

Progress on 2-Loop Amplitude Reduction

Dhimiter D. Canko

In collaboration with: G. Bevilacqua, A. Kardos & C. Papadopoulos.

*Institute of Nuclear and Particle Physics, NCSR "Demokritos"
Nuclear and Particle Physics Department, NKUA*

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Motivation for Precision Physics

Although many cosmological (and not only) results indicate the need of BSM physics:

- *Dark Matter and Dark Energy*
- *Neutrino Oscillations*
- *Matter - Antimatter asymmetry*
- *Quantum Gravity incorporation*

there is no striking manifestation of New Physics beyond the SM at the LHC, as this can be found by the comparison of the measurements with the theoretical predictions!

The upgrading of LHC and the establish of future experiments will result to more accurate measurements and will require more accurate theoretical predictions!

Thus *Precision pQCD*, which provides highly precise theoretical predictions for the SM, becomes a necessity for the discovery of BSM physics via deviation from the Standard Model! Current frontier:

- *NNLO for $2 \rightarrow 3$ processes*
- *N^3LO for $2 \rightarrow 2$ processes*
- *N^4LO for $2 \rightarrow 1$ processes*



Moreover the study of *Precision pQCD* has led to many developments to other domains such as *Theory of Special Functions*, *Computer Algebra*, *Algebraic Geometry* etc.



Cross sections

- The cross section for the collision of 2 initial hadrons (h_1, h_2) to some final state X is

$$d\sigma_{h_1 h_2 \rightarrow X} = \sum_{a,b=q,\bar{q},g} \int_{x_{1,min}}^1 dx_1 \int_{x_{2,min}}^1 dx_2 \mathcal{F}_{a/h_1}(x_1, \mu^2) \mathcal{F}_{b/h_2}(x_2, \mu^2) \hat{\sigma}_{ab \rightarrow X}(\mu^2)$$

where \mathcal{F}_{a/h_1} and \mathcal{F}_{b/h_2} are the Parton Distribution Functions, $\hat{\sigma}_{ab \rightarrow X}$ is the hard-part cross section, and μ^2 is the factorization scale.

- $\hat{\sigma}_{ab \rightarrow X}$ at NNLO receives contributions from three different sources (virtual, mixed real-virtual, and doubly-real corrections)

$$d\hat{\sigma}_{ab \rightarrow X}^{NNLO} \sim |\mathcal{M}_{tree}|^2 + \alpha_S \left(\text{Re} \left[\mathcal{M}_{tree} \mathcal{M}_{loop}^* \right] + |\mathcal{M}_{+1up}|^2 \right) \\ + \alpha_S^2 \left(|\mathcal{M}_{loop}|^2 + \text{Re} \left[\mathcal{M}_{tree} \mathcal{M}_{2loops}^* \right] + |\mathcal{M}_{+2up}|^2 + \text{Re} \left[\mathcal{M}_{+1up+loop} \mathcal{M}_{+1up}^* \right] \right)$$

Each of these contributions is individually divergent, and the divergences cancel in the sum (after renormalization for the UV and IR divergences) leaving behind the finite result for the cross section.



A lot of effort is needed in all the steps for the calculation of the cross section. Nonetheless, from the above expression the most difficult part to be calculated has been proved to be the 2-loop amplitude, \mathcal{M}_{2loops} !



Recent Results for 2-loop Amplitudes

Impressively in the last years, some results have appeared in the literature for different 2-loop $2 \rightarrow 3$ scattering amplitudes, applying different approaches for the amplitude reduction (Numerical Unitarity, Projectors etc)

- Leading color: $gg \rightarrow ggg$, $q\bar{q} \rightarrow ggg$, $q\bar{q} \rightarrow q\bar{q}g$, $q\bar{q} \rightarrow \gamma\gamma\gamma$, and $q\bar{q} \rightarrow g\gamma\gamma$.
- Full color: $gg \rightarrow ggg$ (all-plus helicities) and $q\bar{q} \rightarrow g\gamma\gamma$.

Some benchmark references:

- B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, arXiv:2105.04585 [hep-ph].
- B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, JHEP 2104 (2021) 201.
- H. A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, [arXiv:2105.06940 [hep-ph]].
- H. A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, arXiv:2103.04319 [hep-ph].
- H. A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, arXiv:2012.13553 [hep-ph].
- H. A. Chawdhry, M. A. Lim and A. Mitov, Phys. Rev. D **99** (2019) no.7, 076011.
- S. Abreu, F. Febres Cordero, H. Ita, B. Page and V. Sotnikov, arXiv:2102.13609 [hep-ph].
- S. Abreu, B. Page, E. Pascual and V. Sotnikov, JHEP 2101 (2021) 078.
- S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, R.S. Ruf, V. Sotnikov, arXiv:2009.11957 [hep-ph].
- S. Badger, H. B. Hartanto, C. Brønnum-Hansen and T. Peraro, JHEP 1909 (2019) 119.
- S. Badger, D. Chicherin, T. Gehrmann, G. Heinrich, J. M. Henn, T. Peraro, P. Wasser, Y. Zhang and S. Zoia, Phys. Rev. Lett. **123** (2019) no.7, 071601.
- S. Badger, C. Brønnum-Hansen, H. B. Hartanto and T. Peraro, JHEP **01** (2019), 186.
- T. Gehrmann, J. M. Henn and N. A. Lo Presti, Phys. Rev. Lett. **116** (2016) no.6, 062001.
- T. Peraro, JHEP **12** (2016), 030.
- G. De Laurentis and D. Maître, JHEP **02** (2021), 016.



Structure of an (2-loop) Amplitude

The construction and calculation of a (2-loop) Amplitude, \mathcal{A} , contains the following steps

- 1 Use of SM Feynman Rules to Generate the Feynman Graphs contributing to the process at hand.
- 2 Collect all the above contributions and create the Amplitude.
- 3 Integrand/Integral Reduce the Amplitude in to a set of Master (Feynman) integrals, determining their coefficients.
- 4 Calculate analytically or numerically the Master Integrals (see talk of Syrrakos).

The final result is of the form

$$\mathcal{A} = \sum_i c_i(\mathbf{s}, \varepsilon) F_i(\mathbf{s}, \varepsilon) \quad (3.1)$$

where c_i are rational/algebraic coefficients obtained by the Amplitude reduction and they depend by the process at hand, F_i can be Master Integrals or special functions (Multiple Polylogarithms¹, Pentagon Functions², Elliptic Integrals, etc) that depend from the kinematics and are process-independent, and \mathbf{s} are the Mandelstam variable



¹A. B. Goncharov, *Math. Res. Lett.* **5** (1998), 497-516

²D. Chicherin and V. Sotnikov, *JHEP* **12** (2020), 167

Quick Review of HELAC-1LOOP

Any 1-loop n -particle (color-stripped) amplitude can be written in the form

$$\mathcal{A} = \int d^d k A = \sum_{I \subset \{1, \dots, n\}} \int \frac{\mu^{(4-d)} N_I(k, p_1, \dots, p_{n-1}, \gamma^\mu, \epsilon^\mu)}{(2\pi)^d \prod_{i \in I} D_i}$$

where N_I is the numerator and $D_i = (k + p_i)^2 + m_i^2$ the propagators. The loop momentum "lives" in d dimensions and can be decomposed as

$$k = \bar{k} + k^* \quad \text{with} \quad \bar{k} : 4 - \text{dimensional} \quad \text{and} \quad k^* : \varepsilon - \text{dimensional}.$$

In order to compute \mathcal{A} we need to cast it in to the following well-known form at $d \rightarrow 4$

$$\mathcal{A} = \sum_i d_i \text{Box}_i + \sum_i c_i \text{Triangle}_i + \sum_i b_i \text{Bubble}_i + \sum_i a_i \text{Tadpole}_i + (R_1) + R_2$$

where Box, ..., Tadpole refer to the one-loop Feynman integrals with 4, ..., 1 external leg, (R_1 is the rational part originating from the reduction process of a 4-dimensional numerator in the OPP method³) and R_2 is the rational part originating by the explicit dependence of the numerator on the ε -dimension and can be reproduced by tree-like Feynman rules involving up to 4 particles⁴.



³G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763** (2007), 147-169

⁴G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP **05** (2008), 004



In HELAC-1LOOP⁵ the OPP method (*integrand level*) is used for the amplitude reduction. The main idea is that for any numerator its 4-dimensional part can be written as

$$\begin{aligned} \bar{N}(\bar{k}) = & \sum_{i_0 < i_1 < i_2 < i_3}^l \left[d(i_0, i_1, i_2, i_3) + \tilde{d}(\bar{k}, i_0, i_1, i_2, i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^l \bar{D}_i \\ & + \sum_{i_0 < i_1 < i_2}^l \left[c(i_0, i_1, i_2) + \tilde{c}(\bar{k}, i_0, i_1, i_2) \right] \prod_{i \neq i_0, i_1, i_2}^l \bar{D}_i \\ & + \sum_{i_0 < i_1}^l \left[b(i_0, i_1) + \tilde{b}(\bar{k}, i_0, i_1) \right] \prod_{i \neq i_0, i_1}^l \bar{D}_i \\ & + \sum_{i_0}^l \left[a(i_0) + \tilde{a}(\bar{k}, i_0) \right] \prod_{i \neq i_0}^l \bar{D}_i \end{aligned}$$

where $d_i = d(i_0, i_1, i_2, i_3)$, $c_i = c(i_0, i_1, i_2)$, $b_i = b(i_0, i_1)$, $a_i = a(i_0)$, and \tilde{d} , \tilde{c} , \tilde{b} , \tilde{a} integrate to zero (*spurious terms*). The coefficients are determined by solving (iteratively) systems of equations by evaluating $\bar{N}(\bar{k})$ for values of \bar{k} , that are solutions of

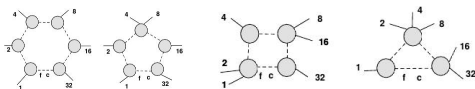
$$\bar{D}_i(\bar{k}) = 0, \text{ for } i = 0, \dots, M-1, \text{ and } M = 1, \dots, 4.$$



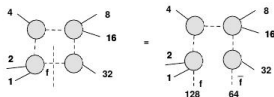
⁵G. Bevilacqua, M. Czakon, M. V. Garzelli, A. van Hameren, A. Kardos, C. G. Papadopoulos, R. Pittau and M. Worek, *Comput. Phys. Commun.* **184** (2013), 986-997

The 4 – dimensional numerator⁶ is numerically calculated by HELAC⁶, which efficiently calculates tree-level amplitudes checking for all possible flavor, spin and color configurations using SM couplings and the color-connection representation!

In this set-up, a binary representation is used for the external particles (lv_1 blobs) and a generation of all topologically inequivalent partitions of $n, n - 1, n - 2, \dots, 1$ blobs attached to the loop is done. For example, for $n = 6$ we could have



The blobs could contain propagators but they do not depend on \vec{k} . Each numerator contribution is calculated (by HELAC) by cutting the propagator-line connecting the first and the last blob and calculating the resulted $n + 2$ tree-level amplitude without using denominators for the internal loop propagators



HELAC-1LOOP: Completely automated framework for the calculation of 1-loop amplitudes for n -particle processes!!!



⁶A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000), 306-315

HELAC-2LOOP on the making

Our intention for the construction of HELAC-2LOOP is to reuse as much as more from the already working framework of HELAC-1LOOP combined with new concepts for the amplitude reduction at 2-loops!

As in the 1-loop case, at 2-loops we expect that at $d \rightarrow 4$ for the Amplitude will hold true the following expression at the *integrand level*

$$A \equiv \sum_{I \subseteq T} \frac{N_I(k_1, k_2, p_1, \dots, p_n, \gamma^\mu, \epsilon^\mu)}{\prod_{\{i_1, i_2, i_3\} \in I} D_{i_1}(k_1) D_{i_2}(k_2) D_{i_3}(k_1, k_2)} = \sum_i c_i(\mathbf{s}) F_i + \sum_j \check{c}_j(\mathbf{s}) S_j + R_1 + R_2$$

where T is the set containing the 2-loop graph topologies of the corresponding process, F_i are the *master integrands* that will integrate to *master integrals*, S_j are the *spurious* terms that will integrate to zero, and $\{R_1, R_2\}$ are the 2-loop generalization of the 1-loop rational terms⁷.

Although some results are obtained using an *master integrand* + surface (spurious) terms approach in d -dimensions, a complete basis is still missing (case-by-case study).

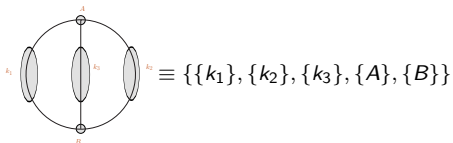
In order to calculate the numerator a generation of the 2-loop amplitude graphs is needed in the "blob"-binary representation that is used internally by HELAC. For this purpose we have created two generators for obtaining two-loop graph topologies for massless particles running within the loop (no tadpole graphs) in a five list format.

⁷The 2-loop R_2 terms have been computed in J. N. Lang, S. Pozzorini, H. Zhang and M. F. Zoller, JHEP **10** (2020), 016 and S. Pozzorini, H. Zhang and M. F. Zoller, JHEP **05** (2020), 077

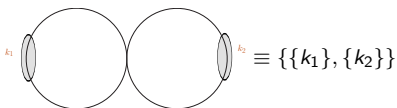


There exist three graph topologies for 2-loop amplitudes, for which we choose the following list parametrization

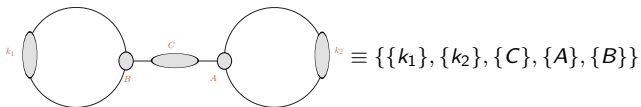
1) *Theta-topologies:*



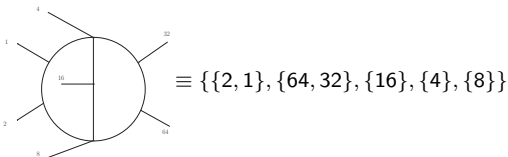
2) *Infinity-topologies:*



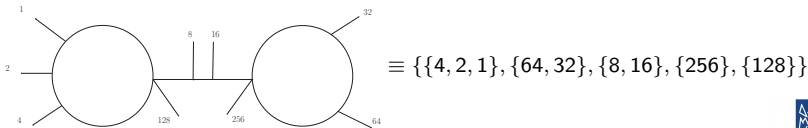
3) *Dumbbell-topologies:*



- In Theta-topologies the particles in the sublists $\{k_1\}$, $\{k_2\}$ and $\{k_3\}$ are en-counted starting from the point B and ending at A . For Example:

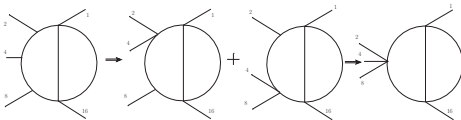


- In Infinity-topologies the particles in the sublists $\{k_1\}$ and $\{k_2\}$ are en-counted using a bottom-up approach.
- In Dumbbell-topologies the particles in the sublists $\{k_1\}$ and $\{k_2\}$ are en-counted using a bottom-up approach, and the particles in the $\{C\}$ sublist are en-counted starting from the point B and ending at A .



We have created two generators, one implemented in Mathematica (BlobMod) and one implemented in Fortran (GENTOOLS), using two different approaches in the generation:

- BlobMod starts by creating all the possible sets of putting the external particles in the sublists. Then in order to create the sub-topologies⁸ if there exist lists with $Length[sublist] \geq 2$ takes for every list all the possible combinations of summing at most 2 neighboring elements of the same sublist⁹. After the generation of all the topologies, graph-symmetries are applied in order to remove identical lists.



$$\{\{8, 4, 2\}, \{\}, \{\}, \{1\}, \{16\}\} \rightarrow \{\{8, 6\}, \{\}, \{\}, \{1\}, \{16\}\} + \{\{12, 2\}, \{\}, \{\}, \{1\}, \{16\}\} \rightarrow \{\{14\}, \{\}, \{\}, \{1\}, \{16\}\}$$

- GENTOOLS generates the topologies exactly in the opposite way! Starts by taking all the possible sets of putting the higher level blobs in the sublists (lower-topologies) and creates the higher topologies by taking all the possible splittings of the blobs. In order to remove the identical lists graph-symmetries are applied also in this case.



Perfect agreement found between the results of the two generators and q-graf (P. Nogueira, *J. Comput. Phys.* **105** (1993)). For $n \geq 6$ GENTOOLS is a lot faster from BlobMod!!!

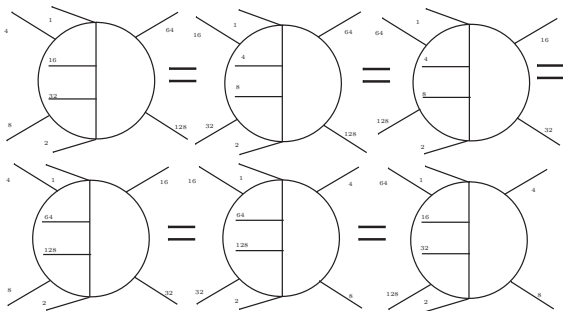
⁸Meaning topologies where more than one particles shrink into a vertex, lv_2, \dots, lv_n blobs.

⁹The elements of $\{A\}$ and $\{B\}$ are always summed from the beginning.



Graph symmetries

The graphs are symmetric on (combined or individual) mirror transformations on the vertical and the horizontal axis (swap of the three loop lines). For example



All the symmetries of the graphs can be expressed in symmetries of the lists using one or both of the following 2 actions:

- **Swap:** corresponds to the swap of two sublists. E.g. $\{\{k_1\}, \{k_2\}\} \rightarrow \{\{k_2\}, \{k_1\}\}$.
- **Reversion:** corresponds to the reversion of the elements of a sublist. E.g. $\{1, 2, 4\} \rightarrow \{4, 2, 1\}$.



See back-up slides for a collection of the list-symmetries of the 3 graph topologies.

Conclusion

- In order to apply an OPP-like Amplitude reduction, at the moment we are working on the upgrade of HELAC code such that to be able to numerically compute the 4-dimensional part of the numerator from the tree-level $n + 4$ amplitude.
- Next step: creation of a general $\{master\ integrand + spurious\ terms\}$ basis, and calculation of the 2-loop R_1 rational terms.
- As already mentioned, hopefully some results are now available for crosschecks!
- In addition, we have also made progress on the second line of the Dyson-Schwinger equations

The diagram shows an equation between two Feynman diagrams. On the left is a tree-level vertex with one external line and two internal lines meeting at a central point. On the right is a sum of five diagrams representing a 2-loop correction to this vertex. The first diagram is a tree-level vertex with a loop on one of the internal lines. The second diagram is a tree-level vertex with a loop on the other internal line. The third diagram is a tree-level vertex with a loop on the external line. The fourth diagram is a tree-level vertex with a loop on the external line, but with a different internal structure. The fifth diagram is a tree-level vertex with a loop on the external line, with yet another internal structure. The diagrams are connected by plus signs, indicating a sum.

and using an Mathematica file, we are able to construct the integrand of the 2-loop Amplitude in d -dimensions!



Thank you!

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Color connection representation

- In the color connection representation, the gluons are represented by a pair of color/anti-color indices (i, j) and the quarks (anti-quarks) by a single color $(i, 0)$ (anti-color $(0, j)$) index, with $i, j \in (1, \dots, N_C)$. All the other particles that do not carry color have $(0, 0)$.
- The amplitude takes the following form

$$\mathcal{M}_{j_1, j_2, \dots, j_k}^{i_1, i_2, \dots, i_k} = \sum_{\sigma} \delta_{i_{\sigma_1} j_1} \delta_{i_{\sigma_2} j_2} \dots \delta_{i_{\sigma_k} j_k} A_{\sigma}$$

with $k = n_g + n_q$ and the sum is running over all the permutations (equal to $k!$). The color-stripped amplitudes, A_{σ} , are calculated using properly defined Feynman rules [A. Cafarella, C. G. Papadopoulos and M. Worek, *Comput. Phys. Commun.* **180** (2009), 1941-1955].

- The total color factor is a product of δ 's, and thus the color summed squared amplitude takes the form

$$\sum_{\{i\}, \{j\}} |\mathcal{M}_{j_1, j_2, \dots, j_k}^{i_1, i_2, \dots, i_k}|^2 = \sum_{\sigma, \sigma'} A_{\sigma'}^* C_{\sigma', \sigma} A_{\sigma}$$

where the color matrix $C_{\sigma', \sigma}$ is given by

$$C_{\sigma', \sigma} = \sum_{\{i\}, \{j\}} \delta_{i_{\sigma'_1} j_1} \delta_{i_{\sigma'_2} j_2} \dots \delta_{i_{\sigma'_k} j_k} \delta_{i_{\sigma_1} j_1} \delta_{i_{\sigma_2} j_2} \dots \delta_{i_{\sigma_k} j_k} = N_C^{m(\sigma', \sigma)}$$

with $m(\sigma', \sigma)$ counting the number of common cycles of the 2 permutations.



Theta–Topology symmetries

The symmetries of the graphs are translated to the following symmetries for the lists of the Theta–Topologies¹⁰

$$\begin{aligned}
 \{\{k_1\}, \{k_2\}, \{k_3\}, \{A\}, \{B\}\} &= \{R[\{k_1\}], R[\{k_2\}], R[\{k_3\}], \{B\}, \{A\}\} \\
 &= \{R[\{k_1\}], R[\{k_3\}], R[\{k_2\}], \{B\}, \{A\}\} \\
 &= \{R[\{k_2\}], R[\{k_1\}], R[\{k_3\}], \{B\}, \{A\}\} \\
 &= \{R[\{k_2\}], R[\{k_3\}], R[\{k_1\}], \{B\}, \{A\}\} \\
 &= \{R[\{k_3\}], R[\{k_1\}], R[\{k_2\}], \{B\}, \{A\}\} \\
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 &= \{\{k_3\}, \{k_1\}, \{k_2\}, \{A\}, \{B\}\} \\
 &= \{\{k_3\}, \{k_2\}, \{k_1\}, \{A\}, \{B\}\}
 \end{aligned}$$



¹⁰We use the notation $R[\{k_i\}] := \text{Reverse}[\{k_i\}]$.

Infinity–Topology and Dumbbell–Topology symmetries

The symmetries of the graphs are translated to the following symmetries for the lists of the Infinity–Topologies:

$$\begin{aligned}
 \{\{k_1\}, \{k_2\}\} &= \{\{k_2\}, \{k_1\}\} \\
 &= \{R[\{k_1\}], \{k_2\}\} = \{\{k_1\}, R[\{k_2\}]\} = \{R[\{k_1\}], R[\{k_2\}]\} \\
 &= \{R[\{k_2\}], \{k_1\}\} = \{\{k_2\}, R[\{k_1\}]\} = \{R[\{k_2\}], R[\{k_1\}]\}
 \end{aligned}$$

The symmetries of the graphs are translated to the following symmetries for the lists of the Dumbbell–Topologies:

$$\begin{aligned}
 \{\{k_1\}, \{k_2\}, \{C\}, \{A\}, \{B\}\} &= \{R[\{k_1\}], \{k_2\}, \{C\}, \{A\}, \{B\}\} \\
 &= \{\{k_1\}, R[\{k_2\}], \{C\}, \{A\}, \{B\}\} \\
 &= \{R[\{k_1\}], R[\{k_2\}], \{C\}, \{A\}, \{B\}\} \\
 &= \{\{k_2\}, \{k_1\}, R[\{C\}], \{B\}, \{A\}\} \\
 &= \{R[\{k_2\}], \{k_1\}, R[\{C\}], \{B\}, \{A\}\} \\
 &= \{\{k_2\}, R[\{k_1\}], R[\{C\}], \{B\}, \{A\}\} \\
 &= \{R[\{k_2\}], R[\{k_1\}], R[\{C\}], \{B\}, \{A\}\}
 \end{aligned}$$

