Gravitational Singularities
and
Modification of Gravity with Perimeter Action

George Savvidy
Demokritos National Research Centre
Greece, Athens

High Energy Physics and Cosmology
HEP 2021, Thessaloniki, 16 - 19 June, 2021
J. R. Oppenheimer and H. Snyder,
*On Continued gravitational contraction*,
Phys. Rev. **56** (1939), 455-459

R. Penrose,
*Gravitational collapse and space-time singularities*,

S. Hawking,
*The Occurrence of Singularities in Cosmology. II. III*,

S. Hawking and R. Penrose, *The singularities of gravitational collapse and cosmology*,
1. The appearance of singularities in General Relativity pointed out to the limitations of the predictive power of the theory and has similarity with the difficulties that appeared in the past in attempts to describe the spectrum of hydrogen atom within the realms of classical electromagnetism and mechanics.

2. The late acceleration of the Universe and the appearance of the positive cosmological constant is another challenging problem within the realms of quantum field theory of vacuum.
Curing Gravitational Singularities. The Limiting Curvature Hypothesis

R. H. Brandenberger, V. F. Mukhanov and A. Sornborger,
A Cosmological theory without singularities,

M. A. Markov, Limiting density of matter as a universal law of nature.,
JETP Lett. 36 (1982) 265

Y. I. Anini, The Limiting Curvature Hypothesis:
Towards a Theory of Gravity Without Singularities,
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Y. I. ANINI

In order to actually implement the limiting curvature hypothesis we are
lead to postulate the following gravitational action 6

$$S_g = k \int \left[ (R - \Lambda/2) + \frac{\Lambda}{2} \sqrt{1 - \frac{R^2}{\Lambda^2}} \right] \sqrt{-g} \; d^4x$$  (1)
Modification of Gravitational Interaction

S. Deser and G. Gibbons,  
Born-Infeld-Einstein Action  

\[ I_{BI} = \frac{-1}{2} \lambda^2 \int d^4x \left\{ -g_{\mu\nu} + \lambda F_{\mu\nu} \right\}^{\frac{1}{2}}. \]

\[ I_{EDD} = \int d^4x |R_{(\mu\nu)}(\Gamma)|^{\frac{1}{2}} \]

\[ I_G = \int d^4x \left\{ -a g_{\mu\nu} + b R_{\mu\nu} + c X_{\mu\nu} \right\}^{\frac{1}{2}}. \quad X_{\mu\nu}(R) \text{ quadratic or higher in curvature.} \]

Modified gravity by the \( f(R) \) term ...
The higher-order curvature corrections to gravitational action

S. Kawai, M.-a. Sakagami, J. Soda
Instability of one loop superstring cosmology

I. Antoniadis, J. Rizos, K. Tamvakis
Singularity-free cosmological solutions of the superstring effective action

R. Brustein, G. Veneziano
The graceful exit problem in string cosmology

R. Easther, K.-i. Maeda
One loop superstring cosmology and the nonsingular universe

I. Antoniadis, C. Bachas, J.R. Ellis, D.V. Nanopoulos
An expanding universe in string theory
Alternative Modification of Gravitational Interaction

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*Gravity with Perimeter Action and Gravitational Singularities*

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*Gravity with a Linear Action and Gravitational Singularities*,
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G. Savvidy,
*Quantum gravity with linear action. Intrinsic rigidity of space-time*,

J. Ambjorn, G. Savvidy and K. Savvidy,
*Alternative actions for quantum gravity and the intrinsic rigidity of the space-time*,

G. Savvidy and K. Savvidy,
*Interaction hierarchy: Gonihedric string and quantum gravity*,
In General Relativity the action is proportional to the Area of the Universe

\[ S_A = -\frac{c^3}{16\pi G} \int R\sqrt{-g}d^4x. \]

R \sim 1/cm^2 \quad \sqrt{-g}d^4x \sim cm^4 \quad \int R\sqrt{-g}d^4x \sim cm^2

The Discrete Area Action of Regge

\[ S_A = \sum_{\langle ijk \rangle} \sigma_{ijk} \cdot \omega_{ijk}^{(2)} \]

the four-dimensional simplex

the area of the triangle \( \sigma_{ijk} \) \quad cm^2

the corresponding deficit angle \( \omega_{ijk}^{(2)} \) \quad dimensionless
The Perimeter Action is proportional to the size of the Universe

\[ S_P = \sum_{<ijk>} \lambda_{ijk} \cdot \omega_{ijk}^{(2)} \]

the perimeter \( \lambda_{ijk} \) of the triangle \( \sim \text{cm} \)

the corresponding deficit angle \( \omega_{ijk}^{(2)} \) \( \text{dimension less} \)

Perimeter action suppresses the lower-dimensional fluctuations that can grow out of the 4D Space-Time manifold

The question is: what is the continuum form of the \( S_P \)?
One can suggest the following form for the Perimeter action

\[ S_P = \gamma \sqrt{\hbar} \sqrt{\frac{c^3}{16\pi G}} \int \sqrt{I_1 + (1 - \epsilon) I_2} \sqrt{-g} d^4x. \]

\[ I_1 = -\frac{1}{80\pi} R_{\mu\nu\lambda\rho;\sigma} R^{\mu\nu\lambda\rho;\sigma}, \quad I_2 = +\frac{1}{16\pi} R_{\mu\nu\lambda\rho} \Box R^{\mu\nu\lambda\rho} \quad \sim 1/cm^6 \]

The dimension of the integrant invariant is 1/cm³

the integral has the dimensions of cm

\( \gamma \) is a dimensionless coupling constant \( \quad 0 \leq \epsilon < 1 \)

\( \gamma \) expressing the mass parameter in terms of Plank mass units,

\[ M = \gamma \ M_P, \quad M_P = \sqrt{\hbar c/16\pi G}, \]
The action we shall consider is a sum

\[ S = -\frac{c^3}{16\pi G} \int R\sqrt{-g}d^4x + \gamma \hbar \sqrt{\frac{c^3}{16\pi G}} \int \sqrt{I_1 + (1 - \epsilon)I_2} \sqrt{-g}d^4x. \]

The full equation has the form

\[ \frac{\delta S}{\delta g^{\mu\nu}} = \frac{\delta S_A}{\delta g^{\mu\nu}} + \gamma \frac{\delta S_P}{\delta g^{\mu\nu}} = -\frac{c^3}{16\pi G} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) + \]

\[ + \gamma \hbar \sqrt{\frac{c^3}{16\pi G}} \left( \frac{1}{2} \frac{1}{\sqrt{I_1 + (1 - \epsilon)I_2}} \left( \frac{\delta I_1}{\delta g^{\mu\nu}} + (1 - \epsilon) \frac{\delta I_2}{\delta g^{\mu\nu}} \right) - \frac{1}{2} \sqrt{I_1 + (1 - \epsilon)I_2} g_{\mu\nu} \right) = 0. \]

We shall solve these equations using perturbation theory in \( \gamma \).
Let us consider the perturbation of the Schwarzschild solution

The Schwarzschild solution has the form

\[ ds^2 = (1 - \frac{r_g}{r})c^2 dt^2 - (1 - \frac{r_g}{r})^{-1} dr^2 - r^2 d\Omega^2 , \]

where \( g_{00} = 1 - \frac{r_g}{r} \), \( g_{11} = -(1 - \frac{r_g}{r})^{-1} \), \( g_{22} = -r^2 \), \( g_{33} = -r^2 \sin^2 \theta \), and

\[ r_g = \frac{2GM}{c^2}, \quad \sqrt{-g} = r^2 \sin \theta. \]

\[ I_0 = \frac{1}{12} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = \left( \frac{r_g}{r^3} \right)^2. \]

and shows the location of the curvature singularity at \( r = 0 \).
On the Schwarzschild solution the invariants are

\[ I_1 = \frac{9\pi r_g^2 (r - r_g)}{4 r^9}, \quad I_2 = \frac{9\pi r_g^3}{4 r^9}, \]

and the perimeter action takes the form

\[ S_P = \gamma \sqrt{\frac{\hbar c^5}{G}} \int \frac{3\pi}{2} \sqrt{1 - \epsilon \frac{r_g}{r}} \frac{r_g}{r} \frac{dr}{dt}, \quad r < \epsilon r_g, \quad 0 \leq \epsilon < 1 \]

the expression under the square root becomes negative at

\[ r < \epsilon r_g, \quad 0 \leq \epsilon < 1 \]
This amplitude is represented by the path integral

$$A_E(r < \varepsilon r_g) = \int_0^{r < \varepsilon r_g} d^3 \vec{r} \int_{x'}^x e^{i \frac{\hbar}{4}(S[g]+S[g,x])} Dg D\mathbf{x} \;.$$ 

![Graph](image)

Figure 2: The graph of the real and imaginary parts of the amplitude in gravity with perimeter action (2.9) $\Delta K = e^{i \gamma \sqrt{\frac{c^5}{\hbar G} \frac{3}{2} \sqrt{1-\varepsilon \frac{r_g}{r} \frac{r_g}{r} \frac{\Delta r}{r}}} \Delta t}$, here $\varepsilon = 0.1$ and $\gamma \sqrt{\frac{c^5}{\hbar G} \frac{3}{2} \frac{\Delta r}{r}} \Delta t = 0.01$.  

Analogy with the Relativistic Particle Action

\[ S = -mc \int ds = -mc^2 \int \sqrt{1 - \frac{\dot{v}^2}{c^2}} \, dt. \]

\[ K(t_b, x_b; t_a, x_a) = \int_{x_a}^{x_b} e^{\frac{i}{\hbar}mc^2} \int_{t_a}^{t_b} \sqrt{1 - \frac{\dot{x}^2}{c^2}} \, dt \, \mathcal{D}x(t). \]

The graphic of the real and imaginary parts of the amplitude \( \Delta K = e^{\frac{i}{\hbar}mc^2 \sqrt{1 - \frac{\dot{v}^2}{c^2}} \Delta t} \).
The first variation of the perimeter

In order to solve the equations in the first order in the Schwarzschild metric is:

and the solution is

We will search the solution of this equations in the following standard spherically symmetric form:

The full equation has the form

\[
\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\delta S_A}{\delta g^{\mu\nu}} + \gamma \frac{\delta S_P}{\delta g^{\mu\nu}} = - \frac{c^3}{16\pi G} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) +
\]

\[
+ \gamma \sqrt{\hbar} \sqrt{\frac{c^3}{16\pi G}} \left( \frac{1}{2} \frac{1}{\sqrt{I_1 + (1 - \epsilon) I_2}} \left( \frac{\delta I_1}{\delta g^{\mu\nu}} + (1 - \epsilon) \frac{\delta I_2}{\delta g^{\mu\nu}} \right) - \frac{1}{2} \sqrt{I_1 + (1 - \epsilon) I_2} g_{\mu\nu} \right) = 0.
\]

We will search the solution of this equations in the following standard spherically symmetric form:

\[
ds^2 = e^{\nu(r)} c^2 dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

the Schwarzschild metric is:

\[
\nu_0(r) = \log(1 - \frac{r_g}{r}), \quad \lambda_0(r) = - \log(1 - \frac{r_g}{r}).
\]

we will represent the solution \( g_{\mu\nu} \) in the form:

\[
g^{\mu\nu} = g_0^{\mu\nu} + \gamma g_1^{\mu\nu}, \quad \nu \to \nu_0 + \gamma \nu_1, \quad \lambda = \lambda_0 + \gamma \lambda_1,
\]
The perturbation of the Schwarzschild solution

\[ g_{11} = e^{\lambda_0 + \lambda_1} = \frac{-1}{1 - \frac{r_g}{r}} e^{\lambda_1}, \]

where

\[ \lambda_1 = \gamma \frac{l_P}{r - r_g} \left[ \log \frac{1 + \sqrt{1 - \varepsilon \frac{r_g}{r}}}{1 - \sqrt{1 - \varepsilon \frac{r_g}{r}}} + \sqrt{1 - \varepsilon \frac{r_g}{r}} - (1 - \varepsilon \frac{r_g}{r})^{3/2} + C \right] \]

\[ g_{00} = e^{\nu_0 + \nu_1} = \left(1 - \frac{r_g}{r}\right) e^{\nu_1} \]

\[ \nu_1 = -\lambda_1 + \gamma \frac{l_P}{r_g} \left[ \sqrt{1 - \varepsilon \frac{r_g}{r}} - (1 - \varepsilon \frac{r_g}{r})^{3/2} \right] \]
the behaviour of the temporal component of the metric at large distances $r \gg r_g$:

$$g_{00} = 1 + 2 \frac{\phi}{c^2} \simeq 1 - \frac{r_g}{r} - \gamma \frac{l_P}{r} \left( a_1 \log \frac{r}{r_g} + a + c_1 \right),$$

The logarithmically growing potential

$$g_{00}(M_\odot) \simeq 1 - 10^{-8} - \gamma 10^{-43},$$

$$g_{00}(M_{MW}) \simeq 1 - 10^{-5} - \gamma 10^{-54},$$

The advance precession of the perihelion $\delta \phi$ expressed in radians per revolution is

$$\delta \phi = \frac{3 \pi m^2 c^2 r_g^2}{2 L^2} \left( 1 + 2 \gamma \frac{l_P}{r_g} (8a_1 + a + c_1) \right) = \frac{6 \pi G M}{c^2 l (1 - e^2)} \left( 1 + 2 \gamma \frac{l_P}{r_g} (8a_1 + a + c_1) \right),$$

$\gamma \frac{l_P}{r_g}$ is tiny

$$l_P = \sqrt{\frac{\hbar 16 \pi G}{c^3}} \simeq 11.3 \times 10^{-33} \text{cm},$$
Thank You!
Figure 1: On the left hand side of the figure there is a trajectory of a structureless point particle with an action which is proportional to the length of its world line. On the right there is a world sheet surface which is swept by a propagating closed string. It is required that the string action $A_{xy}$ should be proportional to the linear size of the space-time surface, measuring it in terms of its length, similar to the action of a point-like particle. This is a natural requirement because when a string collapses to a point its world sheet will degenerate into a world line and both actions will coincide.

Extension of Feynman Path Integral

Feynman path integral over trajectories describes quantum-mechanical behaviour of point-like particles, and it is an important problem to extend the path integral to an integral which describes a quantum-mechanical motion of strings. A string is a one dimensional extended object which moves through the space-time. As string moves through the space-time it sweeps out a two-dimensional surface, and in order to describe its quantum-mechanical behavior one should define an appropriate action and the corresponding functional integral over two-dimensional surfaces.

In string theory the action is defined by using Nambu-Goto area action. The area action suffers from spike instabilities, because the zero-area spikes can easily grow on a surface. Indeed, the spikes have zero area, and there is no suppression of the spike fluctuations in the functional integral. Different modifications of the area action have been suggested in the literature to cure these instabilities, which are based on the addition of extrinsic curvature terms to the area action.
Figure 2: On the left there is a discrete trajectory with an action which is proportional to the sum of the lengths of its edges. On the right there is a discretized surface and the action is a sum of the lengths of its edges multiplied by the corresponding deficit angles.

The alternative principle to cure surface instabilities was put forward in [5, 6, 7]. In its essence there is a new requirement which should be imposed on the string action. The string action should be defined in such a way that when a string shrinks to a point-like object its action should reduce to an action of point-like particle [5]. In other words, when a surface shrinks to a space-time curve, its action should reduce to the length of the curve (see Fig. 1). It is almost obvious that now the spikes cannot easily grow on a surface because in the functional integral such fluctuations will be suppressed exponentially exponentially $e^{-m\text{spike}}$, where $\text{spike}$ is the length of the spike.

One can consider smooth surfaces, as well as discretized random surfaces which are represented by polyhedral surfaces build from triangles (see Fig. 2). For smooth surfaces the proposed action has the form [8, 9, 10, 11, 12]

$$A(M) = m \sum_{<ij>} \lambda_{ij} <ij> \sum_{<ij>} g_{\alpha_{ij}}$$

(1.1)

where $g_{ab} = @a X_\mu @b X_\mu$ is the induced metric and $(g)$ is a Laplace operator. For discretized surfaces the action is defined as a sum over links and deficit angles [5, 6, 7]:

$$A(M) = m \sum_{<ij>} \lambda_{ij} \zeta <ij>$$

(1.2)
**Tensionless string action, there is no Nambu area term**

\[ A(M) = m \sum_{<ij>} \lambda_{ij} |\pi - \alpha_{ij}| \quad \rightarrow \quad m \sum_{<ij>} \lambda_{ij}. \quad \sim \text{cm} \]

**Tensionless string action for the smooth surfaces**

\[ A(M) = m \int d^2 \zeta \sqrt{g} \sqrt{(\Delta(g)X^\mu)^2} \quad \rightarrow \quad m \int ds, \]

where \( g_{ab} = \partial_a X^\mu \partial_b X^\mu \) is the induced metric and \( \Delta(g) \) is a Laplace operator.

**Linear string action suppresses spices that can grow out of the surface**
The perturbation of the Schwarzschild solution

\[ S_P = \gamma \sqrt{\frac{\hbar c^5}{G}} \int \frac{3\pi}{2} \sqrt{1 - \epsilon \frac{r_g}{r} \frac{r_g}{r} \frac{dr}{dt}}, \quad r < \epsilon r_g, \quad 0 \leq \epsilon < 1 \]

it is similar of the relativistic particle action

\[ S = -mc \int ds = -mc^2 \int \sqrt{1 - \frac{\bar{v}^2}{c^2}} \, dt. \]

the region where the action became complex seems unreachable by the test particles. The size of the region depends on the parameter \( \epsilon \) and is smaller than the gravitational radius \( r_g \). This observation may have a profound consequences on the gravitational singularity at \( r = 0 \). In the suggested scenario it seems possible to eliminate the singularities from the theory based on the fundamental principles of quantum mechanics.
Both expressions contain the geometrical invariants which are not in general positive
definite under the square root operation. In the relativistic particle case the expression
under the root becomes negative for a particle moving with a velocity which exceeds the
velocity of light. In that case the action develops an imaginary part and quantum mechan-
ical superposition of amplitudes prevents a particle from exceeding the velocity of light.
A similar mechanism was implemented in the Born-Infeld modification of electrodynamics
with the aim to prevent the appearance of infinite electric fields.

One can expect that in the case of the linear action there may appear a space-time
regions which are unreachable by the test particles as in that regions the action develop an
imaginary part. If these ”locked” space-time regions happen to appear and if that space-
time regions include singularities, then one can expect that the gravitational singularities
are naturally excluded from the theory due to the fundamental principles of quantum
mechanics. The question of consistency of the new action principle, if it is the right one,
can only be decided by their physical consequences.
1 Appendix

The general form of the linear action has the form:

\[ S_L = -m_p c \int \frac{3}{8\pi} \sqrt{\sum_1^3 \eta_i K_i + \sum_1^4 \chi_i J_i + \sum_1^9 \gamma_i I_i} \sqrt{-g} d^4 x, \]

where the curvature invariants have the form

\[ I_0 = \frac{1}{12} R_{\mu \nu \lambda \rho} R^{\mu \nu \lambda \rho}, \quad I_1 = -\frac{1}{180} R_{\mu \nu \lambda \rho; \sigma} R^{\mu \nu \lambda \rho; \sigma}, \quad I_2 = +\frac{1}{36} R_{\mu \nu \lambda \rho} \Box R^{\mu \nu \lambda \rho}, \]

\[ I_3 = -\frac{1}{72} \Box (R_{\mu \nu \lambda \rho} R^{\mu \nu \lambda \rho}), \quad I_4 = -\frac{1}{90} R_{\mu \nu \lambda \rho; \alpha} R^{\alpha \nu \lambda \rho; \mu}, \quad I_5 = -\frac{1}{18} (R^{\alpha \nu \lambda \rho} R^\mu_{\nu \lambda \rho}; \mu; \alpha), \]

\[ I_6 = -\frac{1}{18} (R^{\alpha \nu \lambda \rho} R^\mu_{\nu \lambda \rho}; \alpha; \mu = I_3), \quad I_7 = \frac{1}{18} R^{\alpha \nu \lambda \rho} R^\mu_{\nu \lambda \rho; \alpha; \mu}, \quad I_8 = R^\mu_{\nu \lambda \rho; \mu} R^{\sigma \nu \lambda \rho}_{\rho; \sigma}, \]

\[ I_9 = R^{\alpha \nu \lambda \rho} R^\mu_{\nu \lambda \rho; \mu; \alpha}, \quad I_3 = I_5 = I_6 = 5 I_1 - I_2, \quad I_4 = I_1, \quad I_7 = I_2 \]

\[ J_0 = R_{\mu \nu} R^{\mu \nu}, \quad J_1 = R_{\mu \nu; \lambda} R^{\mu \nu; \lambda}, \quad J_2 = R^{\mu \nu} \Box R_{\mu \nu}, \quad J_3 = \Box (R^{\mu \nu} R_{\mu \nu}), \quad J_4 = R_{\mu \sigma} R^{\nu \sigma; \nu} \]

\[ K_0 = R^2, \quad K_1 = R_{; \mu} R^{; \mu}, \quad K_2 = R \Box R, \quad K_3 = \Box R^2. \]

The \( \eta_i, \chi_i \) and \( \gamma_i \) are free parameters. Some of the invariants can be expressed through others using Bianchi identities.