Vacuum currents near the horizon of a cylindrical black hole

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Introduction

In quantum field theory the vacuum state depends on the choice of the complete set of mode functions used for the expansion of the field operator. Among widely known examples are

Minkowski vacuum	Fulling-Rindler (FR) vacuum in Minkowski spacetime
• Is realized by the plane-wave modes most frequently used in considerations of quantum field-theoretical effects on background of flat spacetime.	• The FR vacuum corresponds to the quantization of fields in the reference frame with the coordinate lines corresponding to the worldlines of uniformly accelerated observers a.k.a. **Rindler coordinates**.

Introduction/Motivations

- On the base of the equivalence principle, we can expect that properties of quantum fluctuations in the FR vacuum will have common qualitative features with those for vacuum states of quantum fields in classical gravitational backgrounds.

In particular, **the Rindler metric is the leading approximation of the near-horizon geometry for most black holes.**

Better understanding of quantum vacuum effects in **Rindler spacetime** serves as a handle in considerations of more complicated background geometries e.g. the **Schwarzschild metric.**

- The study of the properties of the FR vacuum is also related to the conformal connection of the **Rindler metric tensor** to the metric tensors of **de Sitter spacetime** and of the

Friedmann-Robertson-Walker spacetime with negative spatial curvature. By using these conformal relations, the expectation values of local physical observables for conformally invariant fields in those spacetimes are obtained from the Rindler expectation values by conformal transformations. **Rindler observers** in **anti-de Sitter (AdS) spacetime** and the related **AdS/CFT correspondence** have been discussed.

Scalar field modes in partially compact Rindler spacetime

- The background geometry is given by (D + 1)-dimensional locally Rindler line element.

$$ds^2 = \xi^2 d\tau^2 - d\xi^2 - \sum_{i=2}^{D} \left(dx^i\right)^2$$

in terms of the Rindler coordinates:

$$x^\mu = \left(x^0 = \tau, x^1 = \xi, \mathbf{x}_p, \mathbf{x}_q\right) \qquad 0 < \xi < \infty$$

- The p-dimensional subspace with Cartesian coordinates has trivial topology. The q-dimensional subspace covered by the coordinates *l = p+2, .., D* is compactified to a torus $\left(S^1\right)^q,\ q = D - p - 1.$
- In the discussion below the length of the compact dimension x^l will be denoted by L_l.

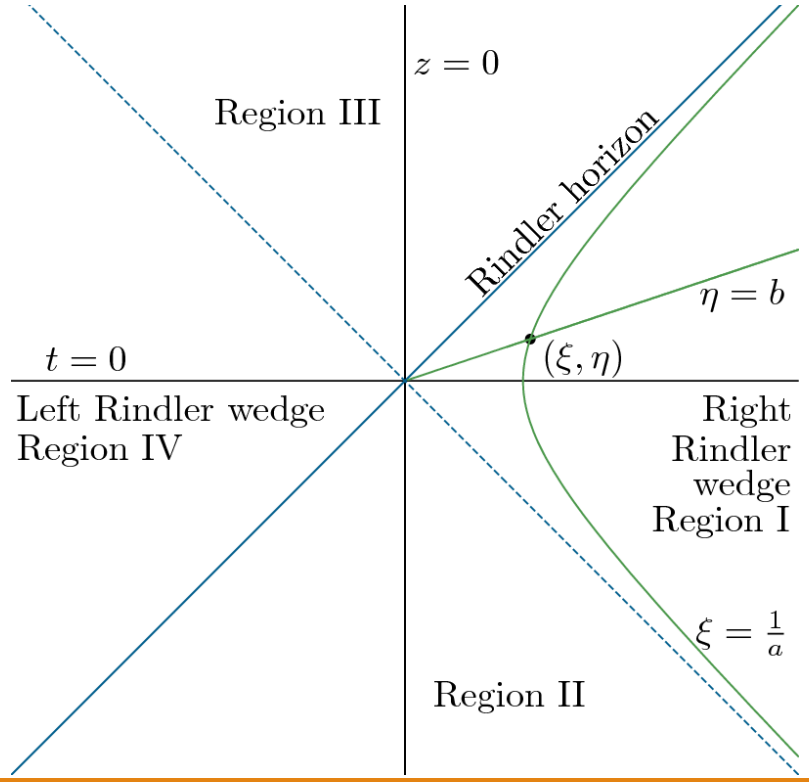
$$0 \leqslant x^l \leqslant L_l \text{ for } l = p+2, \ldots, D$$

Scalar field modes in partially compact Rindler spacetime

- Thus, the subspace with the coordinates $(x_2,\ x_3,\ .\ .\ .\ .\ ,\ x_D)$ has spatial topology $R^p \times \left(S^1\right)^q$.
- By introducing appropriate coordinate transformations in (τ, ξ) one can represent the line element in it's locally Minkowskian form.

$$t_{\mathrm{M}} = \xi \sinh \tau, \qquad x^1_{\mathrm{M}} = \xi \cosh \tau,$$

$$ds^2 = \eta_{\mu\nu} dx^\mu_{\mathrm{M}} dx^\nu_{\mathrm{M}} \quad \eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$$



- It is evident now, that the coordinates x^μ cover the so called **right Rindler wedge:** $x^1_{\mathrm{M}} > |t_{\mathrm{M}}|$
- The worldline for given $(\xi,\ x^2,\ .\ .\ .\ .\ ,\ x^D)$ corresponds to an observer with constant proper acceleration $1/\xi$. The proper time for that observer is expressed in terms of the dimensionless time coordinate τ as $t = \xi\tau$.

Scalar field modes in partially compact Rindler spacetime/Field content

- We consider a quantum charged scalar field $\varphi(x)$, in the presence of an external classical gauge field with the vector potential A_μ.

$$\left(g^{\mu\nu} D_\mu D_\nu + m^2\right)\varphi = 0 \qquad D_\mu = \nabla_\mu + ieA_\mu$$

- The spatial topology is nontrivial and for the theory to be defined in addition to the field equation we need to specify the periodicity conditions on the field operator along compact dimensions. In what follows generic quasiperiodic conditions will be imposed for $l = p + 2, \ldots, D$.

$$\varphi(t, \xi, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\alpha_l}\varphi(t, \xi, \mathbf{x}_p, \mathbf{x}_q),$$

- e_l - the unit vector along the compact dimension x^l
- α_l - arbitrary constant phases

Scalar field modes in partially compact Rindler spacetime/The gauge field

- We will consider a simple case of a gauge field, with constant components only nonzero only along the compact dimensions.
- By using an appropriate gauge transformation, one can omit the new gauge field from the transformed field equation.

$$\varphi'(x) = e^{ie\chi}\varphi(x),\ A'_\mu = A_\mu - \partial_\mu \chi \qquad \chi = A_\mu x^\mu$$

$$A'_\mu = 0$$

- However, the gauge transformation modifies the periodicity conditions for the new field

$$\varphi'(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\tilde{\alpha}_l}\varphi'(t, \mathbf{x}_p, \mathbf{x}_q), \qquad \tilde{\alpha}_l = \alpha_l + eA_l L_l.$$

- The physical observables depend on the phases α_l and on the constant components A_l through the gauge invariant combination. The effect of a constant gauge field is related to the nontrivial spatial topology and is of **the Aharonov-Bohm-type**. The consideration below will be presented in terms of the field $\varphi'(x)$, prime omitted.

The Mode functions

- The properties of the vacuum state in our problem are completely determined by **the two-point functions**. The evaluation of these functions can be conveniently carried on by using the summation representation over the complete set of mode functions.

$$\varphi^{(\pm)}_{\omega,\mathbf{k}}(x) = C_{\omega,\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}\mp i\omega\tau} K_{i\omega}(\lambda\xi),\ \lambda = \sqrt{k^2+m^2}$$

- ω – energy
- $\mathbf{k}$ – momentum
- $(\pm)$-positive/negative energy mode
- $\mathbf{x} = (\mathbf{x}_p, \mathbf{x}_q)$
- $k = |\mathbf{k}|$
- $K_\nu(x)$ is **the modified Bessel function**

- The momentum can be decomposed into it's uncompact and compact components.

$$\mathbf{k} = (\mathbf{k}_p, \mathbf{k}_q) \qquad \mathbf{k}_p = (k_2, \ldots, k_{p+1}) \text{ and } \mathbf{k}_q = (k_{p+2}, \ldots, k_D)$$

- The components along the compact dimensions are quantized by the periodicity conditions.

$$k_l = (2\pi n_l + \tilde{\alpha}_l)/L_l, \quad n_l = 0, \pm1, \pm2, \ldots, \text{ and } l = p+2, \ldots, D$$

The Mode functions

- Below is the normalization condition for the mode functions.

$$\int d\mathbf{x} \int_0^\infty \frac{d\xi}{\xi} \varphi^{(s)}_{\omega,\mathbf{k}} \overleftrightarrow{\partial}_\tau \varphi^{(s')*}_{\omega',\mathbf{k}'} = i\delta_{ss'}\delta\left(\omega-\omega'\right)\delta\left(\mathbf{k}_p-\mathbf{k}'_p\right)\delta_{\mathbf{k}_q\mathbf{k}'_q},$$

$$\text{where } \delta_{\mathbf{k}_q\mathbf{k}'_q} = \delta_{n_{p+2},n'_{p+2}}....\delta_{n_D,n'_D}$$

- One can use the properties of second order Bessel functions to calculate the coefficient in the mode formula.

$$\int_0^\infty \frac{dy}{y} K_{i\omega}(y) K_{i\omega'}(y) = \frac{\pi^2\delta\left(\omega-\omega'\right)}{2\omega\sinh(\omega\pi)} \quad \Rightarrow \quad |C_{\omega,\mathbf{k}}|^2 = \frac{\sinh\left(\pi\omega\right)}{(2\pi)^p\pi^2 V_q}$$

The Hadamard function

- The Hadamard function for a charged scalar field $\varphi(x)$ is defined as the VEV

$$G(x,x') = \langle 0| \varphi(x)\varphi^\dagger(x') + \varphi^\dagger(x')\varphi(x) |0\rangle$$

  - $|0\rangle$-**the vacuum state**, which in our case is the ***Fulling-Rindler vacuum***.

- Having the mode functions, the Hadamard function for the FR vacuum can be written in the form of the mode sum

$$G(x,x') = \int d\mathbf{k}_p \int_0^\infty d\omega \sum_{\mathbf{n}_q} \sum_{s=\pm} \varphi_{\mathbf{k}}^{(s)}(x)\varphi_{\mathbf{k}}^{(s)*}(x') \qquad \sum_{\mathbf{n}_q} = \sum_{n_{p+2}=-\infty}^{+\infty} \cdots \sum_{n_D=-\infty}^{+\infty}$$

$$\mathbf{n}_q = (n_{p+2}, n_{p+3}, \ldots, n_D)$$

Substituting the mode function formulas

$$G(x,x') = \frac{2^{1-p}}{\pi^{p+2}V_q} \int d\mathbf{k}_p \sum_{\mathbf{n}_q} e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \int_0^\infty d\omega \sinh(\pi\omega)\cos(\omega\Delta\tau) K_{i\omega}(\lambda_{\mathbf{k}}\xi) K_{i\omega}(\lambda_{\mathbf{k}}\xi')$$

$$\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}', \ \Delta\tau = \tau - \tau'$$

$$\lambda_{\mathbf{k}} = \sqrt{\mathbf{k}_p^2 + \mathbf{k}_q^2 + m^2}, \ \mathbf{k}_q^2 = \sum_{l=p+2}^{D} \left(\frac{2\pi n_l + \tilde{\alpha}_l}{L_l}\right)^2$$

The Hadamard function

In order to compare the effects of compactification on the FR and Minkowski vacua let us consider the Hadamard function for the Minkowski vacuum $\left|0\right\rangle_M$ in the locally Minkowski spacetime with the line element $ds^2 = \xi^2 d\tau^2 - d\xi^2 - \sum_{i=2}^{D} \left(dx^i\right)^2$ and the spatial topology $R^{p+1} \times \left(S^1\right)^q$.

- It can be shown that the **Minkowskian Hadamard function** can be presented in the form more convenient for comparison.

$$G_{\mathrm{M}}(x,x') = \frac{2^{1-p}}{\pi^{p+2}V_q} \int d\mathbf{k}_p \sum_{\mathbf{n}_q} e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \int_0^\infty d\omega \, \cosh\left(\pi\omega\right) \cos\left(\omega\Delta\tau\right) K_{i\omega}(\lambda_{\mathbf{k}}\xi) K_{i\omega}(\lambda_{\mathbf{k}}\xi')$$

The Hadamard function/Comparison

$$G(x,x') - G_{\rm M}(x,x') = -\frac{2^{1-p}}{\pi^{p+2}V_q}\int d\mathbf{k}_p \sum_{\mathbf{n}_q} e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \int_0^\infty d\omega\, e^{-\pi\omega} \times \cos(\omega\Delta\tau)\, K_{i\omega}(\lambda_{\mathbf{k}}\xi) K_{i\omega}(\lambda_{\mathbf{k}}\xi').$$

$$K_\nu(X)K_\nu(x) = \frac{1}{2}\int_0^\infty dT \int_0^\infty \frac{du}{u}\cosh(\nu T)\, e^{-(xX/u)\cosh T} \exp\left[-\left(\frac{u}{2}+\frac{x^2+X^2}{2u}\right)\right]$$

$$G(x,x') = G_{\rm M}(x,x') - \frac{2^{-p/2}}{\pi^{p/2+1}V_q}\sum_{\mathbf{n}_q}\omega_{\mathbf{n}_q}^p e^{i\mathbf{k}_q\cdot(\mathbf{x}_q-\mathbf{x}'_q)}\int_0^\infty dT \sum_{s=\pm1}\frac{1}{(T-s\Delta\tau)^2+\pi^2} \times f_{p/2}\left(\omega_{\mathbf{n}_q}\sqrt{|\mathbf{x}_p-\mathbf{x}'_p|^2+\xi^2+\xi'^2+2\xi\xi'\cosh T}\right),$$

$$f_\nu(x) = K_\nu(x)/x^\nu$$

$$\omega_{\mathbf{n}_q} = \sqrt{\mathbf{k}_q^2+m^2}$$

The Current density/General formula

Having the Hadamard function we can evaluate the expectation value for the current density (14) in the FR vacuum by using the formula

$$\langle 0| j_\mu(x) |0\rangle \equiv \langle j_\mu \rangle = \frac{i}{2} e \lim_{x' \to x} (\partial_\mu - \partial'_\mu) G(x, x')$$

First we can see that the VEVs for the charge density and for the components along the uncompact directions vanish, $\left\langle j_\mu \right\rangle = 0, \mu = 0, 1, \ldots, p+1$. Thus only the

$$\langle j^l \rangle = \langle j^l \rangle_{\mathrm{M}} - \frac{2^{-p/2} e}{\pi^{p/2+1} V_q} \sum_{\mathbf{n}_q} k_l \omega_{\mathbf{n}_q}^p \int_0^\infty dx \, \frac{f_{p/2}\left(2\xi \omega_{\mathbf{n}_q} \cosh x\right)}{x^2 + \pi^2/4} \qquad k_l = \left(2\pi n_l + \tilde{\alpha}_l\right)/L_l$$

$$\langle j^l \rangle_{\mathrm{M}} = \frac{4 e L_l^2 V_q^{-1}}{(2\pi)^{(p+3)/2}} \sum_{n_l=1}^{\infty} n_l \sin(n_l \tilde{\alpha}_l) \sum_{\mathbf{n}_{q-1}^l} \omega_{\mathbf{n}_{q-1}}^{p+3} f_{\frac{p+3}{2}}(n_l L_l \omega_{\mathbf{n}_{q-1}})$$

$$\mathbf{n}_{q-1}^l = (n_{p+2}, \ldots, n_{l-1}, , n_{l+1}, \ldots, n_D) \qquad \omega_{\mathbf{n}_{q-1}} = \sqrt{\mathbf{k}_{q-1}^2 + m^2} \qquad \mathbf{k}_{q-1}^2 = \mathbf{k}_q^2 - k_l^2$$

The Current density/General formula

- The general formula can be further simplified for a massless field, by taking into account asymptotic expressions for the second order Bessel functions.

$$f_\nu(x) \approx 2^{\nu-1}\Gamma(\nu)x^{-2\nu},\ x \ll 1$$

$$f_\nu(x) = K_\nu(x)/x^\nu$$

$$\langle j^l \rangle = \langle j^l \rangle_{\mathrm{M}} - \frac{e\Gamma((D+1)/2)}{\pi^{(D+1)/2}} L_l \sum_{\mathbf{n}_q} n_l \sin\left(\mathbf{n}_q \cdot \tilde{\boldsymbol{\alpha}}\right) \times \int_0^\infty \frac{dx}{x^2+\pi^2/4} (4\xi^2 \cosh^2 x + \sum\nolimits_{i=p+2}^{D} L_i^2 n_i^2)^{-\frac{D+1}{2}}$$

Numerical results/Asymptotes for the case of a single compact dimension

- The numerical results below will be given for the geometry with a single compact dimension x_D of the length L. In this special case one has $q = 1, p = D - 2$ and the general formulas are simplified to

$$\langle j^D \rangle = \langle j^D \rangle_{\mathrm{M}} - \frac{2^{1-D/2} e}{\pi^{D/2} L} \sum_{n=-\infty}^{+\infty} k_D \omega_D^{D-2} \int_0^\infty dx \, \frac{f_{D/2-1}\left(2\xi\omega_D \cosh x\right)}{x^2 + \pi^2/4}$$

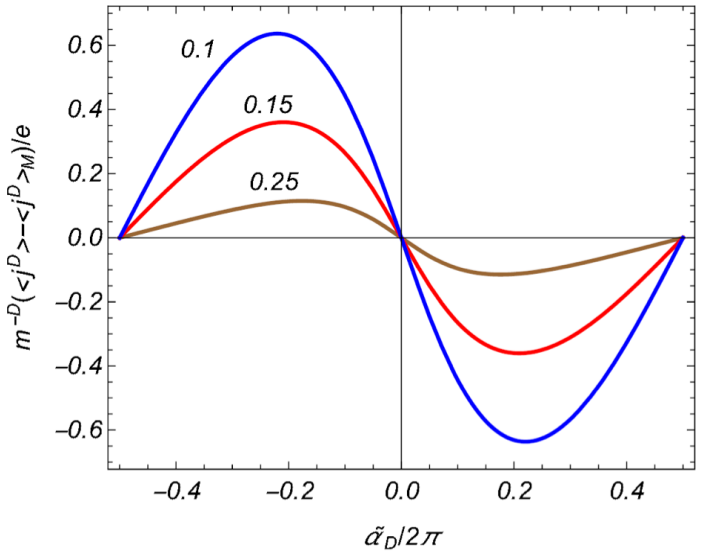
$k_D = \left(2\pi n + \tilde{\alpha}_D\right)/L$ $\qquad$ $\omega_D = \sqrt{k_D^2 + m^2}$

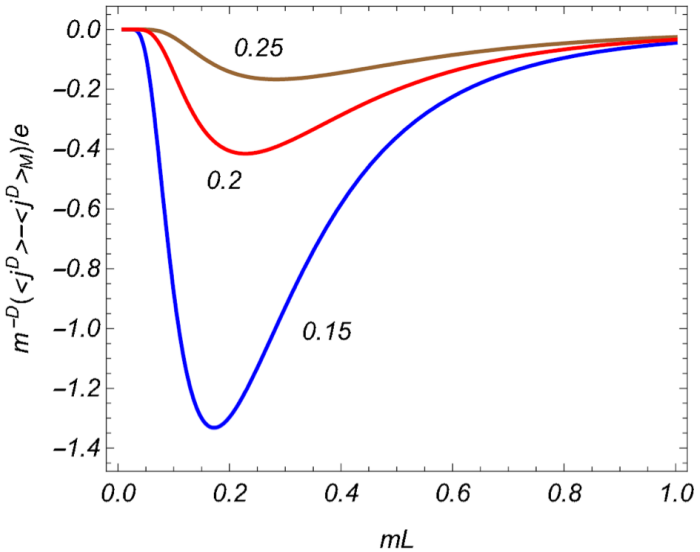
Here the Minkowskian part takes the form

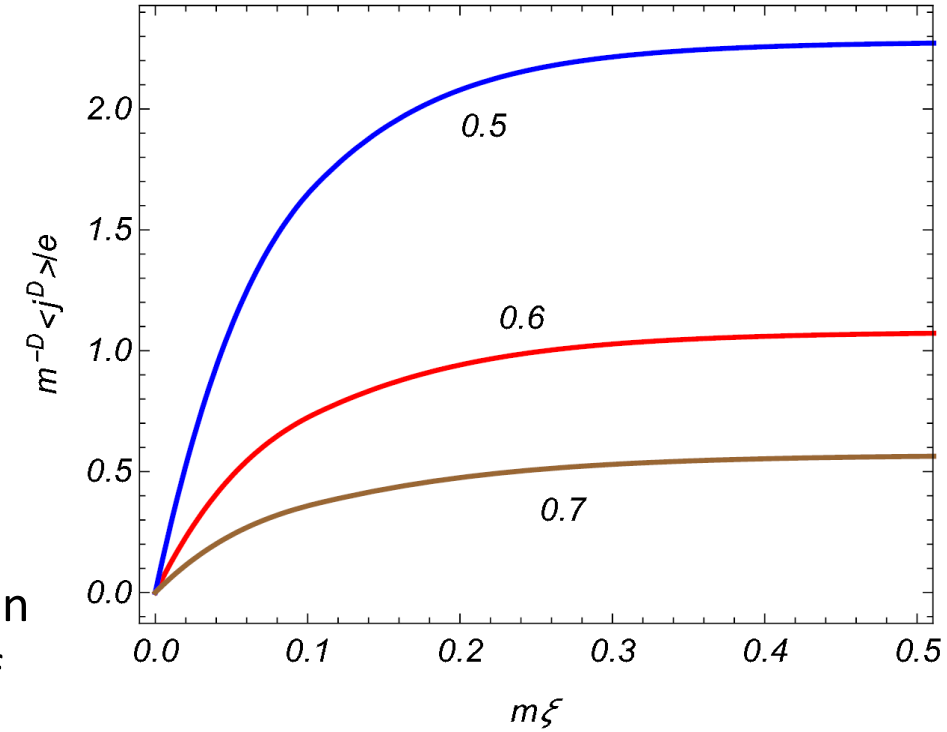
$$\langle j^D \rangle_{\mathrm{M}} = \frac{4 e m^{D+1} L}{(2\pi)^{(D+1)/2}} \sum_{n=1}^{\infty} n \sin(n\tilde{\alpha}_D) f_{\frac{D+1}{2}}\left(nmL\right)$$

Numerical results/Asymptotes for the case of a single compact dimension

In figures below the graphs are plotted for the model $D = 4$ with a single compact dimension of the length L. Figure 1 presents the dependence of the current density on m for different values of mL (the numbers near the curves) and for $\tilde{\alpha}_D/2\pi = 0.2$. In the limit $m \to \infty$ the current density tends to $\langle j_D \rangle_M$. As shown by the asymptotic analysis, the current density vanishes on the Rindler horizon $\xi = 0$.







The difference in the current densities for the FR and Minkowski vacua (in units of em^D) versus the parameter $\tilde{\alpha}_D/2\pi$ (left panel) and the length of the compact dimension (right panel). (D = 4, single compact dimension).

Near-horizon and large-distance vacuum currents around cylindrical black holes

- **Cylindrical black holes** are axially symmetric solutions of the Einstein equations with negative cosmological constant.
- They have also been studied in the context of cosmic strings, supergravity and low energy string theories.

The line element for the exterior geometry of a non-rotating and uncharged cylindrical black hole in (D + 1) dimensional spacetime:

$$ds^2_{\rm bh} = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2\sum_{i=1}^{p}\left(dy^i\right)^2 - r^2\sum_{i=1}^{q}\left(d\phi_i\right)^2 \qquad f(r) = \frac{r^2}{a^2} - \frac{M}{r^{D-2}}$$

$p + q + 1 = D,\ 0 \le \phi_i < 2\pi,\ -\infty < y^i < +\infty$

The parameters a and M are expressed in terms of negative cosmological constant Λ and mass $\mathscr{M}$ per unit volume of the subspace (y1, . . . , yp) as

$$a = \sqrt{D\frac{1-D}{2\Lambda}},\ M = \frac{16\pi G_{D+1}}{D-1}\mathcal{M}$$

G_{D+1}-the Newton gravitational constant in (D +1)-dimensional spacetime

Near-horizon and large-distance vacuum currents around cylindrical black holes

- The gravitational field is characterized by two length scales: a and $r_G = M^{\frac{1}{D-2}}$.
- For the event horizon one has $r = r_H$ with $r_H = \left(a^2 M\right)^{1/D} = r_G (a/r_G)^{2/D}$
- It can be shown that in the near-horizon limit the line element takes a locally Rindler form with the lengths of the compact dimensions determined by the radius of the horizon r_H:

$$L_l = 2\pi r_H \text{ for } l = p+2, \ldots, D$$

- From here we conclude that near the horizon the current density along the direction ϕ_i is expressed by the formulas obtained earlier with $l = i + p + 1$ and

$$\xi = \frac{2a}{\sqrt{D}} \sqrt{r/r_H - 1}$$

Near-horizon and large-distance vacuum currents around cylindrical black holes

- The asymptotic geometry at large distances from the horizon, $r \gg r_H$.

- Expanding the metric tensor and introducing a new coordinate $z = \frac{a^2}{\ }$, the line element is expressed as:

$$ds^2_{\rm bh} \approx \frac{a^2}{z^2}\left[dt^2 - dz^2 - a^2\sum_{i=1}^{p}\left(dy^i\right)^2 - a^2\sum_{i=1}^{q}\left(d\phi_i\right)^2\right]$$

- With the new rescaled coordinates:

$$ds^2_{\rm bh} \approx \frac{a^2}{z^2}\left[dt^2 - dz^2 - \sum_{i=2}^{D}\left(dx^i\right)^2\right]$$

- The expression in the right-hand side of describes a **locally AdS spacetime** with a part of the coordinates compactified to a torus.
- The values $z = 0$ and $z = \infty$ correspond to **the AdS boundary and horizon**.

Near-horizon and large-distance vacuum currents around cylindrical black holes

- In the special case under consideration with $L_l = 2\pi a$ the VEV of the current density for a scalar field is given by the expression

$$\langle j^l \rangle \approx \frac{2ea^{-D}}{(2\pi)^{(D-1)/2}} \sum_{\mathbf{n}_q} n_l \sin(\mathbf{n}_q \cdot \tilde{\boldsymbol{\alpha}})\, q_{\nu-1/2}^{(D+1)/2}(1 + 2(\pi r/a)^2 \sum_{i=p+2}^{D} n_i^2)$$

$$\nu = \sqrt{D^2/4 - D(D+1)\zeta + m^2a^2} \qquad q_\alpha^\mu(x) = (x^2-1)^{-\mu/2} e^{-i\pi\mu} Q_\alpha^\mu(x)$$

- Here, ζ is the curvature coupling parameter and $Q_\mu(x)$ is the associated Legendre function of the second kind. From $r \gg r_H$ we get the constraint on the ratio r/a in: $r/a \gg (r_G/a)^{1-2/D}$. If the length scales r_G and a are of the same order of magnitude, one has $r/a \gg 1$ and, by using the large argument asymptotic for the function $q_\alpha^\mu(x)$ we get

$$\langle j^l \rangle \approx \frac{4ea^{-D}\Gamma(\nu + D/2 + 1)}{\pi^{D/2-1}\Gamma(\nu+1)(2\pi r/a)^{D+2\nu+2}} \sum_{\mathbf{n}_q}{}' \frac{n_l \sin(\mathbf{n}_q \cdot \tilde{\boldsymbol{\alpha}})}{(n_{p+2}^2 + \cdots + n_D^2)^{\frac{D}{2}+\nu+1}}$$

Conclusion

- We have investigated the vacuum currents for a charged scalar field in Rindler spacetime with a toroidally compact subspace. We assume that the field is prepared in the FR vacuum state and obeys quasi-periodicity conditions along l^{th} compact dimension. For an external gauge field A_μ a simple configuration is considered with constant components in the compact subspace.
- Those components and the phases in the quasi-periodicity conditions appear in the expressions for the VEVs of physical observables. The latter is interpreted in terms of the magnetic flux threading the compact dimension. The complete set of scalar mode functions realizing the FR vacuum is given. The components of the momentum in the compact subspace are quantized by the periodicity conditions and the corresponding eigenvalues are given.
- We have started the investigation by evaluating the mode sum for the Hadamard function. The latter is presented in terms of the corresponding function for the Minkowski vacuum in flat space-time with spatial topology $R^{p+1} \times \left(S^1\right)^q$ and the difference in the correlations of the vacuum fluctuations in the FR and Minkowski vacua.

Conclusion

- The VEV of the current density is obtained from the Hadamard function. The charge and current densities along uncompact dimensions vanish. The current density along the lth compact dimension is presented in two equivalent forms. In both the representations the corresponding current density in the Minkowski vacuum is explicitly extracted.
- We have shown that the current density vanishes on the Rindler horizon. The large values of _ correspond to small accelerations and the difference in the current densities for the FR and Minkowski vacua is exponentially small. For small values of the length of compact dimension the difference of the current densities between the FR and Minkowski vacua is exponentially suppressed and the separate current densities along the lth compact dimension behave like $1/L_l^D$. As an application of the obtained results we have considered the near-horizon vacuum currents around cylindrical black holes. Near the horizon the exterior geometry is approximated by a Rindler-like metric.
- At large distances from the horizon the geometry of cylindrical black holes is approximated by a locally AdS spacetime with a toroidally compact subspace. The lengths of the corresponding compact dimensions are expressed in terms of the AdS curvature scale a as $L_l = 2\pi a$. The vacuum currents in that geometry have been investigated.

Thank you for your attention.