Non-analytic relativistic r-modes of slowly rotating neutron stars

Kraav K. Y., Gusakov M. E., Kantor E. M.

Ioffe Physical-Technical Institute of the Russian Academy of Sciences

28 September 2021

Yerevan, Armenia

Outline

- Introduction
- New approach
- Results
- Conclusion

Introduction

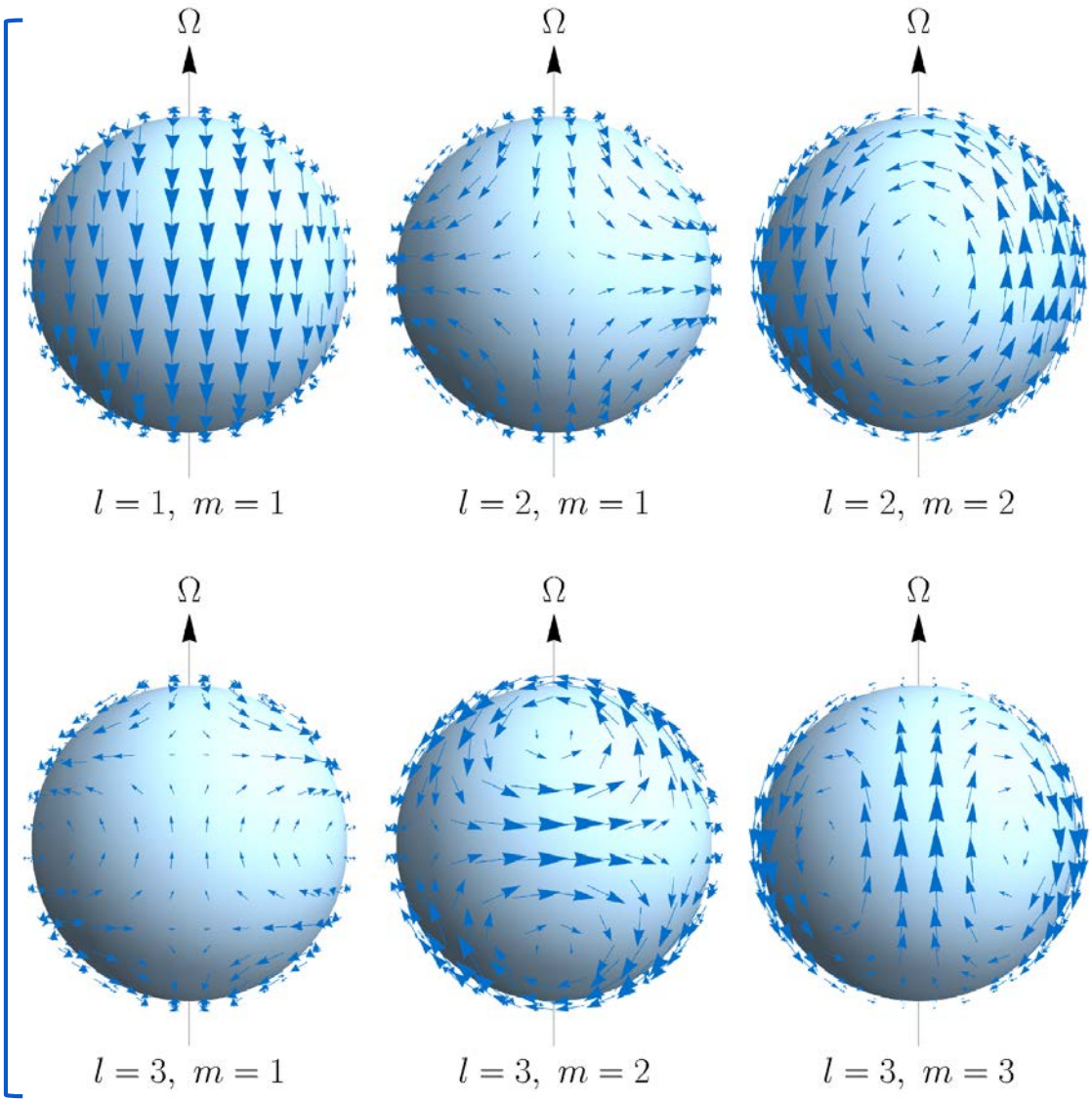
What is an r-mode?

r-modes:

- exist only in rotating stars
- for slow rotation rates the oscillation frequency $\sigma \sim \Omega$
- for slow rotation rates the fluid element displacement is predominantly toroidal:

$$\boldsymbol{\xi} \approx \boldsymbol{\xi}^{(0)} = \frac{1}{\sin\theta}\frac{\partial T}{\partial \varphi}\boldsymbol{e}_\theta - \frac{\partial T}{\partial \theta}\boldsymbol{e}_\varphi$$

$$T \sim P_l^m(\cos\theta)e^{im\varphi + i\sigma t}$$



Notations & terminology:

Ω – stellar angular velocity
$\boldsymbol{\xi}$ – the fluid element displacement
$\boldsymbol{\xi}^{(0)}$ – purely toroidal vector
T – toroidal function

r-mode observations

Search for the gravitational-wave signatures (CFS instability):

1. Abbott et al 2021: [LIGO, VIRGO, KAGRA] «DIVING BELOW THE SPIN-DOWN LIMIT CONSTRAINTS ON THE GRAVITATIONAL WAVES FROM THE ENERGETIC YOUNG PULSAR PSR J0537-6910»

2. Abbott et al 2021: [LIGO, VIRGO, KAGRA] «SEARCHES FOR CONTINUOUS GRAVITATIONAL WAVES FROM YOUNG SUPERNOVA REMNANTS IN THE EARLY THIRD OBSERING RUN OF ADVANCED LIGO AND VIRGO»

Possible interpretations of the electromagnetic signals:

1. Strohmayer & Mahmoodifar 2013: «A NON-RADIAL OSCILLATION MODE IN AN ACCRETING MILLISECOND PULSAR?»

2. Strohmayer & Mahmoodifar 2014: «DISCOVERY OF A NEUTRON STAR OSCILLATION MODE DURING A SUPERBURST»

r-mode calculation in Newtonian theory

Slow-rotation approximation:
(traditional approach)

consider Ω as a small parameter in equations and then follow the ordinary perturbation theory with respect to this parameter

Dependency on t and φ:

$$\boldsymbol{\xi} \sim e^{im\varphi + i\sigma t}$$

[further we omit the $e^{im\varphi+i\sigma t}$ factor]

Expansions in associated Legendre polynomials:

$$f(r,\theta) = \sum_{L \geq m} f_L(r) P_L^m(\cos\theta)$$

[m is fixed]

r-mode definition:

1. frequency $\sigma \sim \Omega$ when $\Omega \to 0$
2. $\boldsymbol{\xi} \approx \boldsymbol{\xi}^{(0)}$ when $\Omega \to 0$

$\boldsymbol{\xi}^{(0)}$ – purely toroidal vector

⇩

r-mode ordering:
(Ω-series)

$$\sigma = \Omega[\sigma^{(0)} + \Omega^2 \sigma^{(1)} + \ldots]$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{(0)} + \Omega^2 \boldsymbol{\xi}^{(1)} + \ldots$$

⇩

r-mode oscillation spectrum:
(e.g., Provost et al. 1981; Andersson & Kokkotas 2001)

$$T = T_l(r) P_l^m(\cos\theta)$$

$$\sigma^{(0)} = \frac{2m}{l(l+1)} - m$$

DISCRETE OSCILLATION SPECTRUM

r-modes in GR

Slow-rotation approximation:

(traditional approach)

consider Ω as a small parameter in equations and then follow the ordinary perturbation theory with respect to this parameter

Gravitational field :

(Hartle 1967)

$$ds^2 = -e^{2\nu_0(r)}c^2dt^2 + e^{2\lambda_0(r)}dr^2 + \\ + r^2[d\theta^2 + \sin^2\theta(d\varphi - \Omega\omega(r)dt)^2] + O(\Omega^2)$$

(frame-dragging effect)

Rotation manifests itself already in the *linear* order in Ω!

r-mode definition:

1. frequency $\sigma \sim \Omega$ when $\Omega \to 0$
2. $\boldsymbol{\xi} \approx \boldsymbol{\xi}^{(0)}$ when $\Omega \to 0$

$\boldsymbol{\xi}^{(0)}$ – purely toroidal vector

⇩

r-mode ordering:

(Ω-series)

$$\sigma = \Omega[\sigma^{(0)} + \Omega^2\sigma^{(1)} + \dots]$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{(0)} + \Omega^2\boldsymbol{\xi}^{(1)} + \dots$$

⇩

r-mode oscillation spectrum:

(Kojima 1997, 1998)

$$T \sim \delta(r - r^*)P_l^m(\cos\theta)$$

$$\sigma^{(0)} = \frac{2m[1 - \omega(r^*)]}{l(l+1)} - m$$

$$r^* \in [0; +\infty)$$

CONTINUOUS OSCILLATION SPECTRUM

The problem of the continuous spectrum

Studies in the slow rotation approximation:

- Kojima 1997, 1998
- Kojima & Hosonuma 1999, 2000

discovery and first investigations of the continuous spectrum problem

- Beyer & Kokkotas 1999
- Beyer 2006

mathematically rigorous proof, that Kojima's equation supports the continuous spectrum

- Lockitch, Andersson & Friedman 2000
- Yoshida 2001

discrete modes; described only beyond the Cowling approximation; most likely do not exist for typical NS parameters

- Ruoff & Kokkotas 2001

discrete modes; isolated modes within the continuous spectrum band; isolated modes have divergent velocity perturbations

- ...

The problem of the continuous spectrum

Spectrum regularization attempts:

- Lockitch, Andersson & Watts 2004
  (accounting for higher order terms might regularize the problem; the exact form of these terms is unknown)

- Yoshida & Futamase 2001
  Ruoff & Kokkotas 2002
  (gravitational radiation does not regularize the spectrum)

- Pons, Gualtieri, Miralles & Ferrari 2005
  Pons, Miralles & Ferrari 2006
  (regularization by the shear viscosity seems to be working; assumed that terms with η are larger than terms with Ω)

Numerical calculations beyond the slow rotation approximation:

- Yoshida & Lee 2002
- Yoshida, Yoshida & Eriguchi 2005
- Gaertig & Kokkotas 2008
- Doneva et al 2013
- Jasiulek & Chirenti 2017

Do not show any signatures of the continuous part in the oscillation spectrum

The problem of the continuous spectrum

Current status of the problem:

- Relativistic r-modes in barotropic stars do not exist, they are replaced by the so-called inertial modes. [Lockitch, Andersson & Friedman 2000]

- The existence of r-modes in non-barotropic (non-isentropic) stars is questionable. Numerical calculations and theoretical analysis contradict each other. The problem of the continuous spectrum has not been solved.

New approach

The model of a neutron star

Parameters of the matter:

p – pressure
ε – energy density
$w = p + \varepsilon$ – enthalpy density
n_k – number density of particle species k
μ_k – chemical potential of particle species k

Thermodynamic relations:

$$d\varepsilon = \mu_k dn_k \quad dp = n_k d\mu_k \quad w = \mu_k n_k$$

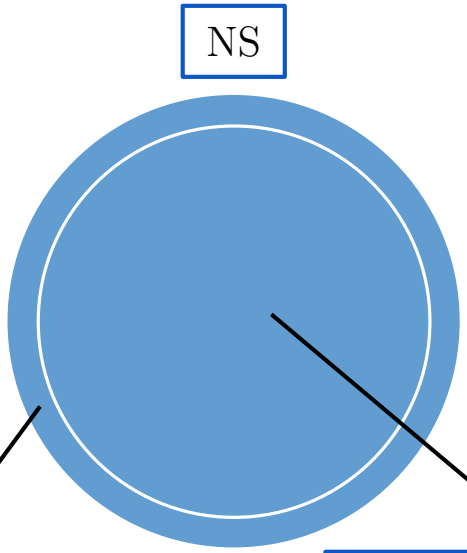
(summation over repeated indices is implied)

General EOS:

$$p = p(n_k) \quad \varepsilon = \varepsilon(n_k)$$
$$w = w(n_k) \quad \mu_m = \mu_m(n_k)$$

$$f(n_k) \equiv f(n_1, n_2, n_3, \dots)$$

A two-layer stellar model:



Crust
(barotropic EOS)
$p = p(\varepsilon)$

relativistic r-modes
do not exist;
we have inertial
modes instead

Core
(non-barotropic EOS)
$p = p(n_k)$

The problem
of the continuous
spectrum

Equations and approximations

Equations:

$$\begin{cases} \delta[wu^\rho \nabla_\rho u^\mu + (g^{\mu\rho} + u^\mu u^\rho)\nabla_\rho p] = 0 \\ \delta[\nabla_\mu (n_k u^\mu)] = 0 \\ \text{thermodynamic relations} \\ \text{equation of state} \end{cases}$$

Approximations and assumptions:

- We ignore the gravitational field perturbations (Cowling approximation)
- We ignore the oblateness of the equilibrium star
- We assume, that the frame-dragging effect is weak

frequency errors 6-11%
(e.g., Jasiulek & Chirenti 2017)

do not affect the problem of the continuous spectrum

New approach

Assume

that the frame-dragging effect is weak:

$$\omega(r) = \epsilon\tilde{\omega}(r) \qquad \epsilon = \omega(0) \text{ – small parameter}$$

We look for the solution to the equations in the form:

$$\sigma = \Omega\left[\sigma^{(0)} + \sigma^{(1)}\right] \qquad \boldsymbol{\xi} = \boldsymbol{\xi}^{(0)} + \boldsymbol{\xi}^{(1)}$$

- $f^{(0)}$ – correspond to $\Omega \to 0$, $\epsilon \to 0$ and $\Omega/\epsilon \to 0$ limit

  [frame-dragging effect is never completely switched off]

- $f^{(1)}$ – small corrections, that can be associated with ϵ as well as with Ω

inspired by the studies of superfluid r-modes:

Kantor, Gusakov & Dommes, Phys.Rev.Lett v.125 №15 id.151101 (2020)
Kantor, Gusakov & Dommes, Phys.Rev.D v.103 №2 id.023013 (2021)

New approach

Leading «order» equations:

$$\Omega \to 0$$
$$\epsilon \to 0$$
$$\frac{\Omega}{\epsilon} \to 0$$

$\Rightarrow$

$$\sigma^{(0)} = \frac{2m}{l(l+1)} - m$$

$T_l(r)$ – still to be found

Thus, we «restore»
the traditional r-mode discrete spectrum

Next «order» equations:

1. Use decompositions in the initial system of equations
2. Use the leading «order» equations
3. Use selection rules:

$$\epsilon f \ll f \qquad \Omega^2 f \ll f$$

$\Rightarrow$

$l = m$ case (schematic):

$$\begin{cases} \left[C_1(r)\frac{d}{dr} + C_2(r)\right]\xi^r_{l+1} + \\ \qquad + \left[\Omega^2 C_3(r) + \sigma^{(1)} + C_4\epsilon\tilde{\omega}(r)\right]T_l = 0 \\ \left[\frac{d}{dr} + G_1(r)\right]T_l + \frac{G_2(r)}{\Omega^2}\xi^r_{l+1} = 0. \end{cases}$$

$C_1(r)$, $C_2(r)$, $C_3(r)$, C_4, $G_1(r)$ and $G_2(r)$
do not depend on Ω or ϵ

Boundary conditions and microphysical input

Boundary conditions:

1. Regularity of the solution near the stellar center
2. Matching with the inertial mode in the crust at the crust-core interface
3. Vanishing pressure at the surface: $p_{\rm surf} = 0$

Microphysical input:

BSk24 equation of state:
(Goriely et al 2013)

- barotropic crust
- non-barotropic outer npe-core
- non-barotropic inner $npe\mu$-core

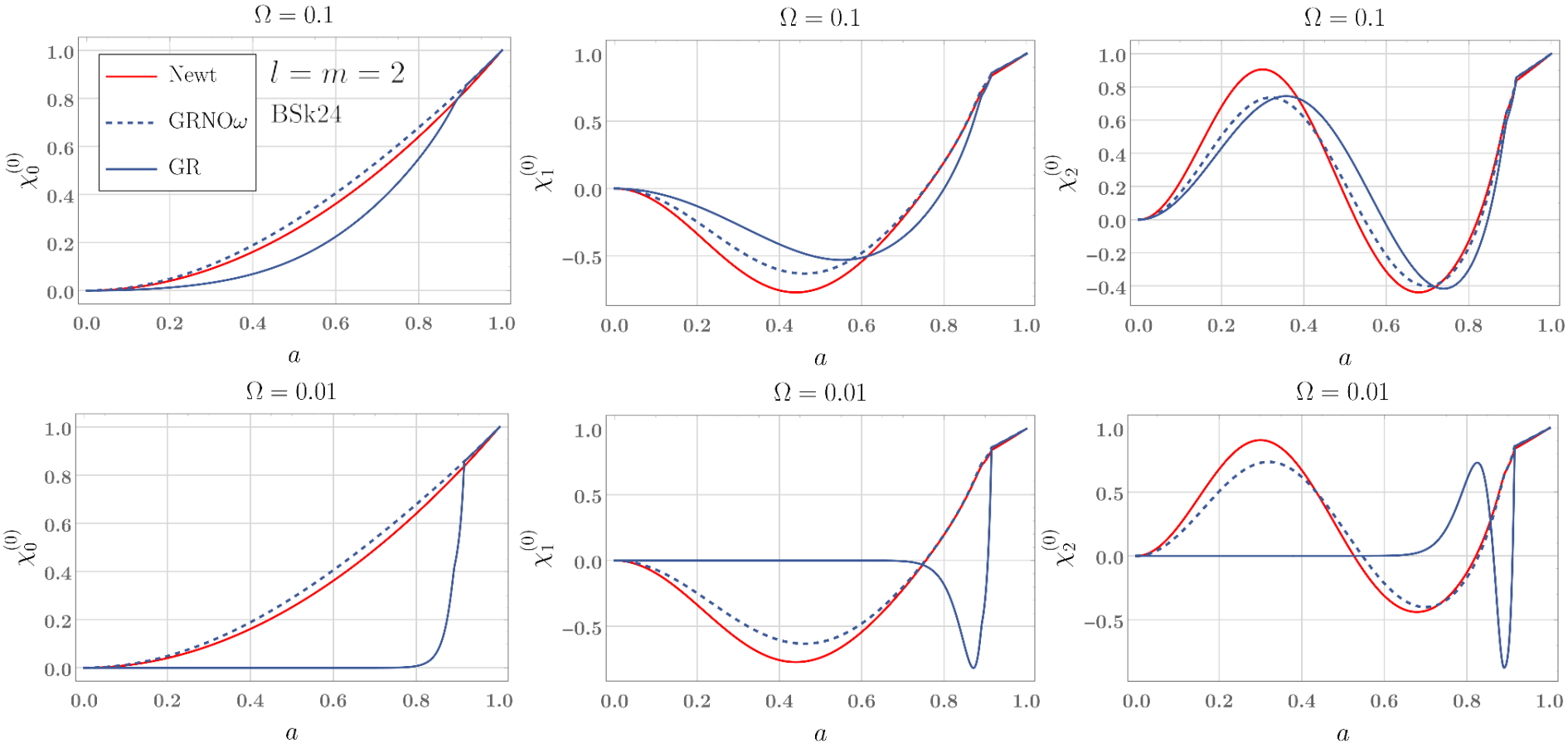
Equilibrium stellar model:

central density: $\rho_c = 0.7 \times 10^{15}$ g/cm^3
mass: $M = 1.4 M_\odot$
radius: $R = 12.6$ km

Results

Numerical results

The frame-dragging effect influence:



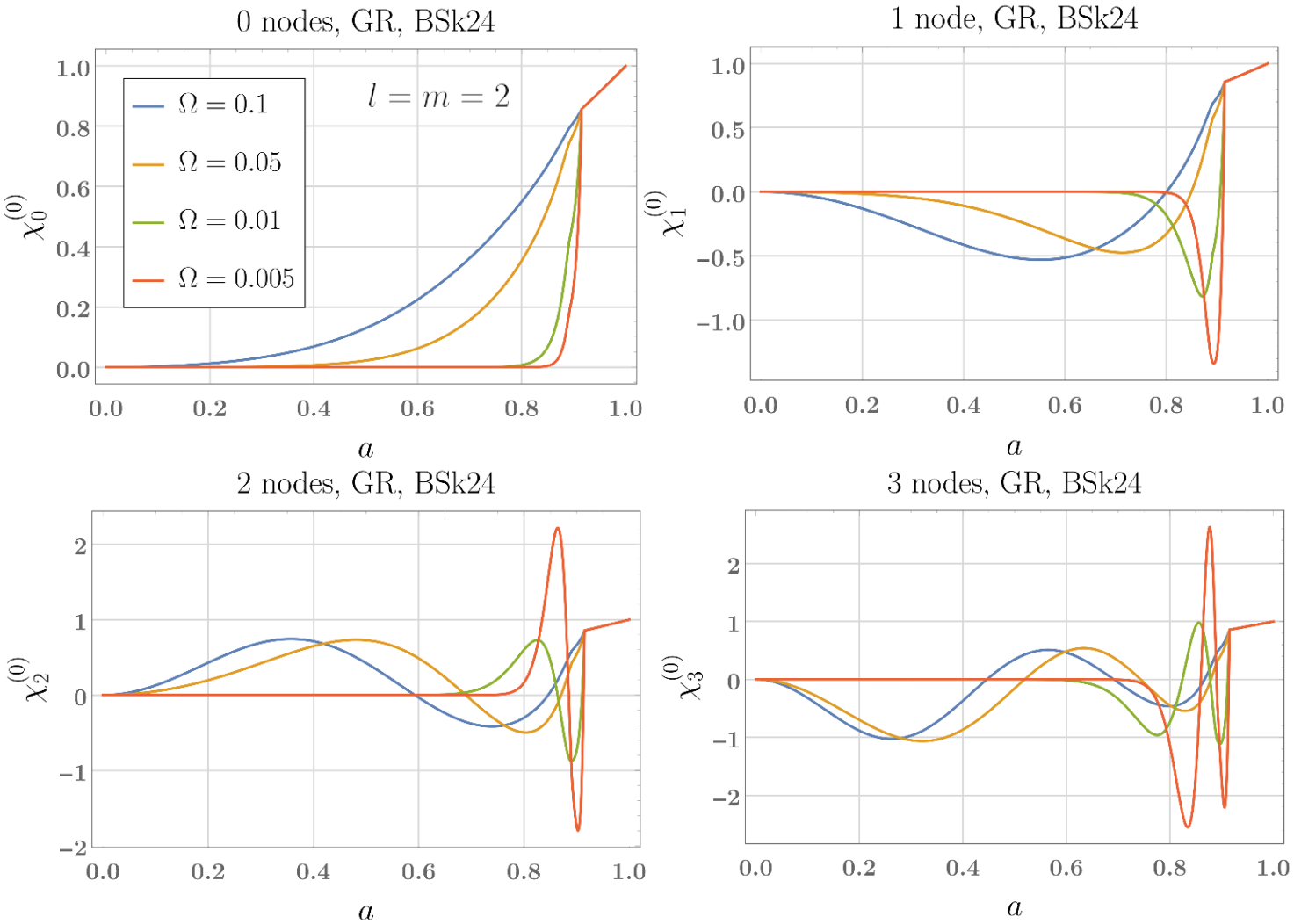
$a = r/R$ – dimensionless radial coordinate

$\chi_n^{(0)}$ – toroidal function T_l with n nodes

Ω is measured in units of the typical Keplerian velocity $\Omega_{\mathrm{K}} = (GM/R^3)^{1/2}$

Numerical results

Mode localization in the $\Omega \to 0$ limit:



$\chi_n^{(0)}$ – toroidal function T_l with n nodes

$a = r/R$ – dimensionless radial coordinate

$a = a_{cc} \approx 0.92$ – crust-core interface

Analytic results

r-mode equations:

$$\begin{cases} \left[C_1(r)\frac{d}{dr} + C_2(r)\right]\xi^r_{l+1} + \left[\Omega^2 C_3(r) + \sigma^{(1)} + C_4\epsilon\tilde{\omega}(r)\right]T_l = 0 \\ \left[\frac{d}{dr} + G_1(r)\right]T_l + \frac{G_2(r)}{\Omega^2}\xi^r_{l+1} = 0. \end{cases}$$

Analysis in the $\Omega \to 0$ limit:

$$\frac{dT_l}{dr} \sim \frac{\sqrt{\epsilon}}{\Omega}T_l \quad \frac{d\xi^r_{l+1}}{dr} \sim \frac{\sqrt{\epsilon}}{\Omega}\xi^r_{l+1}$$

$$\xi^r_{l+1} \sim \sqrt{\epsilon}\Omega\, T_l \quad \sigma^{(1)} \sim \epsilon$$

$$\Rightarrow \boxed{\begin{cases} C_1(a)\frac{d}{dr}\xi^r_{l+1} + \left[\sigma^{(1)} + C_4\epsilon\tilde{\omega}(r)\right]T_l = 0 \\ \Omega\frac{d}{dr}T_l + \frac{G_2(r)}{\Omega}\xi^r_{l+1} = 0. \end{cases}}$$

$$\Rightarrow \boxed{\frac{\Omega^2}{\epsilon}\frac{d^2}{dr^2}T_l - q_\sigma(r)T_l = 0}$$

$q_\sigma(r)$ – monotonically decreasing function

Analytic results

r-mode non-analyticity: toy model

$$\frac{\Omega^2}{\epsilon}\frac{d^2}{dr^2}T_l - q_\sigma T_l = 0$$

$$q_\sigma = \text{const}$$

$$\Downarrow$$

$$T_l(r) \sim \exp\left(\pm\frac{\sqrt{\epsilon}}{\Omega}\sqrt{q_\sigma}r\right)$$

$$\Downarrow$$

$$\frac{dT_l}{dr} = \pm\frac{\sqrt{\epsilon}}{\Omega}\sqrt{q_\sigma}T_l$$

$$\boxed{\frac{dT_l}{dr} \sim \frac{\sqrt{\epsilon}}{\Omega}T_l}$$

Analytic results

Defining equation:

$$\boxed{\frac{\Omega^2}{\epsilon}\frac{d^2}{dr^2}T_l - q_\sigma(r)T_l = 0} \qquad \frac{\Omega^2}{\epsilon} \ll 1$$

Analogy – Schrödinger equation:

$$\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + [E - V(x)]\psi(x) = 0 \qquad \hbar \ll 1$$

WKB approximation: treat Planck constant as a small parameter

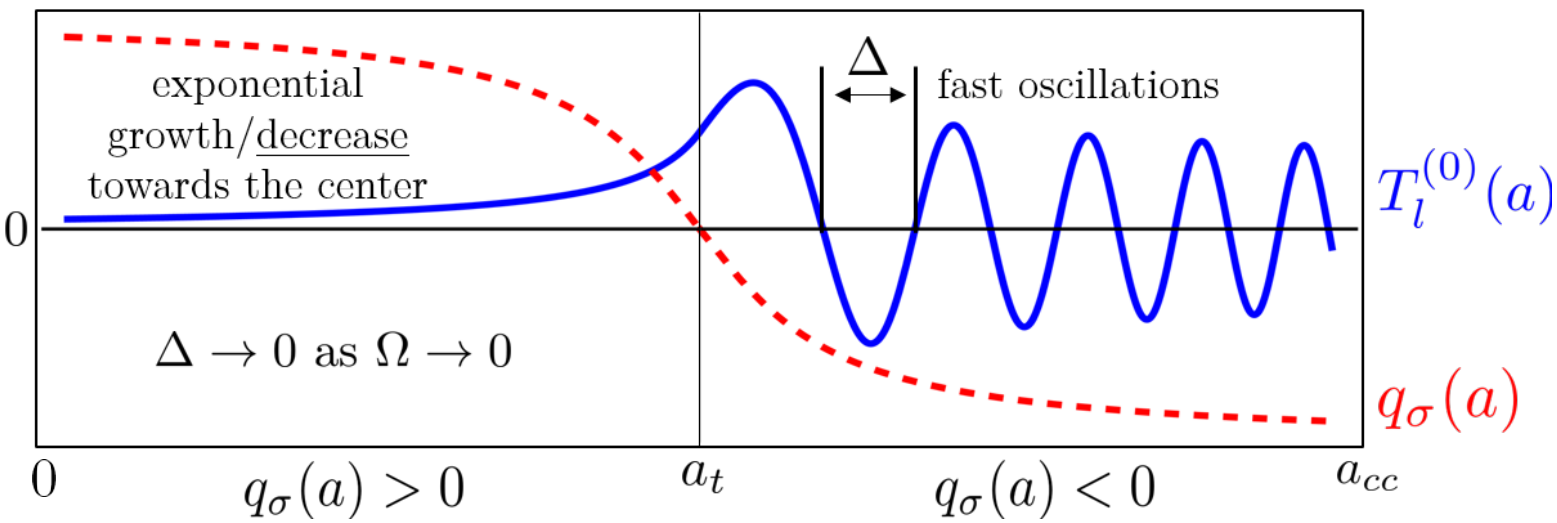
Analytic results

Actually [dimensionless form]:

$$\frac{\Omega^2}{\epsilon}\frac{d^2}{da^2}T_l - q_\sigma(a)T_l = 0$$

$$q_\sigma(a_t) = 0$$

a_t – turning point

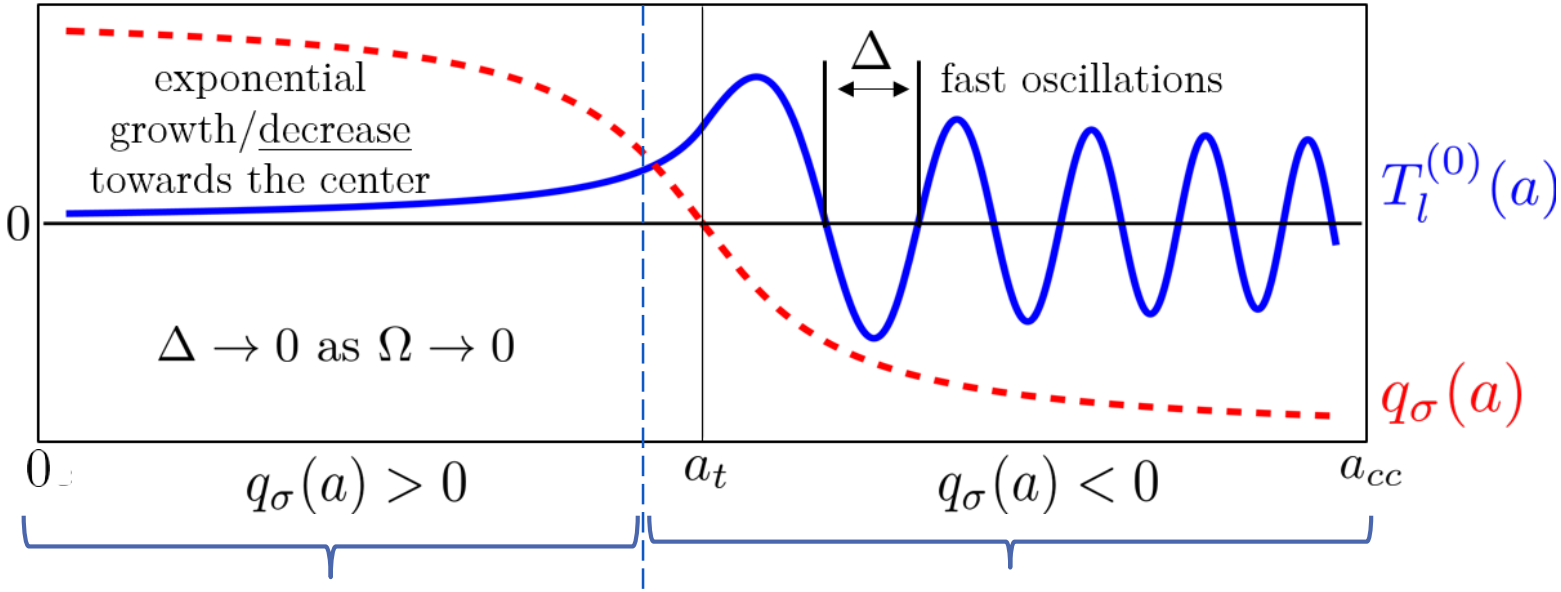


$a = r/R$ – dimensionless radial coordinate

finite number of nodes ⇨ $a_t \to a_{cc}$ ⇨ $q_\sigma(a_{cc}) \to 0$ ⇨ $\boxed{\sigma^{(1)} \to -\frac{2\epsilon\tilde{\omega}(a_{cc})}{l+1}}$

Analytic results

Typical behaviors:



region far from the turning point

actually is a tiny region near the turning point $a_t \to a_{cc}$

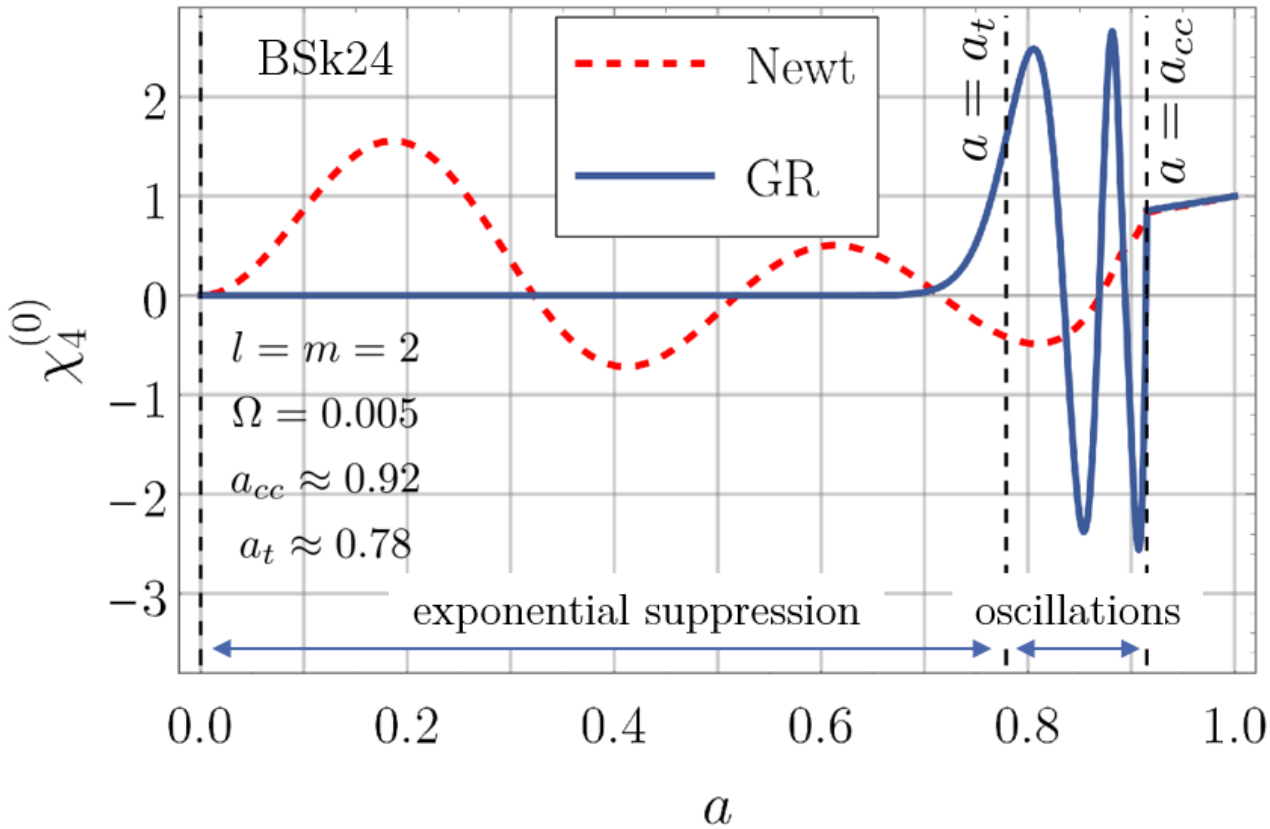
$$T_{l,\mathrm{I}}(a) = \frac{A_\mathrm{I}}{q_\sigma^{1/4}} \exp\left(\frac{\sqrt{\epsilon}}{\Omega} \int_0^a \sqrt{q_\sigma} da \right)$$

$$q_\sigma(a) \approx \alpha^2 (a_t - a) \qquad z = (a_t - a) \left(\frac{\alpha \sqrt{\epsilon}}{\Omega} \right)^{2/3}$$

$$T_{l,\mathrm{II}}(z) = A_\mathrm{II}\ \mathrm{Ai}(z)$$

Analytic results

Explaining the eigenfunction behavior in the $\Omega \to 0$ limit:



Analytic results

Explicit formula
for the r-mode oscillation spectrum:

$$\sigma = \frac{2\Omega}{l+1}\left[1 - \omega(a_{cc})\left\{1 + z_n \frac{\omega'(a_{cc})}{\omega(a_{cc})}\left(\frac{\Omega}{\alpha\sqrt{\epsilon}}\right)^{2/3}\right\}\right] - l\Omega$$

$$\mathrm{Ai}(z_n) = 0 \qquad n \in \mathbb{N}$$

[in good accordance with numerically obtained eigenfrequencies]

Interesting features:

- discrete oscillation spectrum
  (no indications of the continuous part in the spectrum)
- one has to know the frame-dragging function
  (and it's derivative) *only in one point* $a = a_{cc}$
- the r-mode oscillation frequencies are
  non-analytic functions of Ω and ϵ
- r-mode eigenfrequencies are defined by the
  zeros of the Airy function

Conclusion

Conclusion

1. We have obtained equations, governing the dynamics of discrete relativistic r-modes

2. We have calculated relativistic r-mode eigenfunctions and eigenfrequencies for different stellar rotation rates

3. The analysis of the r-mode equations shows, that eigenfunctions and eigenfrequencies are non-analytic functions of Ω and ϵ:

$$\sigma = \frac{2\Omega}{l+1}\left[1 - \omega(a_{cc})\left\{1 + z_n \frac{\omega'(a_{cc})}{\omega(a_{cc})}\left(\frac{\Omega}{\alpha\sqrt{\epsilon}}\right)^{2/3}\right\}\right] - l\Omega \qquad T_l(a) \sim \exp\left(\frac{\sqrt{\epsilon}}{\Omega}\int_0^a \sqrt{q_\sigma}\,da\right)$$

$$\mathrm{Ai}(z_n) = 0 \qquad n \in \mathbb{N}$$

4. Why does the continuous spectrum emerges in other investigations? The reason is, that traditional approach implicitly relies on the assumption, that r-modes are analytic functions of the angular velocity, which is not the case for slow rotation rates