{\eta}_{\alpha} \rangle = 0$$
\[
\langle \tau^\eta_{\alpha} \rangle = \frac{1}{2\chi} \left[ (k_1)^\eta (k_1)_\alpha \left( C^{\mu\nu} C^{*\mu\nu} - \frac{1}{2} |C^\chi_\lambda|^2 \right) \right] \\
+ \frac{1}{2\chi} \left[ \left( -\frac{1}{6} \right) \sum_{j=2}^{p+2} \left( (k_j)^\eta (k_j)_\alpha - \frac{1}{2} k_j^2 \delta^\eta_\alpha \right) |A_j|^2 \right] \\
+ \frac{1}{2\chi} \left\{ \sum_{l=0}^{p} (l + 2) (-1)^l a_l \sum_{j=2}^{p+2} (k_j)^\eta (k_j)_\alpha (k_j^2)^{l+1} |A_j|^2 \right\} \\
- \frac{1}{2} \sum_{l=0}^{p} (-1)^l a_l \sum_{j=2}^{p+2} (k_j^2)^{l+2} |A_j|^2 \delta^\eta_\alpha \right\}
\]
Power emitted by a gravitational radiating source

Let us calculate the emitted power per solid angle $\Omega$ radiated in the direction $\hat{x}^i$ at fixed $k$. Under a suitable gauge, we have:

$$\frac{dP}{d\Omega} = r^2 \hat{x}^i \langle \tau_0^i \rangle$$

Assuming the TT gauge for the first oscillation mode $k_1$ and the harmonic gauge for the other modes $k_m$

$$\begin{cases}
(k_1)_\mu C^\mu\nu = 0 \wedge C^\lambda_\lambda = 0 & \text{if } m = 1 \\
(k_m)_\mu (B_m)^\mu\nu = \frac{1}{2} (B_m)^\lambda_\lambda k^\nu & \text{if } m \geq 2
\end{cases}$$

Considering gravitational waves propagating along the $+z$ direction with fixed $k$, with the 4D-wave vector given by $k^\mu = (\omega, 0, 0, k_z)$ where $\omega_1^2 = k_z^2$ if $k_1^2 = 0$ and $k_m^2 = m^2 = \omega_m^2 - k_z^2$ otherwise with $k_z > 0$, the averaged tensor components are:

$$\langle \tau_0^3 \rangle = \frac{c^4}{8\pi G} \omega_1^2 (C_{11}^2 + C_{12}^2) + \frac{c^4}{16\pi G} \left[ \left( -\frac{1}{6} \right) \sum_{j=2}^{p+2} \omega_j k_z |A_j|^2 
+ \sum_{l=0}^{p} (l+2) (-1)^l a_l \sum_{j=2}^{p+2} \omega_j k_z m_{j}^{2(l+1)} |A_j|^2 \right]$$
Power emitted by a gravitational radiating source

Let us choose:

- $p = 0$, $L_g = (\overline{R} + a_0 R^2) \sqrt{-g}$, fourth-order gravity, with the two modes $\omega_1, \omega_2$, it is:

$$\langle \tau_0^3 \rangle = \frac{c^4 \omega_1^2}{8\pi G} [C_{11}^2 + C_{12}^2] + \frac{c^4}{16\pi G} \left\{ \left( -\frac{1}{6} \right) \omega_2 |A_2|^2 k_z + 2a_0 \omega_2 m_2^2 |A_2|^2 k_z \right\}$$

- $p = 1$, $L_g = (\overline{R} + a_0 R^2 + a_1 R\Box R) \sqrt{-g}$, sixth order gravity, with the three modes $\omega_1, \omega_2, \omega_3$, it is:

$$\langle \tau_0^3 \rangle = \frac{c^4 \omega_1^2}{8\pi G} [C_{11}^2 + C_{12}^2] + \frac{c^4}{16\pi G} \left\{ \left( -\frac{1}{6} \right) (\omega_2 |A_2|^2 + \omega_3 |A_3|^3) k_z \right. + 2a_0 \left[ (\omega_2 m_2^2 |A_2|^2 + \omega_3 m_3^3 |A_3|^2) k_z \right] - 3a_1 \left[ (\omega_2 m_2^4 |A_2|^2 + \omega_3 m_3^4 |A_3|^2) k_z \right] \right\}$$
\[ p = 2, \quad L_g = \left( \bar{R} + a_0 R^2 + a_1 R \Box R + a_2 R \Box^2 R \right) \sqrt{-g}, \] eighth-order gravity, with the four modes \( \omega_1, \omega_2, \omega_3, \omega_4 \), it is:

\[
\langle \tau_0^3 \rangle = \frac{c^4 \omega_1^2}{8\pi G} \left[ C_{11}^2 + C_{12}^2 \right] + \frac{c^4}{16\pi G} \left\{ \left( -\frac{1}{6} \right) \left( \omega_2 |A_2|^2 + \omega_3 |A_3|^3 + \omega_4 |A_4|^2 \right) k_z \right.
\]
\[ + 2a_0 \left[ \left( \omega_2 m_2^2 |A_2|^2 + \omega_3 m_3^2 |A_3|^2 + \omega_4 m_4^2 |A_4|^2 \right) k_z \right] \]
\[ - 3a_1 \left[ \left( \omega_2 m_2^4 |A_2|^2 + \omega_3 m_3^4 |A_3|^2 + \omega_4 m_4^4 |A_4|^2 \right) k_z \right] \]
\[ + 4a_2 \left[ \left( \omega_2 m_2^6 |A_2|^2 + \omega_3 m_3^6 |A_3|^2 + \omega_4 m_4^6 |A_4|^2 \right) \right] \}
\]

As we go up by two with the order of gravity, through the d’Alembert operator \( \Box \), we increase by an oscillation mode \( \omega \) which corresponds to the conformal equivalence of the theories \( \Box^k R \) to General Relativity with \( k + 1 \) scalar fields. See S. Gottlober, H. J. Schmidt and A. A. Starobinsky, Class. Quant. Grav. 7 (1990) 893.
Using the Noether theorem for rigid translations, it is possible to derive the gravitational energy-momentum pseudo-tensor in curvature based gravity theories of any order.

In the same way, it is possible to obtain the gravitational energy-momentum pseudo-tensor in non-local gravity including $\Box^{-1} R$ terms (see S. Capozziello, M. Capriolo and S. Nojiri, Phys. Lett. B 810, 135821 (2020)).

It is also possible to derive the gravitational energy-momentum pseudo-tensor in Metric-Affine Gravity, as in Palatini Formalism.

In general, the method can be used to obtain the gravitational energy-momentum pseudo-tensor in non-metric and teleparallel theories of gravity. See S. Capozziello, M. Capriolo and M. Transirico, Int. J. Geom. Methods Mod. Phys. 15 1850164 (2018).

According to this approach, it is possible to calculate the power emitted by any gravitational radiating source.

New gravitational modes can be derived with respect to GR.