(g-2)_{\mu}: Is the end of the Standard Model nigh?

- Introduction
- SM prediction from the Muon g-2 Theory Initiative:
  - Hadronic Vacuum Polarisation & Light-by-Light contributions
- Discussion, outlook & paths to further progress
• **Muons** are like electrons, but about 200 times heavier, and they **decay**: $\mu^- \rightarrow e^- \nu_\mu \bar{\nu}_e$

• Like other matter particles, they have intrinsic angular momentum, **spin** = $\frac{1}{2}$

• As they are also **charged**, they have a magnetic moment: $\vec{\mu} = g \frac{Q_e}{2m} \vec{s}$

• The **Dirac equation** (1928) not only implied antiparticles, but also tells us that the gyromagnetic factor $g = 2$

• If put in a magnetic field, muons precess (like a spinning top)

• This **$g$-2 precession** can be **measured very precisely** (BNL & now E989 @ FNAL) and can be **calculated very precisely** (this talk)
Introduction

• 1947: small deviations from predictions in hydrogen and deuterium hyperfine structure; Kusch & Foley propose explanation with \( g = 2.00229 \pm 0.00008 \)

• 1948: Schwinger calculates the famous radiative correction:

\[
\Rightarrow g = 2 (1+a), \text{ with the anomaly}
\]

\[
a = \frac{g - 2}{2} = \frac{\alpha}{2\pi} \approx 0.001161
\]

This explained the discrepancy and was a crucial step in the development of perturbative QFT and QED

``If you can’t join ‘em, beat ‘em”

• In terms of an effective Lagrangian, the anomaly is from the Pauli term:

\[
\delta L_{\text{off}}^{\text{AMM}} = -\frac{Qe}{4m} a \bar{\psi}L \sigma^{\mu\nu} \psi R F_{\mu\nu} + (L \leftrightarrow R)
\]

Note: This is a dimension 5 operator and NOT part of the fundamental (QED) Lagrangian, but occurs through radiative corrections and is calculable in (Standard Model) theory:

\[
a^{\text{SM}}_\mu = a^{\text{QED}}_\mu + a^{\text{weak}}_\mu + a^{\text{hadronic}}_\mu
\]
\( a_e \) vs. \( a_\mu \)

\[ a_e = 1 159 \, 652 \, 180.73 \, (0.28) \times 10^{-12} \quad [0.24\text{ppb}] \]

Hanneke, Fogwell, Gabrielse, PRL 100(2008)120801

\[ a_\mu = 116 \, 592 \, 089(63) \times 10^{-11} \quad [0.54\text{ppm}] \]

Bennet et al., PRD 73(2006)072003

BNL!

One-electron quantum cyclotron

- \( a_e^{\text{EXP}} \) more than 2000 times more precise than \( a_\mu^{\text{EXP}} \), but for e\(^-\) loop contributions come from very small photon virtualities, whereas muon `tests’ higher scales

- dimensional analysis: sensitivity to NP (at high scale \( \Lambda_{\text{NP}} \)):  
  \[ a_\ell^{\text{NP}} \sim C \frac{m_\ell^2}{\Lambda_{\text{NP}}^2} \]

\( \mu \) wins by  
  \[ m_\mu^2/m_e^2 \sim 43000 \] for NP, but \( a_e \) determines \( \alpha \), tests QED & low scales

[Note: \( \tau \) too short-lived for storage-rings]
\(a_\mu\): back to the future

- CERN started it nearly 40 years ago

- Brookhaven delivered 0.5ppm precision

- E989 at FNAL and J-PARC’s g-2/EDM experiments are happening and should give us certainty

\[g-2\text{ history plot and motto from Fred Jegerlehner’s book:}

\text{‘The closer you look the more there is to see’}

\[\text{Anomalous Magnetic Moment (}\mu\text{-11659000)} \times 10^{-10}\text{]}\]
SM theory vs. Experiment (before 7.4.2021)

- If the two don’t match, something may be missing in the SM
- Precision measurements + precision theory
  ➤ discovery potential for New Physics
  ➤ need for consolidated & reliable SM prediction

\[ a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{hadronic}} + a_\mu^{\text{NP?}} \]
SM theory vs. Experiment (after FNAL on 7.4.2021)

Measurement of the Positive Muon Anomalous Magnetic Moment to 0.46 ppm

Unblinding of Run 1 analyses: 25 February ’21

FNAL confirms BNL

Release of result: 7 April ’21

As of today, PRL has 158 citations (most of them BSM)

Run 1 is only 6% of total expected statistics

But what about the Standard Model prediction?
...map out strategies for obtaining the **best theoretical predictions for these hadronic corrections** in advance of the experimental result."

- Organised 6 int. workshops in 2017-2020, (virtual) plenary workshop June 28 – July 2, 2021 hosted by KEK (Japan)
- **White Paper** posted 10 June 2020 (132 authors, from 82 institutions, in 21 countries)

``The anomalous magnetic moment of the muon in the Standard Model’’


Group photo from the Seattle workshop in September 2019
SM WP20 prediction from the TI White Paper (0.37 ppm)

\[
\begin{aligned}
\text{QED} &+ \ldots & 116584718.9 (1) \times 10^{-11} & 0.001 \text{ ppm} \\
\text{Weak} &+ \ldots & 153.6 (1.0) \times 10^{-11} & 0.01 \text{ ppm} \\
\text{Hadronic…} & & & \\
\ldots \text{Vacuum Polarization (HVP)} & & 6845 (40) \times 10^{-11} & 0.34 \text{ ppm} \\
\alpha^2 &+ \ldots & [0.6\%] & \\
\ldots \text{Light-by-Light (HLbL)} & & 92 (18) \times 10^{-11} & 0.15 \text{ ppm} \\
\alpha^3 &+ \ldots & [20\%] &
\end{aligned}
\]

➤ Uncertainty dominated by hadronic contributions, now \( \delta \text{HVP} > \delta \text{HLbL} \)
**QED**: Kinoshita et al. + many tests
- \( g-2 \) @ 1, 2, 3, 4 & 5 loops
- Subset of 12672 5-loop diagrams:
  - code-generating code, including
  - renormalisation
  - multi-dim. numerical integrations

\[
a_{\mu}^{QED} = 116\,584\,718.9\, (1) \times 10^{-11}\, \checkmark
\]

**Weak**: (several groups agree)
- done to 2-loop order, 1650 diagrams
- the first full 2-loop weak calculation

\[
a_{\mu}^{\text{weak}} = 153.6\, (1.0) \times 10^{-11}\, \checkmark
\]

SM weak 1-loop diagrams
\( a_\mu^{\text{hadronic}} \): non-perturbative, the limiting factor of the SM prediction

- **Q:** What’s in the hadronic (Vacuum Polarisation & Light-by-Light scattering) blobs?
  
  **A:** Anything `hadronic` the virtual photons couple to, i.e. quarks + gluons + photons
  
  **But:** low \( q^2 \) photons dominate loop integral(s) \( \Rightarrow \) cannot calculate blobs with perturbation theory

- **Two very different strategies:**
  
  1. **use wealth of hadronic data, `data-driven dispersive methods`:**
     
     - data combination from many experiments, radiative corrections required
  
  2. **simulate the strong interaction (+photons) w. discretised Euclidean space-time, `lattice QCD`:**
     
     - finite size, finite lattice spacing, artifacts from lattice actions, QCD + QED needed
     
     - numerical Monte Carlo methods require large computer resources
**$a_\mu^{\text{HVP}}$: WP20 Status/Summary of Hadronic VP contributions**

**HVP from:**
- LM20
- BMW20
- ETM18/19
- Mainz/CLS19
- FHM19
- PACS19
- RBC/UKQCD18
- BMW17
- RBC/UKQCD data/lattice
- BDJ19
- J17
- DHMZ19
- KNT19
- WP20

**BNL**

**Lattice QCD + QED**
- impressive progress, but...
- large spread between results
- tensions when looking at ‘Euclidean time window’ comparisons
- large systematic uncertainties (e.g. from non-trivial extrapolation to continuum limit, finite size)

**Dispersive/lattice hybrid**
(‘window’ method)

**For WP20:** Dispersive data-driven from DHMZ and KNT

TI White Paper 2020 value:

$$a_\mu^{\text{HVP}} = 6845 (40) \times 10^{-11}$$

- **TI WP20** prediction uses dispersive data-driven evaluations with minimal model dependence
- $a_\mu^{\text{HVP}}$ value and error obtained by merging procedure accounts for tensions in input data and differences in data treatment & combination (going beyond usual $\chi^2_{\text{min}}$ inflation)

![Diagram of HVP values from various sources](image)
\[ a_\mu^{HVP} : \text{Basic principles of dispersive method} \]

One-loop diagram with hadronic blob = integral over \( q^2 \) of virtual photon, 1 HVP insertion

Causality \( \Rightarrow \) analyticity \( \Rightarrow \) dispersion integral:

obtain HVP from its imaginary part only

Unitarity \( \Rightarrow \) Optical Theorem:

imaginary part (‘cut diagram’) = sum over \(|\text{cut diagram}|^2\), i.e.

\( \propto \) sum over all total hadronic cross sections

\[ a_\mu^{\text{had,LO}} = \frac{m_\mu^2}{12\pi^3} \int_{s_{\text{th}}}^{\infty} ds \frac{1}{s} \hat{K}(s)\sigma_{\text{had}}(s) \]

- Total hadronic cross section \( \sigma_{\text{had}} \) from >100 data sets for \( e^+e^- \rightarrow \text{hadrons} \) in >35 final states
- Uncertainty of \( a_\mu^{HVP} \) prediction from statistical & systematic uncertainties of input data
- Pert. QCD used only at large \( s \), \textbf{no modelling} of \( \sigma_{\text{had}}(s) \) \textbf{required}, direct data integration
\( a_{\mu}^{\text{HLbL}} : \) Hadronic Light-by-Light: Dispersive approach

For HVP \( \Rightarrow \quad 2 \, \text{Im} \sum_{\text{had.}} \int d\Phi \left| \frac{s}{4\pi\alpha} \right|^2 \quad \Rightarrow \quad \text{Im} \Pi_{\text{had}}(s) = \left( \frac{s}{4\pi\alpha} \right) \sigma_{\text{had}}(s) \)

For HLbL \( \Rightarrow \quad \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\text{pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma}^{\bar{\text{box}}} + \ldots \)

\( \Rightarrow \quad \text{Dominated by pole (pseudoscalar exchange) contributions} \)

\( \Pi_{\mu\nu\lambda\sigma}^{\text{pole}} = \ldots \)

\( \Rightarrow \quad \text{Sum all possible diagrams to get} \ a_{\mu}^{\text{HLbL}} \)

- See also review by Danilkin+Redmer+Vanderhaeghen using dispersive techniques estimates \((8.7 \pm 1.3) \times 10^{-10}\) [Prog. Part. Nucl. Phys. 107 (2019) 20]

- With new results & progress, L-by-L can now be reliably predicted! √
\[ a_\mu^{HLbL} : \text{WP Status/Summary of Hadronic Light-by-Light contributions} \]

- data-driven dispersive & lattice results have confirmed the earlier model-based predictions
- uncertainty much better under control and at 0.15 ppm already sub-leading compared to HVP
- lattice predictions now competitive, good prospects for combination and error reduction to \( \leq 10\% \)

\[ a_\mu^{HLbL} = 92 (18) \times 10^{-11} \checkmark \]
\( a_\mu^{\text{HVP}}: \) Higher orders & QED power counting; WP20 values in 10\(^{-11}\)

- All hadronic blobs also contain photons, i.e. real + virtual corrections in \( \sigma_{\text{had}}(s) \)
  - LO: 6931(40)
  - NLO: -98.3(7)
    - from three classes of graphs:
      - -207.7(7) + 105.9(4) + 3.4(1) [KNT19]
        (photonic, extra e-loop, 2 h-loops)
  - NNLO: 12.4(1) [Kurz et al, PLB 734(2014)144, see also F Jegerlehner]
    - from five classes of graphs:
      - 8.0 - 4.1 + 9.1 - 0.6 + 0.005

\[ \Rightarrow \text{good convergence, iterations of hadronic blobs very small} \]
HVP disp.: cross section (in terms of R-ratio) input

\[ a_{\mu}^{\text{had, LO VP}} = \frac{\alpha^2}{3\pi^2} \int_{s_{\text{th}}}^{\infty} \frac{ds}{s} R(s) K(s), \text{ where } R(s) = \frac{\sigma_{\text{had, } \gamma}(s)}{4\pi \alpha^2/3s} \]

\[ \frac{\rho}{\omega}, J/\psi, \psi(2s), \Upsilon(1s-6s) \]

**Non-perturbative** (Experimental data, isopsin, ChPT...)

**Non-perturbative/perturbative**

**Perturbative** (pQCD)

**Must build full hadronic cross section/R-ratio...**
$\alpha_{\mu}^{\text{HVP}}$: Recent (of 25+ years) experiments providing input $\sigma_{\text{had}}(s)$ data

S. Serednyakov (for SND) @ HVP KEK workshop

- Different methods: `Direct Scan` (tunable $e^+e^-$ beams) & `Radiative Return` (Initial State Radiation scan at fixed cm energy)
- Over last decades detailed studies of radiative corrections & Monte Carlo Generators for $\sigma_{\text{had}}(s)$
HVP dispersive: cross section compilation

How to get the most precise $\sigma^0_{\text{had}}$? Use of $\text{e}^+\text{e}^- \rightarrow \text{hadrons (+\gamma)}$ data:

- **Low energies:** sum $\sim 35$ exclusive channels, $2\pi$, $3\pi$, $4\pi$, $5\pi$, $6\pi$, KK, KK$\pi$, KK$\pi\pi$, $\eta\pi$, ..., [now very limited use iso-spin relations for missing channels]

- **Above $\sim 1.8$ GeV:** use of inclusive data or pQCD (away from flavour thresholds), supplemented by narrow resonances ($J/\Psi$, $\Upsilon$)

- **Challenge of data combination** (locally in $\sqrt{s}$, with error inflation if tensions):
  - many experiments, different energy ranges and bins,
  - statistical + systematic errors from many different sources,
  - use of correlations; must avoid inconsistencies, bias

  ➤ Significant differences between DHMZ and KNT in use of correlated errors:
    - KNT allow non-local correlations to influence mean values,
    - DHMZ restrict this but retain correlations for errors and also betw. channels

- $\sigma^0_{\text{had}}$ means the `bare' cross section, i.e. excluding `running coupling' (VP) effects, 
  but including Final State ($\gamma$) Radiation: data subject to Radiative Corrections
Table from KNT18, PRD 97(2018)114025

**Update: KNT19**

LO+NLO HVP for $a_{e,\mu,\tau}$ & hyperfine splitting of muonium

PRD101(2020)014029

**Breakdown of HVP contributions in ~35 hadronic channels**

From 2-11 GeV, use of inclusive data, pQCD only beyond 11 GeV

<table>
<thead>
<tr>
<th>Channel</th>
<th>Energy range [GeV]</th>
<th>$\alpha_\mu^{\text{had,LO+VP}} \times 10^{10}$</th>
<th>$\Delta a_\mu^{(5)}(M_\mu^2) \times 10^4$</th>
<th>New data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0\gamma$</td>
<td>$m_\pi \leq \sqrt{s} \leq 0.600$</td>
<td>0.12 ± 0.01</td>
<td>0.00 ± 0.00</td>
<td>...</td>
</tr>
<tr>
<td>$\pi^+\pi^-$</td>
<td>$2m_\pi \leq \sqrt{s} \leq 0.305$</td>
<td>0.87 ± 0.02</td>
<td>0.01 ± 0.00</td>
<td>...</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>$3m_\pi \leq \sqrt{s} \leq 0.660$</td>
<td>0.01 ± 0.00</td>
<td>0.00 ± 0.00</td>
<td>...</td>
</tr>
<tr>
<td>$\eta\gamma$</td>
<td>$m_\eta \leq \sqrt{s} \leq 0.660$</td>
<td>0.00 ± 0.00</td>
<td>0.00 ± 0.00</td>
<td>...</td>
</tr>
<tr>
<td><strong>Data based channels ($\sqrt{s} \leq 1.937$ GeV)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi^0\gamma$</td>
<td>$0.600 \leq \sqrt{s} \leq 1.350$</td>
<td>4.46 ± 0.10</td>
<td>0.36 ± 0.01</td>
<td>[65]</td>
</tr>
<tr>
<td>$\pi^+\pi^-$</td>
<td>$0.305 \leq \sqrt{s} \leq 1.937$</td>
<td>502.97 ± 1.97</td>
<td>34.26 ± 0.12</td>
<td>[34,35]</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>$0.660 \leq \sqrt{s} \leq 1.937$</td>
<td>47.79 ± 0.89</td>
<td>4.77 ± 0.08</td>
<td>[36]</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>$0.613 \leq \sqrt{s} \leq 1.937$</td>
<td>14.87 ± 0.20</td>
<td>4.02 ± 0.05</td>
<td>[40,42]</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>$0.850 \leq \sqrt{s} \leq 1.937$</td>
<td>19.39 ± 0.78</td>
<td>5.00 ± 0.20</td>
<td>[44]</td>
</tr>
<tr>
<td>$(\pi^+\pi^-\pi^0)_{\text{noq}}$</td>
<td>$1.013 \leq \sqrt{s} \leq 1.937$</td>
<td>0.99 ± 0.09</td>
<td>0.33 ± 0.03</td>
<td>...</td>
</tr>
<tr>
<td>$K^+K^-$</td>
<td>$1.313 \leq \sqrt{s} \leq 1.937$</td>
<td>0.23 ± 0.01</td>
<td>0.00 ± 0.00</td>
<td>[66]</td>
</tr>
<tr>
<td>$(\pi^+\pi^-\pi^0)_{\text{noqq}}$</td>
<td>$1.322 \leq \sqrt{s} \leq 1.937$</td>
<td>1.35 ± 0.17</td>
<td>0.51 ± 0.06</td>
<td>...</td>
</tr>
<tr>
<td>$KK$</td>
<td>$0.988 \leq \sqrt{s} \leq 1.937$</td>
<td>23.03 ± 0.22</td>
<td>3.37 ± 0.03</td>
<td>[45,46,49]</td>
</tr>
<tr>
<td>$KK_L$</td>
<td>$1.004 \leq \sqrt{s} \leq 1.937$</td>
<td>13.04 ± 0.19</td>
<td>1.77 ± 0.03</td>
<td>[50,51]</td>
</tr>
<tr>
<td>$KK\pi$</td>
<td>$1.260 \leq \sqrt{s} \leq 1.937$</td>
<td>2.71 ± 0.12</td>
<td>0.89 ± 0.04</td>
<td>[53,54]</td>
</tr>
<tr>
<td>$KK2\pi$</td>
<td>$1.350 \leq \sqrt{s} \leq 1.937$</td>
<td>1.93 ± 0.08</td>
<td>0.75 ± 0.03</td>
<td>[50,53,55]</td>
</tr>
<tr>
<td>$\eta\gamma$</td>
<td>$0.660 \leq \sqrt{s} \leq 1.760$</td>
<td>0.70 ± 0.02</td>
<td>0.15 ± 0.00</td>
<td>[67]</td>
</tr>
<tr>
<td>$\eta\pi^+$</td>
<td>$0.101 \leq \sqrt{s} \leq 1.937$</td>
<td>1.29 ± 0.06</td>
<td>0.39 ± 0.02</td>
<td>[68,69]</td>
</tr>
<tr>
<td>$(\eta\pi^-\pi^0)_{\text{noqq}}$</td>
<td>$1.333 \leq \sqrt{s} \leq 1.937$</td>
<td>0.60 ± 0.15</td>
<td>0.21 ± 0.05</td>
<td>[70]</td>
</tr>
<tr>
<td>$\eta\pi^0\pi^0$</td>
<td>$1.338 \leq \sqrt{s} \leq 1.937$</td>
<td>0.08 ± 0.01</td>
<td>0.03 ± 0.00</td>
<td>...</td>
</tr>
<tr>
<td>$\eta\phi$</td>
<td>$3.333 \leq \sqrt{s} \leq 1.937$</td>
<td>0.31 ± 0.03</td>
<td>0.10 ± 0.01</td>
<td>[70,71]</td>
</tr>
<tr>
<td>$\omega \rightarrow \pi^0\gamma\pi^0$</td>
<td>$0.920 \leq \sqrt{s} \leq 1.937$</td>
<td>0.88 ± 0.02</td>
<td>0.19 ± 0.00</td>
<td>[72,73]</td>
</tr>
<tr>
<td>$\eta\phi$</td>
<td>$1.569 \leq \sqrt{s} \leq 1.937$</td>
<td>0.42 ± 0.03</td>
<td>0.15 ± 0.01</td>
<td>...</td>
</tr>
<tr>
<td>$\phi \rightarrow \text{un accounted}$</td>
<td>$0.988 \leq \sqrt{s} \leq 1.029$</td>
<td>0.04 ± 0.04</td>
<td>0.01 ± 0.01</td>
<td>...</td>
</tr>
<tr>
<td>$\eta\phi\pi^0$</td>
<td>$1.550 \leq \sqrt{s} \leq 1.937$</td>
<td>0.35 ± 0.09</td>
<td>0.14 ± 0.04</td>
<td>[74]</td>
</tr>
<tr>
<td>$\eta \rightarrow \text{nn}p\bar{K}_{\text{noqq}} \rightarrow KK$</td>
<td>$1.569 \leq \sqrt{s} \leq 1.937$</td>
<td>0.01 ± 0.02</td>
<td>0.00 ± 0.01</td>
<td>[53,75]</td>
</tr>
<tr>
<td>$p\bar{p}$</td>
<td>$1.890 \leq \sqrt{s} \leq 1.937$</td>
<td>0.03 ± 0.00</td>
<td>0.01 ± 0.00</td>
<td>[76]</td>
</tr>
<tr>
<td>$n\bar{n}$</td>
<td>$1.912 \leq \sqrt{s} \leq 1.937$</td>
<td>0.03 ± 0.01</td>
<td>0.01 ± 0.00</td>
<td>[77]</td>
</tr>
</tbody>
</table>

| Estimated contributions ($\sqrt{s} \leq 1.937$ GeV) | | | | |
| $(\pi^+\pi^-\pi^0)_{\text{noqq}}$ | $1.013 \leq \sqrt{s} \leq 1.937$ | 0.50 ± 0.04 | 0.16 ± 0.01 | ... |
| $(\pi^+\pi^-\pi^0)_{\text{noqq}}$ | $1.313 \leq \sqrt{s} \leq 1.937$ | 0.21 ± 0.21 | 0.08 ± 0.08 | ... |
| $KK\pi\pi$ | $1.569 \leq \sqrt{s} \leq 1.937$ | 0.03 ± 0.02 | 0.02 ± 0.01 | ... |
| $\omega \rightarrow \text{nn}p2\pi$ | $1.285 \leq \sqrt{s} \leq 1.937$ | 0.10 ± 0.02 | 0.03 ± 0.01 | ... |
| $\omega \rightarrow \text{nn}p3\pi$ | $1.322 \leq \sqrt{s} \leq 1.937$ | 0.17 ± 0.03 | 0.06 ± 0.01 | ... |
| $\omega \rightarrow \text{nn}pKK$ | $1.569 \leq \sqrt{s} \leq 1.937$ | 0.00 ± 0.00 | 0.00 ± 0.00 | ... |
| $\eta\phi\pi^0\pi^0$ | $1.338 \leq \sqrt{s} \leq 1.937$ | 0.08 ± 0.04 | 0.03 ± 0.02 | ... |

| Other contributions ($\sqrt{s} > 1.937$ GeV) | | | | |
| **Inclusive channel** | $1.937 \leq \sqrt{s} \leq 11.199$ | 43.67 ± 0.67 | 82.82 ± 1.05 | [56,62,63] |
| $J/\psi$ | | 6.26 ± 0.19 | 7.07 ± 0.22 | ... |
| $\psi'$ | | 1.58 ± 0.04 | 2.51 ± 0.06 | ... |
| $\Upsilon(1S-4S)$ | | 0.09 ± 0.00 | 1.06 ± 0.02 | ... |
| pQCD | $11.199 \leq \sqrt{s} \leq \infty$ | 2.07 ± 0.00 | 124.79 ± 0.10 | ... |
| **Total** | $m_\pi \leq \sqrt{s} \leq \infty$ | 693.26 ± 2.46 | 276.11 ± 1.11 | ... |
$a_{\mu}^{\text{HVP}}$: Landscape of $\sigma_{\text{had}}(s)$ data & most important $\pi^+\pi^-$ channel

- Combination of >30 data sets, >1000 points, contributing >70% of total HVP
- Precise measurements from 6 independent experiments with different systematics and different radiative corrections
- Data sets from Radiative Return dominate
- Some tension in data accounted for by local $\chi^2_{\text{min}}$ inflation and via WP merging procedure

$\pi^+\pi^-$: 
- Hadronic channels for energies below 2 GeV
- Dominance of $2\pi$

$\sqrt{s}$ [GeV]
• Tension between different sets, especially between the most precise 4 sets from BaBar and KLOE

• Inflation of error with local $\chi^2_{\text{min}}$ accounts for tensions, leading to a $\sim 15\%$ error inflation

• Important role of correlations; their treatment in the data combination is crucial and can lead to significant differences between different combination methods

- In addition they employ a fit, based on analyticity + unitarity + crossing symmetry, similar to Colangelo et al. and Ananthanarayan+Caprini+Das, leading to stronger constraints/lower errors at low energies.

- For $2\pi$, based on difference between result for $a_\mu^{\pi\pi}$ w/out KLOE and BaBar, sizeable additional systematic error is applied and mean value adjusted.

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**Figure 5:**

![Diagram showing SM predictions for $a_\mu^{\pi\pi}$, 0.6 - 0.9 GeV, with error bars for different experiments: CLEO, SND, BESIII, CMD-2, BABAR, and KLOE.]  

**Figure 6:**

![Graph showing relative difference to fit (all data) with comparison to fit (without KLOE) and fit (without BABAR) for $\sqrt{s}$ in GeV]
**HVP: KK channels**  [KNT18, PRD97, 114025]

### $K^+ K^-$

- **New data:**
  - **BaBar:** [Phys. Rev. D 88 (2013), 032013.]
  - **SND:** [Phys. Rev. D 94 (2016), 112006.]
  - **CMD-3:** [arXiv:1710.02989.]


- \( a_{K^+ K^-} = 23.03 \pm 0.22_{\text{tot}} \)

- **HLMNT11:** 22.15 ± 0.46_{\text{tot}}

- Large increase in mean value

### $K_S^0 K_L^0$

- **New data:**
  - **BaBar:** [Phys. Rev. D 89 (2014), 092002.]

- **Note:**

- \( a_{K_S^0 K_L^0} = 13.04 \pm 0.19_{\text{tot}} \)

- **HLMNT11:** 13.33 ± 0.16_{\text{tot}}

- Large changes due to new precise measurements on $\phi$

KEDR data improves the inclusive data combination below $c\bar{c}$ threshold.

$R_b$ resolves the resonances of the $\Upsilon(5S - 6S)$ states.

Choose to adopt entirely data driven estimate from threshold to 11.2 GeV

$$a_{\mu}^{\text{Inclusive}} = 43.67 \pm 0.17_{\text{stat}} \pm 0.48_{\text{sys}} \pm 0.01_{\text{vp}} \pm 0.44_{\text{fsr}} = 43.67 \pm 0.67_{\text{tot}}$$
History plot of $a_{\mu}^{\text{HVP}}$ w. min. model dep. Pies.

- Stability and consolidation over two decades thanks to more and better data input and improved compilation procedures

- Compare with `merged’ DHMZ & KNT WP20 value:

$$a_{\mu}^{\text{had, LO VP}}(\text{WP20}) = 693.1(4.0) \times 10^{-10}$$

Pie diagrams [KNT]:

- error still dominated by two pion channel

- significant contribution to error from additional uncertainty from radiative corrections
Muon $g-2$ SM prediction from the TI WP vs. FNAL

This is experiment vs. theory with the new FNAL $g-2$ Run-1 result announced 7th April

SM prediction:
$$a_{\mu}^{SM} = 116591810(43) \times 10^{-11}$$

FNAL E989 (2021):
$$a_{\mu}^{E989} = 116592040(54) \times 10^{-11}$$

Combined with BNL E821 (2004):
$$a_{\mu}^{exp} = 116592061(41) \times 10^{-11}$$

$$a_{\mu}^{SM} - a_{\mu}^{exp} = 251(59) \times 10^{-10} \quad (4.2 \sigma)$$

HVP from:
BMW20
WP20(lattice)
J17
not used in WP20

HVP from:
DHMZ19
KNT19
WP20

-40 -30 -20 -10 0 10

$$(a_{\mu}^{SM} - a_{\mu}^{exp}) \times 10^{10}$$

Projected final Fermilab uncertainty

4.2 $\sigma$
HVP from electron-muon scattering in the space-like

\[ a^\text{HLO}_\mu = \frac{\alpha}{\pi} \int_0^1 dx \, (1 - x) \, \Delta \alpha_{\text{had}}[t(x)] \]

\[ t(x) = \frac{x^2 m^2_\mu}{x - 1} < 0 \]

\( \Delta \alpha_{\text{had}}(t) \) is the hadronic contribution to the running of \( \alpha \) in the space-like region. It can be extracted from scattering data!

- use CERN M2 muon beam (150 GeV)
- Physics beyond colliders program @ CERN
- LOI June 2019
- Jan 2020: SPSC recommends pilot run in 2021
- goal: run with full apparatus in 2023-2024
• The still **unresolved muon $g-2$ discrepancy** has triggered new experiments and a lot of theory activities, including and helped by the Muon $g-2$ Theory Initiative

• **Much progress** has been made for **HLbL** which previously was seen as the bottleneck. **New data driven dispersive approaches & lattice** have confirmed earlier model estimates and now allow a **reliable error estimate**, and **more work is in progress**

• For **HVP dispersive**, the **TI published a conservative & robust consensus**. Soon **new hadronic data for $2\pi$** will come from **BaBar, CMD-3, BESIII and Belle-II**

• Longer term: direct HVP measurement planned with $e-\mu$ scattering: **MUonE** at **CERN**

• **Lattice** has started to deliver impressive results with **high precision**. **Further work is needed** and ongoing to scrutinize, check and improve different approaches, and lattice is expected to play an important role in the future

• The **Muon $g-2$ Theory Initiative** will continue to facilitate this work and to publish **agreed & conservative SM predictions** for $g-2$ prior to new experimental results

• With the **WP20 SM** prediction and the new first $g-2$ result from **FNAL**, the discrepancy stands at $4.2\sigma$ and is more intriguing than ever.
Lattice HVP: Tension betw. BMW & data-driven. Systematics

**BNL-E821**

- **HVP from:**
  - BMW20
  - WP20(lattice)
  - DHMZ19
  - KNT19
  - WP20 (not used in WP20)

- **Projected final Fermilab uncertainty**

- **3.7σ**

**BMW20**

- Large systematics from *continuum limit*,
  - Large taste-breaking corrections ('SRHO')

- **Upper right panel:** Limit and uncertainty estimation

- **Lower right panel:** Limit for central 'window' compared to other lattice and data-driven results

---


- **Graphs**
  - V1, V3
  - SRHO(>0.4fm), SRHO(>1.3fm)
  - SRHO(0.4-1.3fm)+NNLO(>1.3fm)
  - None

- **Equation:**
  \[
  (a_\mu^{\text{SM}} - a_\mu^{\text{exp}}) \times 10^{10}
  \]
Lattice HVP: Cross checks, window method (I)

\[ a_{\mu}^{\text{HVP,LO}} = \left( \frac{\alpha}{\pi} \right)^2 \int_0^{\infty} dt \, \tilde{w}(t) \, C(t) \]

- Use windows in Euclidean time to consider the different time regions separately.
  
  Short Distance (SD) \( t : 0 \rightarrow t_0 \)
  
  Intermediate (W) \( t : t_0 \rightarrow t_1 \)
  
  Long Distance (LD) \( t : t_1 \rightarrow \infty \)

- Compute each window separately (in continuum, infinite volume limits,…) and combine

\[ a_{\mu} = a_{\mu}^{\text{SD}} + a_{\mu}^{\text{W}} + a_{\mu}^{\text{LD}} \]
Lattice HVP: Cross checks, window method (II)

H. Wittig @ Lattice HVP workshop

\[ a_\mu = a_\mu^{SD} + a_\mu^W + a_\mu^{LD} \]
\[ t_0 = 0.4 \text{ fm}, \ t_1 = 1.0 \text{ fm} \]
\[ \Delta = 0.15 \text{ fm} \]

"Window" quantities

\((t_0, \Delta) = (0.4, 0.15) \text{ fm}\)

- Aubin et al. 19
- Aubin et al. 19 - finest :s
- LM 20
- BMW 20
- FHM 20 (prelim., stat only)
- RBC/UKQCD 18
- ETMC 20 (prelim.)
- Mainz/CLS 20 (prelim.)
- R-ratio & lattice

\[(t_0, \Delta) = (0.4, 0.15) \text{ fm}\]

- FHM 20 (prelim., stat. only)
- RBC/UKQCD 20 (prelim., stat. only)
- ETMC 20 (prelim.)
- Mainz/CLS 20 (prelim.)

\[(t, \Delta) = (1.0, 0.15) \text{ fm}\]

- FHM 20 (prelim., stat. only)
- ETMC 20 (prelim.)
- Mainz/CLS 20 (prelim.)

- \(a_\mu^W (ud, \text{ conn, iso}) \times 10^{10}\)
- \(a_\mu^{SD} (ud, \text{ conn, iso}) \times 10^{10}\)
- \(a_\mu^{LD} (ud, \text{ conn, iso}) \times 10^{10}\)

- Straightforward reference quantities
- Can be applied to individual contributions (light, strange, charm, disconnected, ...)
- Large discrepancies between different results, also with data-driven: **BMW vs KNT: 3.7\sigma**
- Individual results must sum up, and different groups & discretisations must agree (universality)
HVP: White Paper comparison & merging procedure

Detailed comparisons by-channel and energy range between direct integration results:

<table>
<thead>
<tr>
<th>Channel</th>
<th>DHMZ19</th>
<th>KNT19</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+\pi^-$</td>
<td>507.85(0.83)</td>
<td>504.23(1.90)</td>
<td>3.62</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>46.21(0.40)</td>
<td>46.63(94)</td>
<td>-0.42</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^+\pi^-$</td>
<td>13.68(0.03)</td>
<td>13.99(19)</td>
<td>-0.31</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0\pi^0$</td>
<td>18.03(0.06)</td>
<td>18.15(74)</td>
<td>-0.12</td>
</tr>
<tr>
<td>$K^+K^-$</td>
<td>23.08(0.20)</td>
<td>23.00(22)</td>
<td>0.08</td>
</tr>
<tr>
<td>$K_SK_L$</td>
<td>12.82(0.06)</td>
<td>13.04(19)</td>
<td>-0.22</td>
</tr>
<tr>
<td>$\pi^0\gamma$</td>
<td>4.41(0.06)</td>
<td>4.58(10)</td>
<td>-0.17</td>
</tr>
</tbody>
</table>

Sum of the above: 626.08(0.95)(3.48)(1.47) 623.62(2.27) 2.46

- [1.8, 3.7] GeV (without $c\bar{c}$) 33.45(71) 34.45(56) -1.00
  - $J/\psi, \psi(2S)$ 7.76(12) 7.84(19) -0.08
  - [3.7, $\infty$] GeV 17.15(31) 16.95(19) 0.20

Total $a_H^{HVP, LO}$: 694.0(1.0)(3.5)(1.6)(0.1)${}_\psi$(0.7)${}_{DV+QCD}$ 692.8(2.4) 1.2

+ evaluations using unitarity & analyticity constraints for $\pi\pi$ and $\pi\pi\pi$ channels

[CHS 2018, HHKS 2019]
Conservative merging procedure developed during 2019 Seattle TI workshop:

- Accounts for the different results obtained by different groups based on the same or similar experimental input
- Includes correlations and their different treatment as much as possible
- Allows to give one recommended (merged) result, which is conservative w.r.t. the underlying (and possibly underestimated) uncertainties
- Note: Merging leads to a bigger error estimate compared to individual evaluations

\[ a_{\mu}^{\text{HVP, LO}} = 693.1 (4.0) \times 10^{-10} \] is the result used in the WP `SM2020' value

- This result does not include lattice, but is compatible with published lattice results apart from the BMW prediction:
\[ a_{\mu}^{\text{HVP, LO (BMW)}} = 707.5 (5.5) \times 10^{-10} \quad \text{[Nature]} \]

Efforts are ongoing in the community to check their result, with a topical online workshop from the g-2 Theory Initiative in November 2020 shedding first light.
Theory vs. Experiment: sensitivity chart

Plot from Fred Jegerlehner

Need to control the hadronic contributions

\[ a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{hadronic}} + a_\mu^{\text{NP}}? \]
Historically, hadronic tau decay data, e.g. \( \tau^- \rightarrow \pi^0 \pi^- \nu_\tau \), were used to improve precision of \( e^+e^- \) based evaluations.

However, with the increased precision of the \( e^+e^- \) data there is now limited merit in this (DHMZ have dropped it), and

The required iso-spin breaking corrections re-introduce a model-dependence and connected systematic uncertainty (there is, e.g., no \( \rho-\omega \) mixing in \( \tau \) decays).

Quote from the WP, where this approach is discussed in detail:

"Concluding this part, it appears that, at the required precision to match the \( e^+e^- \) data, the present understanding of the IB corrections to \( \tau \) data is unfortunately not yet at a level allowing their use for the HVP dispersion integrals. It remains a possibility, however, that the alternate lattice approach, discussed in Sec. 3.4.2, may provide a solution to this problem.”
Rad. Corrs.: HVP for running $\alpha(q^2)$. Undressing

- Dyson summation of Real part of one-particle irreducible blobs $\Pi$ into the effective, real running coupling $\alpha_{\text{QED}}$:

$$\Pi = \begin{array}{c}
\gamma^\ast_q \\
\gamma^\ast_q \\
\gamma^\ast_q \\
\end{array}$$

Full photon propagator $\sim 1 + \Pi + \Pi \cdot \Pi + \Pi \cdot \Pi \cdot \Pi + \ldots$

$$\leadsto \alpha(q^2) = \frac{\alpha}{1 - \text{Re} \Pi(q^2)} = \alpha \left(1 - \Delta \alpha_{\text{lep}}(q^2) - \Delta \alpha_{\text{had}}(q^2)\right)$$

- The Real part of the VP, $\text{Re} \Pi$, is obtained from the Imaginary part, which via the Optical Theorem is directly related to the cross section, $\text{Im} \Pi \sim \sigma(e^+e^- \rightarrow \text{hadrons})$:

$$\Delta \alpha_{\text{had}}^{(5)}(q^2) = -\frac{q^2}{4\pi^2\alpha} \mathcal{P} \int_{m^2}^{\infty} \frac{\sigma_{\text{had}}^{0}(s) \, ds}{s - q^2}, \quad \sigma_{\text{had}}(s) = \frac{\sigma_{\text{had}}^{0}(s)}{|1 - \Pi|^2}$$

$$\leadsto \sigma^0 \text{ requires ‘undressing’, e.g. via } \left(\alpha/\alpha(s)\right)^2 \leadsto \text{iteration needed}$$

- Observable cross sections $\sigma_{\text{had}}$ contain the $|\text{full photon propagator}|^2$, i.e. $|\text{infinite sum}|^2$.

$$\leadsto$$ To include the subleading Imaginary part, use dressing factor $\frac{1}{|1 - \Pi|^2}$. 


Rad. Corrs.: HVP for running $\alpha(q^2)$. Undressing

- $\Delta \alpha(q^2)$ in the time-like: HLMNT compared to Fred Jegerlehner’s new routines

$\Delta \alpha(q^2)$

For demonstration only, results >10 years old!

Different groups use their own HVP routines:
- Fred Jegerlehner,
- DHMZ,
- KNT,
- Novosibirsk (Fedor Ignatov)

$\Delta \alpha_{\text{had}}(s)/\alpha$

$\sqrt{s}$ (GeV)

$\rightarrow$ with new version big differences (with 2003 version) gone

- smaller differences remain and reflect different choices, smoothing etc.
HVP from electron muon scattering in the space-like

C. Carloni @ g-2 INT workshop [A. Abbiendi et al, arXiv:1609.08987, EPJC 2017]

Time-like

Space-like

- requires calculations of radiative corrections [M. Fael @ g-2 INT workshop]
- complement region not accessible to experiment with LQCD calculation [M. Marinkovic @ g-2 INT workshop]
Rad. Corrs.: Final State $\gamma$ Radiation

- Real + virtual, **must be included** in $\sigma^0_{\text{had}}$ as part of the hadronic dynamics,

- but some events with real radiation will have been cut-off by experimental analyses (no problem if $\gamma$ just missed but event counted. Possible problem of mis-identifies)

- Experiments (or compilations) account for this and add FSR back;
  - based on MC and scalar QED for pions (detailed studies, checked to work well)
  - contributes to systematic uncertainties
  - intrinsic part of Radiative Return analyses of many recent data sets

- Notes:
  - at low energies and at resonances, hard radiation is limited by phase space
  - different compilations apply **additional uncertainty** to cover possible problems of the FSR (& VP/undressing) treatment, e.g.

➤ KNT: $\delta a_\mu^\text{had, FSR} = 7.0 \times 10^{-11}$, and also $\delta a_\mu^\text{had, VP} = 2.1 \times 10^{-11}$

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Value $\times 10^{11}$</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experiment (E821)</td>
<td>116 592 089(63)</td>
<td>Ref. [1]</td>
</tr>
<tr>
<td>HVP LO ($e^+e^-$)</td>
<td>6931(40)</td>
<td>Refs. [2–7]</td>
</tr>
<tr>
<td>HVP NLO ($e^+e^-$)</td>
<td>−98.3(7)</td>
<td>Ref. [7]</td>
</tr>
<tr>
<td>HVP NNLO ($e^+e^-$)</td>
<td>12.4(1)</td>
<td>Ref. [8]</td>
</tr>
<tr>
<td>HVP LO (lattice, $udsc$)</td>
<td>7116(184)</td>
<td>Refs. [9–17]</td>
</tr>
<tr>
<td>HLVbL (phenomenology)</td>
<td>92(19)</td>
<td>Refs. [18–30]</td>
</tr>
<tr>
<td>HLVbL NLO (phenomenology)</td>
<td>2(1)</td>
<td>Ref. [31]</td>
</tr>
<tr>
<td>HLVbL (lattice, $uds$)</td>
<td>79(35)</td>
<td>Ref. [32]</td>
</tr>
<tr>
<td>HLVbL (phenomenology + lattice)</td>
<td>90(17)</td>
<td>Refs. [18–30, 32]</td>
</tr>
<tr>
<td>QED</td>
<td>116 584 718.931(104)</td>
<td>Refs. [33, 34]</td>
</tr>
<tr>
<td>Electroweak</td>
<td>153.6(1.0)</td>
<td>Refs. [35, 36]</td>
</tr>
<tr>
<td>HVP ($e^+e^-$, LO + NLO + NNLO)</td>
<td>6845(40)</td>
<td>Refs. [2–8]</td>
</tr>
<tr>
<td>HLVbL (phenomenology + lattice + NLO)</td>
<td>92(18)</td>
<td>Refs. [18–32]</td>
</tr>
<tr>
<td>Total SM Value</td>
<td>116 591 810(43)</td>
<td>Refs. [2–8, 18–24, 31–36]</td>
</tr>
<tr>
<td>Difference: $\Delta a_\mu := a_\mu^{\exp} - a_\mu^{\text{SM}}$</td>
<td>279(76)</td>
<td></td>
</tr>
</tbody>
</table>

w.r.t. BNL

References:
- [1]: [reference 1]
- [2–7]: References 2 to 7
- [8]: Reference 8
- [9–17]: References 9 to 17
- [18–30]: References 18 to 30
- [31]: Reference 31
- [32]: Reference 32
- [33, 34]: References 33 and 34
- [35, 36]: References 35 and 36
- [2–8]: References 2 to 8
- [18–24]: References 18 to 24
- [31–36]: References 31 to 36
HVP: Connection between g-2 and $\Delta \alpha(M_Z^2)$

Precision observable $\alpha(M_Z^2) = \alpha/(1-\Delta \alpha(M_Z^2))$ as a sensitive test of HVP

- Can $\Delta a_\mu$ be due to *hypothetical mistakes* in the hadronic $\sigma(s)$?
- An upward shift of $\sigma(s)$ also induces an increase of $\Delta \alpha_{\text{had}}^{(5)}(M_Z)$.
- Consider:

\[
a_{\text{HLO}} \rightarrow a = \int_{4m^2_\pi}^{s_u} ds \, f(s) \sigma(s),
\quad f(s) = \frac{K(s)}{4\pi^3}, \quad s_u < M_Z^2,
\]

\[
\Delta \alpha_{\text{had}}^{(5)} \rightarrow b = \int_{4m^2_\pi}^{s_u} ds \, g(s) \sigma(s),
\quad g(s) = \frac{M_Z^2}{(M_Z^2 - s)(4\alpha\pi^2)},
\]

and the increase

\[
\Delta \sigma(s) = \epsilon \sigma(s)
\]

$\epsilon > 0$, in the range:

\[
\sqrt{s} \in [\sqrt{s_0} - \delta/2, \sqrt{s_0} + \delta/2]
\]

Note the very different energy-dependent weighting of the integrands…
HVP: Connection between g-2 and $\Delta\alpha(M_Z^2)$

Marciano, Passera, Sirlin (2008):

- changing the hadronic cross section at higher energies significantly upwards leads to tensions in EW precision fits of the SM.
- not easy to reconcile g-2 without running into problems with $\Delta\alpha(M_Z^2)$

Recent studies by several groups, e.g.

- Crivellin et al, PRL125(2020)9,091801: Shifts in HVP make fit based on HEPFitter worse, but they can not rule out shifts at low energies as obtained by the BMW lattice analysis.
- Keshavarzi et al, PRD102(2020)3,033002: updating Marciano et al, again find significant tensions with Gfitter if shifts in HVP were to explain g-2, unless they are below $\sim$0.7 GeV.
- However, the low energies hadronic cross section measurements (mainly 2pi) are most precise there.