Non-analytic relativistic r-modes of slowly rotating neutron stars

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Outline

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Introduction
What is an r-mode?

r-modes:
• exist only in rotating stars
• for slow rotation rates the oscillation frequency \( \sigma \sim \Omega \)
• for slow rotation rates the fluid element displacement is predominantly toroidal:

\[
\xi \approx \xi^{(0)} = \frac{1}{\sin \theta} \frac{\partial T}{\partial \varphi} e_\theta - \frac{\partial T}{\partial \theta} e_\varphi
\]

\[
T \sim P_l^m(\cos \theta) e^{im\varphi + i\sigma t}
\]

Notations & terminology:
\( \Omega \) – stellar angular velocity
\( \xi \) – the fluid element displacement
\( \xi^{(0)} \) – purely toroidal vector
\( T \) – toroidal function
Possible interpretations of the electromagnetic signals:

1. Strohmayer & Mahmoodifar 2013: «A NON-RADIAL OSCILLATION MODE IN AN ACCRETING MILLISECOND PULSAR?»

2. Strohmayer & Mahmoodifar 2014: «DISCOVERY OF A NEUTRON STAR OSCILLATION MODE DURING A SUPERBURST»

Search for the gravitational-wave signatures (CFS instability):


r-mode calculation in Newtonian theory

Slow-rotation approximation: (traditional approach)

consider $\Omega$ as a small parameter in equations and then follow the ordinary perturbation theory with respect to this parameter.

Dependency on $t$ and $\varphi$:

$$\xi \sim e^{im\varphi+i\sigma t}$$

[further we omit the $e^{im\varphi+i\sigma t}$ factor]

Expansions in associated Legendre polynomials:

$$f(r, \theta) = \sum_{L \geq m} f_L(r) P_L^m(\cos \theta)$$

|$m$ is fixed|

r-mode definition:

1. frequency $\sigma \sim \Omega$ when $\Omega \to 0$
2. $\xi \approx \xi^{(0)}$ when $\Omega \to 0$

$\xi^{(0)}$ – purely toroidal vector

r-mode ordering: ($\Omega$-series)

$$\sigma = \Omega[\sigma^{(0)} + \Omega^2 \sigma^{(1)} + \ldots]$$

$$\xi = \xi^{(0)} + \Omega^2 \xi^{(1)} + \ldots$$

r-mode oscillation spectrum:

(e.g., Provost et al. 1981; Andersson & Kokkotas 2001)

$$T = T_l(r) P_l^m(\cos \theta)$$

$$\sigma^{(0)} = \frac{2m}{l(l+1) - m}$$

DISCRETE OSCILLATION SPECTRUM
Slow-rotation approximation: (traditional approach)

Gravitational field: (Hartle 1967)

\[ ds^2 = -e^{2\nu_0(r)}c^2dt^2 + e^{2\lambda_0(r)}dr^2 + r^2[d\theta^2 + \sin^2\theta(d\varphi - \Omega\omega(r)dt)^2] + O(\Omega^2) \]

(frame-dragging effect)

r-mode definition:
1. frequency \( \sigma \sim \Omega \) when \( \Omega \to 0 \)
2. \( \xi \approx \xi^{(0)} \) when \( \Omega \to 0 \)
   \( \xi^{(0)} \) – purely toroidal vector

r-mode ordering: (\( \Omega \)-series)
\[ \sigma = \Omega[\sigma^{(0)} + \Omega^2\sigma^{(1)} + \ldots] \]
\[ \xi = \xi^{(0)} + \Omega^2\xi^{(1)} + \ldots \]

r-mode oscillation spectrum: (Kojima 1997, 1998)
\[ T \sim \delta(r - r^*)P_l^m(\cos \theta) \]
\[ \sigma^{(0)} = \frac{2m[1 - \omega(r^*)]}{l(l + 1)} - m \]
\[ r^* \in [0; +\infty) \]

Rotation manifests itself already in the linear order in \( \Omega \)!

Consider \( \Omega \) as a small parameter in equations and then follow the ordinary perturbation theory with respect to this parameter.

CONTINUOUS OSCILLATION SPECTRUM
The problem of the continuous spectrum

Studies in the slow rotation approximation:

- Kojima & Hosonuma 1999, 2000
- Beyer & Kokkotas 1999
- Beyer 2006
- Lockitch, Andersson & Friedman 2000
- Yoshida 2001
- Ruoff & Kokkotas 2001
- ...

- discovery and first investigations of the continuous spectrum problem
- mathematically rigorous proof, that Kojima’s equation supports the continuous spectrum
- discrete modes; described only beyond the Cowling approximation; most likely do not exist for typical NS parameters
- discrete modes; isolated modes within the continuous spectrum band; isolated modes have divergent velocity perturbations
The problem of the continuous spectrum

Spectrum regularization attempts:

- Lockitch, Andersson & Watts 2004
  (accounting for higher order terms might regularize the problem; the exact form of these terms is unknown)

- Yoshida & Futamase 2001
  Ruoff & Kokkotas 2002
  (gravitational radiation does not regularize the spectrum)

- Pons, Gualtieri, Miralles & Ferrari 2005
  Pons, Miralles & Ferrari 2006
  (regularization by the shear viscosity seems to be working; assumed that terms with $\eta$ are larger than terms with $\Omega$)

Numerical calculations beyond the slow rotation approximation:

- Yoshida & Lee 2002
- Yoshida, Yoshida & Eriguchi 2005
- Gaertig & Kokkotas 2008
- Doneva et al 2013
- Jasiulek & Chirenti 2017

Do not show any signatures of the continuous part in the oscillation spectrum
The problem of the continuous spectrum

Current status of the problem:

• Relativistic r-modes in barotropic stars do not exist, they are replaced by the so-called inertial modes. [Lockitch, Andersson & Friedman 2000]

• The existence of r-modes in non-barotropic (non-isentropic) stars is questionable. Numerical calculations and theoretical analysis contradict each other. The problem of the continuous spectrum has not been solved.
New approach
The model of a neutron star

Parameters of the matter:

\[ p - \text{pressure} \]
\[ \varepsilon - \text{energy density} \]
\[ w = p + \varepsilon - \text{enthalpy density} \]
\[ n_k - \text{number density of particle species} \ k \]
\[ \mu_k - \text{chemical potential of particle species} \ k \]

Thermodynamic relations:

\[ d\varepsilon = \mu_k dn_k \quad dp = n_k d\mu_k \quad w = \mu_k n_k \]

(summation over repeated indices is implied)

General EOS:

\[ p = p(n_k) \quad \varepsilon = \varepsilon(n_k) \]
\[ w = w(n_k) \quad \mu_m = \mu_m(n_k) \]
\[ f(n_k) \equiv f(n_1, n_2, n_3, \ldots) \]

A two-layer stellar model:

Crust

Barotropic EOS

\[ p = p(\varepsilon) \]

relativistic r-modes do not exist;
we have inertial modes instead

Core

Non-barotropic EOS

\[ p = p(n_k) \]

The problem of the continuous spectrum
Equations and approximations

Equations:
\[
\begin{align*}
\delta [wu^\rho \nabla_\rho u^\mu + (g^{\mu\rho} + u^\mu u^\rho) \nabla_\rho p] &= 0 \\
\delta [\nabla_\mu (n_\kappa u^\kappa)] &= 0 \\
\text{thermodynamic relations} \\
\text{equation of state}
\end{align*}
\]

Approximations and assumptions:

- We ignore the gravitational field perturbations (Cowling approximation)
- We ignore the oblateness of the equilibrium star
- We assume, that the frame-dragging effect is weak
- Frequency errors 6-11\% (e.g., Jasiulek & Chirenti 2017)
- Do not affect the problem of the continuous spectrum
New approach

Assume that the frame-dragging effect is weak:

\[ \omega(r) = \epsilon \tilde{\omega}(r) \quad \epsilon = \omega(0) \text{ -- small parameter} \]

We look for the solution to the equations in the form:

\[ \sigma = \Omega \left[ \sigma^{(0)} + \sigma^{(1)} \right] \quad \xi = \xi^{(0)} + \xi^{(1)} \]

- \( f^{(0)} \) -- correspond to \( \Omega \to 0, \epsilon \to 0 \) and \( \Omega/\epsilon \to 0 \) limit
  [frame-dragging effect is never completely switched off]

- \( f^{(1)} \) -- small corrections, that can be associated with \( \epsilon \) as well as with \( \Omega \)

inspired by the studies of superfluid r-modes:

Kantor, Gusakov & Dommes, Phys.Rev.D v.103 №2 id.023013 (2021)
New approach

Leading «order» equations:

\[ \Omega \to 0 \]
\[ \epsilon \to 0 \]
\[ \frac{\Omega}{\epsilon} \to 0 \]

\[ \sigma^{(0)} = \frac{2m}{l(l + 1)} - m \]

\[ T_l(r) - \text{still to be found} \]

Thus, we «restore»
the traditional r-mode discrete spectrum

Next «order» equations:

1. Use decompositions in the initial
   system of equations

2. Use the leading «order» equations

3. Use selection rules:
   \[ \epsilon f \ll f \quad \Omega^2 f \ll f \]

   \[ l = m \text{ case (schematic):} \]
   \[ \begin{align*}
   \left[ C_1(r) \frac{d}{dr} + C_2(r) \right] \xi_{l+1}^r + \right. \\
   &+ \left[ \Omega^2 C_3(r) + \sigma^{(1)} + C_4 \epsilon \bar{\omega}(r) \right] T_l = 0 \\
   \left[ \frac{d}{dr} + G_1(r) \right] T_l + \frac{G_2(r)}{\Omega^2} \xi_{l+1}^r = 0.
   \end{align*} \]

   \( C_1(r), C_2(r), C_3(r), C_4, G_1(r) \) and \( G_2(r) \)
do not depend on \( \Omega \) or \( \epsilon \)
Boundary conditions and microphysical input

Boundary conditions:

1. Regularity of the solution near the stellar center

2. Matching with the inertial mode in the crust at the crust-core interface

3. Vanishing pressure at the surface: \( p_{\text{surf}} = 0 \)

Microphysical input:

**BSk24 equation of state:**
(Goriely et al 2013)

- barotropic crust
- non-barotropic outer \( npe \)-core
- non-barotropic inner \( npe\mu \)-core

**Equilibrium stellar model:**

- central density: \( \rho_c = 0.7 \times 10^{15} \text{ g/cm}^3 \)
- mass: \( M = 1.4M_\odot \)
- radius: \( R = 12.6 \text{ km} \)
Results
Numerical results

The frame-dragging effect influence:

\[ a = \frac{r}{R} \] - dimensionless radial coordinate

\[ \chi_n^{(0)} \] - toroidal function \( T_1 \) with \( n \) nodes

\( \Omega \) is measured in units of the typical Keplerian velocity \( \Omega_K = (GM/R^3)^{1/2} \)
Numerical results

Mode localization in the $\Omega \to 0$ limit:

- $\chi_{n}^{(0)}$ - toroidal function $T_l$ with $n$ nodes

- $a = r/R$ - dimensionless radial coordinate

- $a = a_{cc} \approx 0.92$ - crust-core interface
Analytic results

r-mode equations:

\[
\begin{align*}
\left[ C_1(r) \frac{d}{dr} + C_2(r) \right] \xi_{l+1}^r + \left[ \Omega^2 C_3(r) + \sigma^{(1)} + C_4 \epsilon \bar{\omega}(r) \right] T_l &= 0 \\
\left[ \frac{d}{dr} + G_1(r) \right] T_l + \frac{G_2(r)}{\Omega^2} \xi_{l+1}^r &= 0.
\end{align*}
\]

Analysis in the \( \Omega \to 0 \) limit:

\[
\begin{align*}
\frac{dT_l}{dr} &\sim \sqrt{\epsilon} \frac{\Omega}{T_l} \\
\frac{d\xi_{l+1}^r}{dr} &\sim \sqrt{\epsilon} \frac{\Omega}{T_l} \\
\xi_{l+1}^r &\sim \sqrt{\epsilon} \Omega T_l \\
\sigma^{(1)} &\sim \epsilon
\end{align*}
\]

\[
\begin{align*}
C_1(a) \frac{d}{dr} \xi_{l+1}^r + \left[ \sigma^{(1)} + C_4 \epsilon \bar{\omega}(r) \right] T_l &= 0 \\
\Omega \frac{dT_l}{dr} + \frac{G_2(r)}{\Omega} \xi_{l+1}^r &= 0.
\end{align*}
\]

\[
\frac{\Omega^2}{\epsilon} \frac{d^2}{dr^2} T_l - q_\sigma(r) T_l = 0
\]

\( q_\sigma(r) \) – monotonically decreasing function
r-mode non-analyticity: toy model

\[ \frac{\Omega^2}{\epsilon} \frac{d^2}{dr^2} T_l - q_\sigma T_l = 0 \]

\[ q_\sigma = \text{const} \]

\[ T_l(r) \sim \exp \left( \pm \frac{\sqrt{\epsilon}}{\Omega} \sqrt{q_\sigma} r \right) \]

\[ \frac{dT_l}{dr} = \pm \frac{\sqrt{\epsilon}}{\Omega} \sqrt{q_\sigma} T_l \]
Analytic results

Defining equation:
\[
\frac{\Omega^2}{\epsilon} \frac{d^2}{dr^2} T_l - q_\sigma(r) T_l = 0 \quad \frac{\Omega^2}{\epsilon} \ll 1
\]

Analogy – Schrödinger equation:
\[
\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + [E - V(x)] \psi(x) = 0 \quad \hbar \ll 1
\]

WKB approximation: treat Planck constant as a small parameter
Analytic results

Actually [dimensionless form]:

\[ \frac{\Omega^2}{\epsilon} \frac{d^2}{da^2} T_l - q_\sigma(a) T_l = 0 \]
\[ q_\sigma(a_t) = 0 \]
\[ a_t - \text{turning point} \]

\[ \Delta \rightarrow 0 \text{ as } \Omega \rightarrow 0 \]

\[ a = \frac{r}{R} - \text{dimensionless radial coordinate} \]

finite number of nodes \[ \Rightarrow a_t \rightarrow a_{cc} \Rightarrow q_\sigma(a_{cc}) \rightarrow 0 \Rightarrow \sigma^{(1)} \rightarrow -\frac{2\epsilon\tilde{\omega}(a_{cc})}{l + 1} \]
Analytic results

Typical behaviors:

- **Exponential growth/decrease towards the center**
- **Fast oscillations**

\[ T^{(0)}_l(a) \]
\[ q_\sigma(a) \]

\[ \Delta \to 0 \text{ as } \Omega \to 0 \]

- **Region far from the turning point**
- **Actually is a tiny region near the turning point**

\[ a_t \to a_{cc} \]

\[ q_\sigma(a) \approx \alpha^2 (a_t - a) \]

\[ z = (a_t - a) \left( \frac{\alpha \sqrt{\epsilon}}{\Omega} \right)^{2/3} \]

\[ T_{l,II}(z) = A_{II} \text{Ai}(z) \]
Analytic results

Explaining the eigenfunction behavior in the $\Omega \to 0$ limit:

\[ \chi_4^{(0)} \]

- $l = m = 2$
- $\Omega = 0.005$
- $a_{cc} \approx 0.92$
- $a_t \approx 0.78$

- BSk24
- Newt
- GR

exponential suppression oscillations
Analytic results

Explicit formula
for the r-mode oscillation spectrum:

\[
\sigma = \frac{2\Omega}{l + 1} \left[ 1 - \omega(a_{cc}) \left\{ 1 + z_n \frac{\omega'(a_{cc})}{\omega(a_{cc})} \left( \frac{\Omega}{\alpha \sqrt{\epsilon}} \right)^{2/3} \right\} \right] - l\Omega
\]

\[\text{Ai}(z_n) = 0 \quad n \in \mathbb{N}\]

[in good accordance with numerically obtained eigenfrequencies]

Interesting features:

• discrete oscillation spectrum
  (no indications of the continuous part in the spectrum)
• one has to know the frame-dragging function
  (and it’s derivative) only in one point \( a = a_{cc} \)
• the r-mode oscillation frequencies are non-analytic functions of \( \Omega \) and \( \epsilon \)
• r-mode eigenfrequencies are defined by the zeros of the Airy function
Conclusion
Conclusion

1. We have obtained equations, governing the dynamics of discrete relativistic r-modes.

2. We have calculated relativistic r-mode eigenfunctions and eigenfrequencies for different stellar rotation rates.

3. The analysis of the r-mode equations shows, that eigenfunctions and eigenfrequencies are non-analytic functions of $\Omega$ and $\epsilon$:

   $$\sigma = \frac{2\Omega}{l+1} \left[ 1 - \omega(a_{cc}) \left\{ 1 + z_n \frac{\omega'(a_{cc})}{\omega(a_{cc})} \left( \frac{\Omega}{\alpha \sqrt{\epsilon}} \right)^{2/3} \right\} \right] - l\Omega$$

   $$T_i(a) \sim \exp\left( \frac{\sqrt{\epsilon}}{\Omega} \int_0^a \sqrt{q\sigma} da \right)$$

   $$\text{Ai}(z_n) = 0 \quad n \in \mathbb{N}$$

4. Why does the continuous spectrum emerges in other investigations? The reason is, that traditional approach implicitly relies on the assumption, that r-modes are analytic functions of the angular velocity, which is not the case for slow rotation rates.