The Gravitational Energy-Momentum Pseudo-Tensor in Higher Order Theories of Gravity

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Summary

We discuss the generalization of gravitational energy-momentum pseudo-tensor to Extended Theories of Gravity, in particular to higher-order theories in curvature invariants. This result is achieved by imposing that the local variation of gravitational action of any order n vanishes under rigid translations. We also prove that this tensor, in general, is not covariant but only affine, that is, it is a **pseudo-tensor**. The pseudo-tensor τ^{μ}_{α} is calculated in the weak-field limit up to a first non-vanishing term of order h^2, where h is the metric perturbation. The average value of the pseudo-tensor, over a suitable spacetime domain, is obtained. Finally, we calculate the emitted power, per unit solid angle Ω, carried by a gravitational wave in the direction $\hat{x}$ for a fixed wave number $\mathbf{k}$ under a suitable gauge.

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References

1 S. Capozziello, M. Capriolo and M. Transirico, *The gravitational energy-momentum pseudo-tensor of higher-order theories of gravity* , Ann. der Phys. **525**, 1600376 (2017)

2 S. Capozziello, M. Capriolo and L. Caso, *Weak field limit and gravitational waves in higher order gravity*, Int. J. Geom. Methods Mod. Phys. **16** No.03, 1950047 (2019)

The Energy-Momentum Pseudo-Tensor in GR

- In GR, there is no unanimously accepted definition of energy-momentum of the gravitational field. Some prescriptions have been given by Einstein, Landau-Lifshitz, Papapetrou, Weinberg, and Möller.
- The "non-tensoriality" and the "affine" character of the gravitational energy-momentum "tensor" make the energy and momentum of the gravitational field non-localizable.
- However, it is possible to define the energy-momentum of total gravitational field in an asymptotically flat spacetime almost independently of the coordinates.
- The continuity equation of Special Relativity $\partial_\mu T^{\mu\nu} = 0$, in GR, becomes

$$\nabla_\mu T^{\mu\nu} = 0 \quad \rightarrow \quad \frac{1}{\sqrt{-g}}\partial_\nu \left(\sqrt{-g}\, T^{\mu\nu}\right) + \Gamma^{\mu}{}_{\nu\lambda} T^{\lambda\nu} = 0 \,.$$

It is not, in principle, a conservation law.

The Energy-Momentum Pseudo-Tensor in GR

Einstein postulated a local conservation law by introducing a pseudo-tensor $\tau^{\mu\nu}$ related to the energy-momentum of the gravitational field

$$\partial_\mu \left(\sqrt{-g}\,(T^{\mu\nu} + \tau^{\mu\nu})\right) = 0$$

$$\sqrt{-g}\tau_\mu{}^\nu = \frac{1}{16\pi}\left(\delta^\nu_\mu L - \frac{\partial L}{\partial g^{\rho\sigma}{}_{,\nu}} g^{\rho\sigma}{}_{,\mu}\right)$$

depending on the metric $g_{\mu\nu}$ and its derivatives, being

$$L = \sqrt{-g}g^{\mu\nu}\left(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\nu\sigma}\right).$$

The pseudo-tensor $\tau^{\mu\nu}$ does not transform as a tensor under generic coordinate transformations but under affine transformations.

The Energy-Momentum Pseudo-Tensor in GR

From the pseudo-tensor character of $\tau^{\mu\nu}$, we have:

- Energy-momentum of the gravitational field, in a given region of the universe, depends on the coordinate system, i.e. it is not localizable.

If we choose a space-time domain Ω, verifying the

- spatial asymptotic flatness condition where the metric asymptotically joins with continuity with the Minkowski one and fields and derivatives rapidly go to zero, by the Gauss theorem, we can
- define the energy-momentum of the gravitational field plus that of non-gravitational fields, contained in V independently of the coordinate choice as

$$P^{\nu} = \int_V \sqrt{-g}\left(T^{0\nu} + \tau^{0\nu}\right) d^3x$$

here V is an infinite spatial hypersurface defined at t constant.

The Energy-Momentum Pseudo-Tensor for Lagrangians of order n

S. Capozziello, M. Capriolo and M. Transirico, *The gravitational energy-momentum pseudo-tensor for higher-order theories of gravity* , Ann. Phys. **525**, 1600376 (2017)

In order to calculate the gravitational pseudo-tensor for fourth-order Lagrangians, let us consider the Noether Theorem for rigid translations. Let us define

$$L = L\left(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\lambda}, g_{\mu\nu,\rho\lambda\xi}, g_{\mu\nu,\rho\lambda\xi\sigma}\right)$$

The variation $\tilde{\delta}$ with respect to the metric $g_{\mu\nu}$ and coordinates x^μ is

$$\mathcal{S} = \int_\Omega d^4xL \to \tilde{\delta}\mathcal{S} = \int_{\Omega'} d^4x'L' - \int_\Omega d^4xL = \int_\Omega d^4x\left[\delta L + \partial_\mu\left(L\delta x^\mu\right)\right] .$$

Here δ represents the variation with fixed coordinates x^μ. An infinitesimal translation is:

$$x'^\mu = x^\mu + \epsilon^\mu\left(x\right)$$

and the variation of metric tensor is

$$\delta g_{\mu\nu} = g'_{\mu\nu}\left(x\right) - g_{\mu\nu}\left(x\right) = -\epsilon^\alpha\partial_\alpha g_{\mu\nu} - g_{\mu\alpha}\partial_\nu\epsilon^\alpha - g_{\nu\alpha}\partial_\mu\epsilon^\alpha$$

The Energy-Momentum Pseudo-Tensor for Lagrangians of order n

The metric variation, under global transformations $\partial_\lambda \epsilon^\mu = 0$, is $\delta g_{\mu\nu} = -\epsilon^\alpha \partial_\alpha g_{\mu\nu}$ and, if we require the action invariant under these transformations, i.e. $\tilde{\delta}\mathcal{S} = 0$, for an arbitrary integration domain Ω, we get:

$$0 = \delta L + \partial_\mu \left(L \delta x^\mu \right) = \left(\frac{\partial L}{\partial g_{\mu\nu}} - \partial_\rho \frac{\partial L}{\partial g_{\mu\nu,\rho}} + \partial_\rho \partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}} - \partial_\rho \partial_\lambda \partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} \right.$$
$$\left. + \partial_\rho \partial_\lambda \partial_\xi \partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} \right) \delta g_{\mu\nu} - \partial_\eta \left(2\chi \sqrt{-g}\, \tau^\eta_\alpha \right) \epsilon^\alpha \,.$$

From the Euler-Lagrange equations, we have:

$$\frac{\partial L}{\partial g_{\mu\nu}} - \partial_\rho \frac{\partial L}{\partial g_{\mu\nu,\rho}} + \partial_\rho \partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda}} - \partial_\rho \partial_\lambda \partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi}} + \partial_\rho \partial_\lambda \partial_\xi \partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\xi\sigma}} = 0 \,.$$

We obtain the continuity equation:

$$\partial_\eta \left(\sqrt{-g}\, \tau^\eta_\alpha \right) = 0$$

for any ϵ^α where τ^η_α is the gravitational energy-momentum pseudo-tensor.

The case of Energy-Momentum Pseudo-Tensor for Lagrangians of order 4

The energy-momentum pseudo-tensor for Lagrangians depending on fourth-order derivatives in the metric $g_{\mu\nu}$ is

$$\tau^{\eta}_{\alpha} = \frac{1}{2\chi\sqrt{-g}}\Bigg[\Bigg(\frac{\partial L}{\partial g_{\mu\nu,\eta}} - \partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\eta\lambda}} + \partial_\lambda\partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\eta\lambda\xi}}$$

$$-\partial_\lambda\partial_\xi\partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\eta\lambda\xi\sigma}}\Bigg) g_{\mu\nu,\alpha} + \Bigg(\frac{\partial L}{\partial g_{\mu\nu,\rho\eta}} - \partial_\xi \frac{\partial L}{\partial g_{\mu\nu,\rho\eta\xi}} + \partial_\xi\partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\eta\xi\sigma}}\Bigg) g_{\mu\nu,\alpha\rho}$$

$$+ \Bigg(\frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\eta}} - \partial_\sigma \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\eta\sigma}}\Bigg) g_{\mu\nu,\rho\lambda\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\rho\lambda\eta\sigma}} g_{\mu\nu,\rho\lambda\xi\alpha} - \delta^{\eta}_{\alpha} L\Bigg]$$

where $\chi = \frac{8\pi G}{c^4}$ is the gravitational coupling and the metric derivatives are up to 7-th order.

The Energy-Momentum Pseudo-Tensor for Lagrangians of order n

Let us consider now a general Lagrangian density depending on the n-th derivatives of $g_{\mu\nu}$

$$L = L\left(g_{\mu\nu}, g_{\mu\nu,i_1}, g_{\mu\nu,i_1 i_2}, g_{\mu\nu,i_1 i_2 i_3}, \cdots, g_{\mu\nu,i_1 i_2 i_3 \cdots i_n}\right)$$

The gravitational pseudo-tensor for Lagrangians of order n is

$$\boxed{\begin{aligned}\tau^{\eta}_{\alpha} =& \frac{1}{2\chi\sqrt{-g}}\left[\sum_{m=0}^{n-1}(-1)^m\left(\frac{\partial L}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)_{,i_0\cdots i_m} g_{\mu\nu,\alpha}\right.\\ &\left.+\Theta_{[2,+\infty[}(n)\sum_{j=0}^{n-2}\sum_{m=j+1}^{n-1}(-1)^j\left(\frac{\partial L}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right)_{,i_0\cdots i_j} g_{\mu\nu,i_{j+1}\cdots i_m\alpha} - \delta^{\eta}_{\alpha}L\right]\end{aligned}}$$

depending up to $2n-1$ derivatives in the metric $g_{\mu\nu}$.

$$()_{,i_0} = 1 \qquad ()_{,i_0\cdots i_m} = \begin{cases} ()_{,i_1} & \text{if} \quad m=1 \\ ()_{,i_1 i_2} & \text{if} \quad m=2 \\ ()_{,i_1 i_2 i_3} & \text{if} \quad m=3 \\ \text{and so on} & \end{cases} \qquad ()_{,i_k\, i_k} = ()_{,i_k}$$

Continuity Equation

The field equations associated to a generic Lagrangian, in presence of matter, are now $P^{\eta\alpha} = \chi T^{\eta\alpha}$ where

$$P^{\eta\alpha} = \frac{1}{\sqrt{-g}} \frac{\delta L_g}{\delta g_{\eta\alpha}}, \qquad \text{with the coupling} \quad \chi = \frac{8\pi G}{c^4}$$

From the Lagrangian invariance for rigid translations and from the symmetry of $T^{\eta\alpha}$, we have

$$\partial_\eta \left[\sqrt{-g}\left(\tau^\eta{}_\alpha + T^\eta{}_\alpha\right)\right] = \sqrt{-g}\, T^\eta{}_{\alpha;\eta}$$

Continuity Equation

$$P^{\eta\alpha}{}_{;\eta} = 0 \leftrightarrow T^{\eta\alpha}{}_{;\eta} = 0 \leftrightarrow \partial_\eta \left[\sqrt{-g}\left(\tau^\eta{}_\alpha + T^\eta{}_\alpha\right)\right] = 0$$

The conserved quantities are not only the energy and momentum associated to the matter and non-gravitational fields but the overall contribution of these fields plus the energy-momentum of the gravitational field.

Let us now integrate the continuity equation on a spatial domain Σ, which is the foliation of the $4D$ space-time at a fixed t, where fields and their derivatives go to zero in a sufficiently rapid way on the boundary $\partial\Sigma$. Using the Gauss theorem, the surface integrals go to zero

$$\partial_0 \int_\Sigma d^3x\sqrt{-g}\left(T^{\mu 0} + \tau^{\mu 0}\right) = -\int_{\partial\Sigma} d\sigma_i\sqrt{-g}\left(T^{\mu i} + \tau^{\mu i}\right) = 0$$

The overall contribution of energy-momentum in the volume Σ is defined as

$$P^\mu = \int_\Sigma d^3x\sqrt{-g}\left(T^{\mu 0} + \tau^{\mu 0}\right)$$

depending on the coordinate choice. These conditions are often realized for isolated systems where it is possible to derive the spatial asymptotic flatness so that P^μ is independent of the coordinates and transforms as a 4-vector.

Non-covariance of gravitational energy-momentum tensor

It is possibile to demonstrate that τ^η_α is not, in general, a covariant tensor but it behaves as a tensor only under affine transformations. This means it is a pseudo-tensor. Let us consider first the particular case with a Lagrangian density of order 2

$$\tau^\eta_\alpha = \frac{1}{2\chi\sqrt{-g}}\left[\left(\frac{\partial L}{\partial g_{\mu\nu,\eta}} - \partial_\lambda \frac{\partial L}{\partial g_{\mu\nu,\eta\lambda}}\right) g_{\mu\nu,\alpha} + \frac{\partial L}{\partial g_{\mu\nu,\eta\xi}} g_{\mu\nu,\xi\alpha} - \delta^\eta_\alpha L\right]$$

In general, under diffeomorphisms $x' = x'(x)$, it is

$$\tau'^\eta_{\ \alpha}(x') \neq \mathrm{J}^\eta_\sigma \mathrm{J}^{-1\tau}_{\ \ \ \alpha} \tau^\sigma_\tau(x)$$

where the Jacobian matrix and the determinant are defined as

$$\mathrm{J}^\eta_\sigma = \frac{\partial x'^\eta}{\partial x^\sigma} \qquad \mathrm{J}^{-1\tau}_{\ \ \ \alpha} = \frac{\partial x^\tau}{\partial x'^\alpha} \qquad \det\left(\mathrm{J}^\alpha_\beta\right) = J$$

On the other hand, under linear affine transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \qquad \mathrm{J}^\mu_\nu = \Lambda^\mu_\nu \qquad |\Lambda| \neq 0$$

the tensor transforms as

$$\tau'^\eta_{\ \alpha}(x') = \Lambda^\eta_\sigma \Lambda^{-1\tau}_{\ \ \ \alpha} \tau^\sigma_\tau(x)$$

Non-covariance of gravitational energy-momentum tensor

$$g'_{\mu\nu,\alpha}(x') = \mathrm{J}^{-1}{}^{a}_{\mu}\mathrm{J}^{-1}{}^{b}_{\nu}\mathrm{J}^{-1}{}^{c}_{\alpha} g_{ab,c}(x) + \partial'_{\alpha}\left[\mathrm{J}^{-1}{}^{a}_{\mu}\mathrm{J}^{-1}{}^{b}_{\nu}\right] g_{ab}(x)$$

$$\tau'^{\eta}_{\alpha}(x') = \mathrm{J}^{\eta}_{\sigma}\mathrm{J}^{-1}{}^{\tau}_{\alpha}\tau^{\sigma}_{\tau}(x) + \left\{\text{containing terms} \frac{\partial^2 x}{\partial x'^2}, \frac{\partial^3 x}{\partial x'^3}\right\}$$

This results derives from the non-covariance of metric tensor $g_{\mu\nu}$ derivatives. These derivatives give rise to the affine tensor. In general,

$$g'_{\mu\nu,i_1\cdots i_m\alpha}(x') = \mathrm{J}^{-1}{}^{\alpha}_{\mu}\mathrm{J}^{-1}{}^{\beta}_{\nu}\mathrm{J}^{-1}{}^{j_1}_{i_1}\cdots\mathrm{J}^{-1}{}^{j_m}_{i_m}\mathrm{J}^{-1}{}^{\tau}_{\alpha} g_{\alpha\beta,j_1\cdots j_m\tau}(x) + \left\{\text{containing terms } \frac{\partial^2 x}{\partial x'^2}, \cdots, \frac{\partial^{m+2} x}{\partial x'^{m+2}}\right\}$$

and

$$\frac{\partial L'}{\partial g'_{\mu\nu,\eta i_0\cdots i_m}} = \mathrm{J}^{-1}\mathrm{J}^{\mu}_{\gamma}\mathrm{J}^{\nu}_{\rho}\mathrm{J}^{\eta}_{\tau}\mathrm{J}^{i_1}_{j_1}\cdots\mathrm{J}^{i_m}_{j_m}\frac{\partial L}{\partial g_{\gamma\rho,\tau j_1\cdots j_m}}$$

tensor densities (m+3,0)

of weight $w = -1$

from which the non-covariance of tensor τ^{η}_{α}.

The Energy-Momentum Pseudo-Tensor of $L_{\Box^k R}$ Lagrangians

Let us calculate now the energy-momentum pseudo-tensor τ_α^η for the gravitational Lagrangian

$$L_{\Box^k R} = (\overline{R} + a_0 R^2 + \sum_{k=1}^{p} a_k R \Box^k R)\sqrt{-g}$$

where $\overline{R}$ is the part of curvature Ricci scalar R depending only on $g_{\mu\nu}$ and its first derivatives. If the variation of action $\mathcal{S}_{\Box^k R}$ is zero for rigid translations $\tilde{\delta}_{g,x}\mathcal{S}_{\Box^k R} = 0$ with $g_{\mu\nu}$ satisfying the Euler-Lagrange equations, we have

$$\tau_\alpha^\eta = \tau_{\alpha|GR}^\eta + \tilde{\tau}_\alpha^\eta \quad \text{with} \quad \tau_{\alpha|GR}^\eta = \frac{1}{2\chi}\left(\frac{\partial \overline{R}}{\partial g_{\mu\nu,\eta}} g_{\mu\nu,\alpha} - \delta_\alpha^\eta \overline{R}\right)$$

which, in the weak field limit and harmonic gauge, becomes, up to the order h^2,

$$\tau_{\alpha|GR}^\eta \stackrel{h^2}{=} \frac{1}{2\chi}\left[\frac{1}{2} h^{\mu\nu,\eta} h_{\mu\nu,\alpha} - h^{\eta\mu,\nu} h_{\mu\nu,\alpha} - \frac{1}{4}\delta_\alpha^\eta \left(h^{\sigma\lambda}{}_{,\rho} h_{\lambda\sigma}{}^{,\rho} - 2h^{\sigma\lambda}{}_{,\rho} h^\rho{}_{\lambda,\sigma}\right)\right]$$

depending quadratically on first derivatives of metric perturbations $h_{\mu\nu}$. If a source is far, it is $h_{\mu\nu} \sim 1/r$, $h_{\mu\nu,\alpha} \sim 1/r^2$, and $\tau^\eta{}_\alpha \sim 1/r^4$.

The Energy-Momentum Pseudo-Tensor for $L_{\Box^k R}$ Lagrangians

$$
\begin{aligned}
\tau^{\eta}_{\alpha} = \tau^{\eta}_{\alpha|GR} + \frac{1}{2\chi\sqrt{-g}}\Bigg\{ & \sqrt{-g}\left(2a_0 R + \sum_{k=1}^{p} a_k \Box^k R\right)\left[\frac{\partial R}{\partial g_{\mu\nu,\eta}} g_{\mu\nu,\alpha}\right. \\
& \left. + \frac{\partial R}{\partial g_{\mu\nu,\eta\lambda}} g_{\mu\nu,\lambda\alpha}\right] - \partial_\lambda\left[\sqrt{-g}\left(2a_0 R + \sum_{k=1}^{p} a_k \Box^k R\right)\frac{\partial R}{\partial g_{\mu\nu,\eta\lambda}}\right] g_{\mu\nu,\alpha} \\
& + \Theta_{[1,+\infty[}(p)\sum_{h=1}^{p}\Bigg\{\sum_{q=0}^{2h+1}(-1)^q\,\partial_{i_0\cdots i_q}\left[\sqrt{-g}\,a_h R\frac{\partial \Box^h R}{\partial g_{\mu\nu,\eta i_0\cdots i_q}}\right] g_{\mu\nu,\alpha} \\
& + \sum_{j=0}^{2h}\sum_{m=j+1}^{2h+1}(-1)^j\,\partial_{i_0\cdots i_j}\left[\sqrt{-g}\,a_h R\frac{\partial \Box^h R}{\partial g_{\mu\nu,\eta i_0\cdots i_m}}\right] g_{\mu\nu,i_{j+1}\cdots i_m\alpha}\Bigg\} \\
& - \delta^{\eta}_{\alpha}\left(a_0 R^2 + \sum_{k=1}^{p} a_k R\Box^k R\right)\sqrt{-g}\Bigg\}
\end{aligned}
$$

Continuity Equation for $\Box^k R$ gravity

$$G^{\eta\alpha}{}_{;\eta} = 0 \leftrightarrow P^{\eta\alpha}{}_{;\eta} = 0 \leftrightarrow T^{\eta\alpha}{}_{;\eta} = 0 \leftrightarrow \partial_\eta \left[\sqrt{-g}\left(\tau^\eta{}_\alpha + T^\eta{}_\alpha\right)\right] = 0$$

In other words, the Bianchi identities imply the conservation of gravitational fields + matter. For a spatial domain where fields and derivatives go to zero at boundaries and, in the asymptotic flatness hypothesis, we have

Energy and momentum for $\Box^k R$ gravity contained in the volume Σ

$$P^\mu = \int_\Sigma d^3x\sqrt{-g}\left(T^{\mu 0} + \tau^{\mu 0}\right)$$

where P^μ is a 4-vector independent of the chosen coordinates.

Weak field limit of Energy-Momentum Pseudo-Tensor for $L_{\Box^k R}$ Lagrangian at the order h^2 in harmonic gauge

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1 \quad \text{in harmonic gauge} \quad \tau^\eta_\alpha \overset{\substack{h^2 \\ \text{h.g.}}}{=} \tau^\eta_{\alpha|GR} + \tilde{\tau}^\eta_\alpha$$

$$\boxed{\begin{aligned}\tilde{\tau}^\eta_\alpha \overset{\substack{h^2 \\ \text{h.g.}}}{=} \frac{1}{2\chi}\Bigg\{ & \frac{1}{4}\left(\sum_{k=0}^{p} a_k \Box^{k+1} h\right) h^{,\eta}{}_{\alpha} + \frac{1}{2}\sum_{t=0}^{p} a_t \Box^{t+1} h_{,\lambda}\left(h^{\eta\lambda} - \eta^{\eta\lambda} h\right)_{,\alpha} \\ & + \frac{1}{2}\sum_{h=0}^{1}\sum_{j=h}^{p}\sum_{m=j}^{p} (-1)^h a_m \Box^{m-j}\left(h^{\eta\lambda} - \eta^{\eta\lambda} h\right)_{,\alpha i_h} \Box^{j+1-h} h_{,\lambda}{}^{i_h} \\ & + \frac{1}{4}\sum_{l=0}^{p} a_l \Box^l \left(h^{,\eta}{}_{\alpha} - \Box h \delta^\eta_\alpha\right)\Box h + \Theta_{[1,+\infty[}(p)\left[(D_p)^\eta_\alpha + (F_p)^\eta_\alpha\right]\Bigg\}\end{aligned}}$$

where we used the conventions:

$$()_{,\alpha i_0} = ()_{,\alpha} \qquad h_{,\lambda}{}^{i_0} = h_{,\lambda}$$

depending up to the $2p+3$ derivatives of metric perturbations $h_{\mu\nu}$ in the hypothesis $h_{\mu\nu} \sim 1/r$. We have $\Box^{p+1} h h^{,\eta}{}_{\alpha} \sim 1/r^{2p+6}$.

Weak field limit of Energy-Momentum Pseudo-Tensor for $L_{\Box^k R}$ Lagrangian at the order h^2 in harmonic gauge

Here Θ is the Haeviside function and $(D_p)^\eta_\alpha$ and $(F_p)^\eta_\alpha$ are two terms containing the partial derivatives of $\Box^h R$ with respect to $g_{\mu\nu}$ derivatives, when the permutations of the $(\mu\nu)$ and $(\eta i_1 \dots i_{2h+1})$ indices are considered, namely

$$\frac{\partial \Box^h R}{\partial g_{\mu\nu,\eta i_1 \cdots i_{2h+1}}} = g^{j_2 j_3} \cdots g^{j_{2h} j_{2h+1}} g^{ab} g^{cd} \Big\{ \delta_a^{(\mu} \delta_d^{\nu)} \delta_c^{(\eta} \delta_b^{i_1} \delta_{j_2}^{i_2} \cdots \delta_{j_{2h}}^{i_{2h}} \delta_{j_{2h+1}}^{i_{2h+1})} - \delta_a^{(\mu} \delta_b^{\nu)} \delta_c^{(\eta} \delta_d^{i_1} \delta_{j_2}^{i_2} \cdots \delta_{j_{2h}}^{i_{2h}} \delta_{j_{2h+1}}^{i_{2h+1})} \Big\}$$

Let us consider the corrections to $\tilde{\tau}^{\eta}_{\alpha}$ where p is 0 and 1.
For $p = 0$ and $L_g = \left(\overline{R} + a_0 R^2\right)\sqrt{-g}$, we get fourth-order gravity where $\tilde{\tau}^{\eta}_{\alpha}$ depends up to the third derivatives of $h_{\mu\nu}$ and $\tilde{\tau}^{\eta}_{\alpha} = \mathcal{O}\left(1/r^6\right)$

$$\tilde{\tau}^{\eta}_{\alpha} \overset{h^2}{\underset{\text{h.g.}}{=}} \frac{a_0}{2\chi}\left(\frac{1}{2}h^{,\eta}{}_{\alpha}\Box h + h^{\eta}{}_{\lambda,\alpha}\Box h^{,\lambda} - h_{,\alpha}\Box h^{,\eta} - \frac{1}{4}\left(\Box h\right)^2\delta^{\eta}_{\alpha}\right)$$

For $p = 1$, that is $L_g = \left(\overline{R} + a_0 R^2 + a_1 R\Box R\right)\sqrt{-g}$, sixth-order gravity where $\tilde{\tau}^{\eta}_{\alpha}$ depends up to the fifth derivatives of $h_{\mu\nu}$ ed $\tilde{\tau}^{\eta}_{\alpha} = \mathcal{O}\left(1/r^6\right) + \mathcal{O}\left(1/r^8\right)$

$$\begin{aligned}\tilde{\tau}^{\eta}_{\alpha} \overset{h^2}{\underset{\text{h.g.}}{=}} \frac{1}{2\chi}\Bigg\{&\frac{1}{4}\left(2a_0\Box h + a_1\Box^2 h\right)h^{,\eta}{}_{\alpha} + \frac{1}{2}\left(2a_0\Box h_{,\lambda} + a_1\Box^2 h_{,\lambda}\right)\left(h^{\eta\lambda} - \eta^{\eta\lambda}h\right)_{,\alpha}\\
&+\frac{1}{2}a_1\Box\left(h^{\eta\lambda} - \eta^{\eta\lambda}h\right)_{,\alpha}\Box h_{,\lambda} + \frac{1}{2}a_1\left(h^{\eta\lambda} - \eta^{\eta\lambda}h\right)_{,\alpha}\Box^2 h_{,\lambda}\\
&-\frac{1}{2}a_1\left(h^{\eta\lambda} - \eta^{\eta\lambda}h\right)_{,\sigma\alpha}\Box h_{,\lambda}{}^{\sigma} + \frac{1}{4}a_1\Box h^{,\eta}{}_{\alpha}\Box h\\
&-\frac{1}{4}\delta^{\eta}_{\alpha}\left[a_0\left(\Box h\right) + a_1\left(\Box^2 h\right)\right]\Box h + \left(D_1\right)^{\eta}_{\alpha} + \left(F_1\right)^{\eta}_{\alpha}\Bigg\}\end{aligned}$$

Average of the energy-momentum pseudo-tensor

In order to derive the emitted power from a radiating gravitational source, we have to average on a space-time region Ω so that $|\Omega| \gg \frac{1}{|k|}$, in short wavelength approximation, to remove integrals containing $e^{i(k_i - k_j)_\alpha x^\alpha}$, in the harmonic gauge $g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu} = 0$. We can use the modified gravitational waves derived in *S. Capozziello, M. Capriolo and L. Caso, Int. J. Geom. Methods Mod. Phys.* **16**, *1950047 (2019)*, namely

$$h_{\mu\nu}(x) = \sum_{m=1}^{p+2} \int_\Omega \frac{d^3\mathbf{k}}{(2\pi)^3} (B_m)_{\mu\nu}(\mathbf{k})\, e^{i(k_m)_\alpha x^\alpha} + c.c. \tag{1}$$

where

$$(B_m)_{\mu\nu}(\mathbf{k}) = \begin{cases} C_{\mu\nu}(\mathbf{k}) & \text{for} \quad m = 1 \\ \frac{1}{3}\left[\frac{\eta_{\mu\nu}}{2} + \frac{(k_m)_\mu (k_m)_\nu}{k^2_{(m)}}\right] \mathrm{A}_m(\mathbf{k}) & \text{for} \quad m \geq 2 \end{cases} \tag{2}$$

Here "c.c." stands for the complex conjugate, $\mathrm{A}_m(\mathbf{k})$ is the amplitude of $m-th$ modified gravitational waves and $C_{\mu\nu}(\mathbf{k})$ is the transverse polarization tensor of the massless gravitational waves predicted by Einstein.
The trace is

$$(B_m)^\lambda_\lambda(\mathbf{k}) = \begin{cases} C^\lambda_\lambda(\mathbf{k}) & \text{for} \quad m = 1 \\ \mathrm{A}_m(\mathbf{k}) & \text{for} \quad m \geq 2 \end{cases} \tag{3}$$

and $k^\mu_m = (\omega_m, \mathbf{k})$ with $k^2_m = \omega^2_m - |\mathbf{k}|^2 = \mathrm{M}^2$ where $k^2_1 = 0$ and $k^2_m \neq 0$ for $m \geq 2$. If we average on Ω spacetime region, the following terms vanish

$$\langle (D_p)^\eta_\alpha \rangle = \langle (F_p)^\eta_\alpha \rangle = 0$$

Average of the energy-momentum pseudo-tensor

$$
\begin{aligned}
\langle \tau^{\eta}_{\alpha} \rangle = & \frac{1}{2\chi}\left[(k_1)^{\eta}(k_1)_{\alpha}\left(C^{\mu\nu}C^{*}_{\mu\nu} - \frac{1}{2}|C^{\lambda}_{\lambda}|^2 \right)\right] \\
& + \frac{1}{2\chi}\left[\left(-\frac{1}{6}\right)\sum_{j=2}^{p+2}\left((k_j)^{\eta}(k_j)_{\alpha} - \frac{1}{2}k_j^2\delta^{\eta}_{\alpha}\right)|A_j|^2 \right] \\
& + \frac{1}{2\chi}\left\{ \left[\sum_{l=0}^{p}(l+2)(-1)^{l}a_l \sum_{j=2}^{p+2}(k_j)^{\eta}(k_j)_{\alpha}\left(k_j^2\right)^{l+1}|A_j|^2 \right] \right. \\
& \left. - \frac{1}{2}\sum_{l=0}^{p}(-1)^{l}a_l\sum_{j=2}^{p+2}\left(k_j^2\right)^{l+2}|A_j|^2\delta^{\eta}_{\alpha} \right\}
\end{aligned}
$$

Power emitted by a gravitational radiating source

Let us calculate the emitted power per solid angle Ω radiated in the direction $\hat{x}^i$ at fixed $\mathbf{k}$. Under a suitable gauge, we have:

$$\frac{dP}{d\Omega} = r^2 \hat{x}^i \left\langle \tau_0^i \right\rangle$$

Assuming the TT gauge for the first oscillation mode k_1 and the harmonic gauge for the other modes k_m

$$\begin{cases} (k_1)_\mu C^{\mu\nu} = 0 \quad \wedge \quad C_\lambda^\lambda = 0 & \text{if} \quad m = 1 \\ (k_m)_\mu (B_m)^{\mu\nu} = \frac{1}{2} (B_m)_\lambda^\lambda k^\nu & \text{if} \quad m \geq 2 \end{cases}$$

Considering gravitational waves propagating along the $+z$ direction with fixed $\mathbf{k}$, with the 4D-wave vector given by $k^\mu = (\omega, 0, 0, k_z)$ where $\omega_1^2 = k_z^2$ if $k_1^2 = 0$ and $k_m^2 = m^2 = \omega_m^2 - k_z^2$ otherwise with $k_z > 0$, the averaged tensor components are:

$$\left\langle \tau_0^3 \right\rangle = \frac{c^4}{8\pi G} \omega_1^2 \left(C_{11}^2 + C_{12}^2 \right) + \frac{c^4}{16\pi G} \left[\left(-\frac{1}{6} \right) \sum_{j=2}^{p+2} \omega_j k_z |A_j|^2 + \sum_{l=0}^{p} (l+2) (-1)^l a_l \sum_{j=2}^{p+2} \omega_j k_z m_j^{2(l+1)} |A_j|^2 \right]$$

Let us choose:

- $p=0$, $L_g = \left(\overline{R} + a_0 R^2\right)\sqrt{-g}$, fourth-order gravity, with the two modes ω_1, ω_2, it is:

$$\boxed{\left\langle \tau_0^3 \right\rangle = \frac{c^4\omega_1^2}{8\pi G}\left[C_{11}^2 + C_{12}^2\right] + \frac{c^4}{16\pi G}\left\{\left(-\frac{1}{6}\right)\omega_2|A_2|^2 k_z + 2a_0\omega_2 m_2^2|A_2|^2 k_z\right\}}$$

- $p=1$, $L_g = \left(\overline{R} + a_0 R^2 + a_1 R\Box R\right)\sqrt{-g}$, sixth order gravity, with the three modes ω_1, ω_2, ω_3, it is:

$$\left\langle \tau_0^3 \right\rangle = \frac{c^4\omega_1^2}{8\pi G}\left[C_{11}^2 + C_{12}^2\right] + \frac{c^4}{16\pi G}\left\{\left(-\frac{1}{6}\right)\left(\omega_2|A_2|^2 + \omega_3|A_3|^3\right) k_z\right.$$

$$\left. + 2a_0\left[\left(\omega_2 m_2^2|A_2|^2 + \omega_3 m_3^2 |A_3|^2|^2\right) k_z\right] - 3a_1\left[\left(\omega_2 m_2^4|A_2|^2 + \omega_3 m_3^4|A_3|^2\right) k_z\right]\right\}$$

- $p=2$, $L_g = \left(\overline{R} + a_0 R^2 + a_1 R\Box R + a_2 R\Box^2 R\right)\sqrt{-g}$, eighth-order gravity, with the four modes $\omega_1, \omega_2, \omega_3$, ω_4, it is:

$$\left\langle \tau_0^3 \right\rangle = \frac{c^4\omega_1^2}{8\pi G}\left[C_{11}^2 + C_{12}^2\right] + \frac{c^4}{16\pi G}\left\{\left(-\frac{1}{6}\right)\left(\omega_2|A_2|^2 + \omega_3|A_3|^3 + \omega_4|A_4|^2\right)k_z \right.$$
$$+ 2a_0\left[\left(\omega_2 m_2^2|A_2|^2 + \omega_3 m_3^2|A_3|^2 + \omega_4 m_4^2|A_4|^2\right)k_z\right]$$
$$- 3a_1\left[\left(\omega_2 m_2^4|A_2|^2 + \omega_3 m_3^4|A_3|^2 + \omega_4 m_4^4|A_4|^2\right)k_z\right]$$
$$\left. + 4a_2\left[\left(\omega_2 m_2^6|A_2|^2 + \omega_3 m_3^6|A_3|^2 + \omega_4 m_4^6|A_4|^2\right)\right]\right\}$$

As we go up by two with the order of gravity, through the d'Alembert operator $\Box$, we increase by an oscillation mode ω which corresponds to the conformal equivalence of the theories $\Box^k R$ to General Relativity with $k+1$ scalar fields. See *S. Gottlober, H. J. Schmidt and A. A. Starobinsky, Class. Quant. Grav.* **7** *(1990) 893.*

Conclusions

- Using the Noether theorem for rigid translations, it is possible to derive the gravitational energy-momentum pseudo-tensor in curvature based gravity theories of any order.
- In the same way, it is possible to obtain the gravitational energy-momentum pseudo-tensor in non-local gravity including $\Box^{-1}R$ terms (see *S. Capozziello, M. Capriolo and S. Nojiri, Phys. Lett. B* **810**, *135821 (2020)*).
- It is also possible to derive the gravitational energy-momentum pseudo-tensor in Metric-Affine Gravity, as in Palatini Formalism.
- In general, the method can be used to obtain the gravitational energy-momentum pseudo-tensor in non-metric and teleparallel theories of gravity. See *S. Capozziello, M. Capriolo and M. Transirico, Int. J. Geom. Methods Mod. Phys.* **15** *1850164 (2018)*.
- According to this approach, it is possible to calculate the power emitted by any gravitational radiating source.
- New gravitational modes can be derived with respect to GR.