The hidden fluxes that control the stochastic closure of scalar fields

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Consistent description of scalars

Scalar fields are considered the "simplest" fields–but such simplicity is, only, apparent, since, while, as classical objects, they are well–understood, how these objects "emerge" from quantum fluctuations is, of course, not understood, beyond perturbation theory–and even then, the understanding is incomplete.
The discovery of the Higgs boson shows that it is possible to have scalar fields (there are more than one in the Standard Model!) appear on the same footing as other fields at a level where quantum fluctuations are relevant.
So how might it be possible to explore the consequences of their presence and the fluctuations they are subject to, in a way that doesn't rely on perturbation theory?

A change of variables: "The world, from another point of view"

How to realize this was found by Parisi and Sourlas (1982).
And has some quite striking consequences, as function of the number of spacetime dimensions.
The idea: Start from the partition function of a Gaussian field, with ultra–local 2–point function

$$Z = \int [\mathscr{D}\eta_I(x)]\, e^{-\int\, d^D x\, \frac{1}{2}\eta_I(x)\eta_J(x)\delta^{IJ}} = 1$$

and perform the following change of variables

$$\eta_I(x) = \sigma^\mu_{IJ}\partial_\mu\phi_J + \frac{\partial W}{\partial\phi_I} \qquad (1)$$

where the σ^μ generate a Clifford algebra

$$\{\sigma^\mu, \sigma^\nu\} = 2\delta^{\mu\nu}$$

Eq. (1) is known in the literature as the Nicolai map (1980), who introduced it for *supersymmetric* theories.

The superpartners hide among the partners

The partition function, after the change of variables, remains equal to 1 and takes the form

$$
\begin{aligned}
Z = & \int [\mathscr{D}\phi_I(x)] \\
& \left|\det \frac{\partial \eta_I(x)}{\partial \phi_J(x')}\right| \; e^{-\int d^Dx \, \frac{1}{2}\left(\sigma^\mu_{IK}\partial_\mu\phi_K + \frac{\partial W}{\partial \phi_I}\right)\left(\sigma^\nu_{JL}\partial_\nu\phi_L + \frac{\partial W}{\partial \phi_J}\right)\delta^{IJ}} = 1 \\
= & \int [\mathscr{D}\phi_I(x)]\,[\mathscr{D}\psi_I(x)]\,[\mathscr{D}\chi_I(x)] \\
& e^{\mathrm{i}\theta_{\det}} \, e^{-\int d^Dx \, \frac{1}{2}\left(\sigma^\mu_{IK}\partial_\mu\phi_K + \frac{\partial W}{\partial \phi_I}\right)\left(\sigma^\nu_{JL}\partial_\nu\phi_L + \frac{\partial W}{\partial \phi_J}\right)\delta^{IJ}} \times \\
& e^{\int d^Dx \, \psi_I(x)\left\{\sigma^\mu_{IJ}\partial_\mu + \frac{\partial^2 W}{\partial\phi_I \partial\phi_J}\right\}\chi_J(x)}
\end{aligned}
\tag{2}
$$

If the determinant is real, $e^{\mathrm{i}\theta_{\det}} = \mathrm{sign}[\det(\partial\eta/\partial\phi)]$ and we may recognize that this expression is nothing more or less than the Witten index, assuming periodic boundary conditions for all fields in all directions.
(There's a subtle issue about the representation of the Pauli/Dirac matrices, that will be discussed shortly.)

The classical action

$$S = \int d^D x \left\{ \frac{1}{2} \left(\sigma^{\mu}_{IK} \partial_\mu \phi_K + \frac{\partial W}{\partial \phi_I} \right) \left(\sigma^{\nu}_{JL} \partial_\nu \phi_L + \frac{\partial W}{\partial \phi_J} \right) \delta^{IJ} - \psi_I(x) \left(\sigma^{\mu}_{IJ} \partial_\mu + \frac{\partial^2 W}{\partial \phi_I \partial \phi_J} \right) \chi_J(x) \right\}$$

looks a bit familiar, but, also, quite weird. Whereas the fermionic action has the "expected" form, the action for the scalars looks decidedly not like the action one is accustomed to write for scalar fields, invariant under global SO(D) transformations. It is, however; and the full action is, also, invariant under supersymmetry transformations, that describe $\mathcal{N} = 2$ SUSY.
It looks, almost, like the action we would write for scalars interacting with gauge fields–but this isn't, quite, the case. Let's take a closer look.

A closer look at the action for the scalars

If we expand out the action for the scalars, we obtain

$$S[\phi_I] = \int d^D x$$
$$\left\{ (\partial_\mu \phi_I \partial_\nu \phi_J)\delta_{IJ}\delta^{\mu\nu} + \frac{1}{2}\frac{\partial W}{\partial \phi_I}\frac{\partial W}{\partial \phi_J}\delta_{IJ} + \sigma^\mu_{IK}\partial_\mu \phi_K \frac{\partial W}{\partial \phi_J}\delta_{IJ} \right\}$$

The first two terms are familiar–and to obtain the first, the property that the σ^μ generate a Clifford algebra is crucial. It's the last term that is of interest here. To get some intuition, let's see how it looks like in $D = 1$ and $D = 2$, first.

The new term in $D = 1$.

In $D = 1$, the worldvolume becomes a worldline and this term then becomes

$$\dot{\phi}\frac{dW}{d\phi}$$

in the simplest case, of a one–dimensional target space. This is a total derivative:

$$\frac{dW}{d\tau}$$

so it doesn't contribute to the equations of motion. On the lattice it becomes an infinite sum of irrelevant terms and total derivatives (cf. arXiv:1405.0820).

The new term in $D = 1$: Higher dimensional target space

The target space can have more than one dimensions, however. In that case

$$\dot{\phi}_I \frac{\partial W}{\partial \phi_I}$$

is the term that describes the coupling to a "gauge potential", $A_I = \partial W / \partial \phi_I$; for the term to be a total derivative, the corresponding field strength must vanish. This does happen, if the superpotential has continuous first derivatives, of course, which implies that the mixed second derivatives are equal.

So the interesting case for particles is that where the vector field, that appears in the Nicolai map, is *not* a gradient, but has a general Clebsch-Monge-Helmholtz-Hodge decomposition, into a "curl–free" and a "div–free" part (cf. arXiv:1912.12925 for a discussion).

The new term in $D > 1$:

The new term when $D > 1$ can be written as:

$$\sigma^{\mu}_{IK}\partial_{\mu}\phi_K\frac{\partial W}{\partial\phi_J}\delta_{IJ} = \partial_{\mu}\left\{\sigma^{\mu}_{JK}\phi_K\frac{\partial W}{\partial\phi_J}\right\} - \phi_K\partial_{\mu}\phi_M\sigma^{\mu}_{JK}\frac{\partial^2 W}{\partial\phi_J\partial\phi_M}$$

The first term is a total derivative–and can contribute flux, in the presence of boundaries; the second term seems to hint that it is of the form $\mathrm{Tr}\left[J\cdot A\right]$, since the current of scalar fields is, indeed, of the form $[J_{\mu}]_{KM} \sim \phi_K\partial_{\mu}\phi_M$. Of course for this statement to really make sense, beyond appearances, it is necessary to establish that what seems to be a gauge field is, indeed, a gauge field, i.e. that it transforms as a gauge field is expected to.

The new term in $D = 2$:

In $D = 2$ spacetime dimensions there exists a Majorana representation for the Pauli matrices–we may choose $\sigma^1 = \sigma_x$ and $\sigma^2 = \sigma_z$, so as to ensure that the action is (a) real and (b) bounded from below. One shouldn't forget that we're working in Euclidian signature! In this case the term takes the form

$$\sigma^{\mu}_{IK}\partial_{\mu}\phi_K\frac{\partial W}{\partial\phi_I} = \partial_1\phi_2\frac{\partial W}{\partial\phi_1} + \partial_1\phi_1\frac{\partial W}{\partial\phi_2} + \partial_2\phi_1\frac{\partial W}{\partial\phi_1} - \partial_2\phi_2\frac{\partial W}{\partial\phi_2}$$

It is here that the special properties of two dimensions enter the picture: If $W(\phi_1, \phi_2) = W(\Phi \equiv \phi_1 + \mathrm{i}\phi_2)$, then the expression displayed above is a total derivative. This is where holomorphicity of the superpotential seems to play a role. (Cf. arXiv:1712.07045 for a discussion of this point).

If W isn't a holomorphic function?

Then the term

$$\sigma^{\mu}_{IK}\partial_{\mu}\phi_{K}\frac{\partial W}{\partial\phi_{I}}$$

isn't a total derivative–unless $\phi_1(x,y)+\mathrm{i}\phi_2(x,y)\equiv\Phi(x,y)$ is a holomorphic function of the worldsheet. In that case, the Cauchy–Riemann equations imply that the term, in fact, vanishes.

So, if the term is a total derivative, or the fields are holomorphic functions of the worldvolume, this extra term doesn't contribute to the equations of motion and, if we choose periodic boundary conditions, its flux vanishes and only the usual terms of the classical action are relevant.

So what's new?

Probing SUSY implicitly

What's new is that the change of variables in the path integral, from the noise to the scalars

$$Z = \int [\mathscr{D}\phi_I(x)] \left|\det\frac{\partial\eta_I(x)}{\partial\phi_J(x')}\right| e^{-\int d^Dx\, \frac{1}{2}\left(\sigma^\mu_{IK}\partial_\mu\phi_K + \frac{\partial W}{\partial\phi_I}\right)\left(\sigma^\nu_{JL}\partial_\nu\phi_L + \frac{\partial W}{\partial\phi_J}\right)\delta^{IJ}} = 1$$

implies that the fluctuations of the scalars are described by

$$\left|\det\left(\sigma^\mu_{IJ}\partial_\mu + \frac{\partial^2 W}{\partial\phi_I\partial\phi_J}\right)\right|$$

that can be introduced in the action–up to the phase of the determinant–in the guise of the superpartners of the scalars and the supersymmetry is extended SUSY. It's the presence of the phase of the determinant, that implies that the partition function is, in fact, an index.
So what? Can we test this concretely? Yes!

A new set of observables

We can test the hypothesis that the fluctuations of the scalars are described by the absolute value of the determinant of the Jacobian from the noise to the scalars, by computing the correlation functions of the Nicolai map, namely

$$\eta_I(x) = \sigma^{\mu}_{IJ}\partial_\mu\phi_J + \frac{\partial W}{\partial \phi_I}$$

using the measure

$$[\mathscr{D}\phi_I(x)]\, e^{-\int d^Dx \left\{ \partial_\mu\phi_I\partial_\nu\phi_J\delta^{IJ}\delta^{\mu\nu} + \frac{1}{2}\frac{\partial W}{\partial\phi_I}\frac{\partial W}{\partial\phi_J}\delta^{IJ}\right\}}$$

that's perfectly suited to Monte Carlo simulations.

Recovering the noise from the scalar

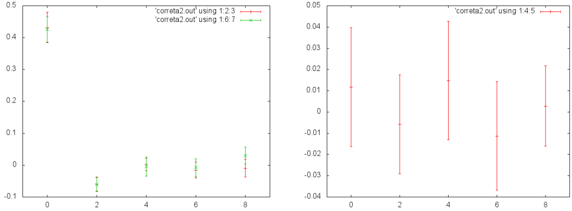
If we find that

$$\langle \eta_I(x) \rangle = 0$$
$$\langle (\eta_I(x) - \langle \eta_I(x) \rangle) (\eta_J(x') - \langle \eta_J(x') \rangle) \rangle = \delta_{IJ} \delta(x - x')$$

we will have strong hints that this idea has a chance of being correct and the study of the higher order cumulants is worth the effort,
What is interesting is that, if $\langle \eta(x) \rangle = 0$ to numerical precision, supersymmetry is realized; if it's non-zero, it's spontaneously broken; and, if the connected 2–point function isn't ultra–local, i.e. a $\delta-$function,up to lattice artifacts, there are anomalies (cf. for a 0–dimensional model in arXiv:1302.2361 with A. Zerkak).
In the present case we find that the noise function behaves as expected:

Typical results in D=2

Figure: Typical results for the cubic superpotential: $\langle \eta_n^I \eta_{n+d}^J \rangle$ for $I = J$ (left panel) and $I \neq J$ (right panel) and $d = 0, 2, 4, 6, 8$, on the 17×17 square lattice. $g^2_{\mathrm{latt}} = 0.7$. The diagonal noise term is a $\delta-$function, while the off-diagonal noise term vanishes, to numerical precision.



Taken from arXiv:1712.07045

How can supersymmetry describe the fluctuations of scalars beyond $D = 2$?

The problem is that, in Euclidian signature, the $\gamma-$matrices, even while Hermitian, have complex entries, unless $D \equiv 2 \bmod 8$. This was described by Parisi and Sourlas (1982) as the fact that it's not possible to define holomorphicity uniquely in $D > 2$. Cecotti and Girardello in subsequent work (1982–1983) found that the term

$$\sigma^{\mu}_{IJ}\partial_{\mu}\phi_J \frac{\partial W}{\partial \phi_I}$$

isn't a total derivative in $D > 2$, therefore contributes to the equations of motion.
However this isn't the real problem.

The real problem for $D > 2$ and how to resolve it

The reason this isn't the real problem is that this, apparently, extra term allows to write the Euclidian action as

$$S = \int d^D x \, \frac{1}{2} \left(\sigma^{\mu}_{IK} \partial_\mu \phi_K + \frac{\partial W}{\partial \phi_I} \right) \left(\sigma^{\nu}_{JL} \partial_\nu \phi_L + \frac{\partial W}{\partial \phi_J} \right) \delta_{IJ}$$

For $D > 2$, however, this action won't be real, since the σ^{μ}_{IJ} won't have exclusively real entries, since the σ^{μ} don't have a Majorana representation, unless $D \equiv 2 \bmod 8$.
The solution is immediate: Double the degrees of freedom!

Doubling

It's clear that

$$\eta_I(x) = \sigma^\mu_{IJ}\partial_\mu\phi_J + \frac{\partial W}{\partial\phi_I}$$

doesn't make much sense, even if the σ^μ are Hermitian, if they have complex entries, since these don't combine to produce a real number for $\eta_I(x)$.
So the solution is to introduce the $\eta_I^\dagger(x)$ and the $\phi_I^\dagger(x)$:

$$\eta_I(x)^\dagger = \sigma^\mu_{IJ}\frac{\partial\phi_J^\dagger}{\partial x^\mu} + \frac{\partial W^\dagger}{\partial\phi_I^\dagger}$$

and perform this change of variables in the path integral for the noise that, now, is defined as

$$Z = \int [\mathscr{D}\eta_I(x)][\mathscr{D}\eta_I^\dagger(x)]\, e^{-\int d^Dx\, \eta_I(x)\eta_J^\dagger(x)\delta^{IJ}}$$

Doubling and the particle spctrum

The consequences for the particle spectrum are:

- $D = 3$: All three Pauli matrices are now needed; doubling means that 4 (real) scalars, instead of 2, are required. And the fact that all three matrices are used, implies that chirality can't be defined.
- $D = 4$: Whereas one would think that 4 (real) scalars, at least, are needed, since the $\gamma-$matrices are four–dimensional, doubling implies that 8 (real) scalars are needed,at minimum. What is interesting is that this counting matches the counting, that appears to come from a totally independent chain of thought, when constructing the Standard Model and its supersymmetric extensions.

Conclusions and outlook

- The Nicolai map was originally proposed for describing supersymmetric theories in terms only of commuting variables; its relevance is much broader, however. For purely scalar theories it defines a new class of observables that can capture the fluctuations of the scalars.
- When using the stochastic approach, a "new" term appears, that isn't always, a total derivative, nor does it vanish identically. Its presence may lead to "explicit" supersymmetry breaking, with respect to the known terms. Its presence in $D > 2$ deserves further study, especially in the different phases the system can be found.
- Regarding further applications, it would be interesting to explore inflationary models in this setting (cf. talk by Skenderis).

Conclusions and outlook

- It should be kept in mind that, upon coupling this system to gauge fields, in the broken phase some of the scalars will become the longitudinal components of the gauge bosons. Therefore the stochastic identities for the scalars can be expressed as relations between the components of the gauge bosons and the Higgs and can lead to new ways of probing effects beyond the Standard Model.

Conclusions and outlook

- How about the fluctuations of the gauge fields? Beyond perturbation theory gauge fields probe the Lie group, not just the tangent manifold, the algebra. So it's necessary to understand how to write the Nicolai map for gauge fields. This is still an open problem, despite recent work by Nicolai et al. and Lechtenfeld et al.
- SUSY can be understood as describing the property that the system is consistently closed; its breaking implies that, either all the degrees of freedom haven't been taken into account; or that the degrees of freedom that do "restore" are different than the degrees of freedom of the system (e.g. they can be extended objects).
- There's still a lot to understand about SUSY!

Navigating in uncertain waters

The lights of India are strange
and they say that you don't see them first time round.
Nikos Kavvadias, "Kuro Siwo"