On the adiabaticity of emittance exchange due to crossing of the coupling resonance

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Introduction

The impact of linear coupling on transverse betatron motion has been extensively studied, as it has a peculiar impact already on the linear dynamics.

In 2001, the phenomenon of emittance exchange due to dynamic crossing of the difference coupling resonance was studied, with further results reported in 2007, in which it is mentioned that the full emittance exchange happens if the resonance crossing is adiabatic and an adiabatic condition is given.

This research has opened a new domain of investigations and a recent paper addressed the same topic with the goal to develop a complete theory to describe the emittance exchange process.

The most appropriate mathematical framework for these studies is the theory of adiabatic invariance for Hamiltonian systems.

This framework provides also the natural way of addressing the analysis of the resonance crossing in the presence of linear coupling and:

- All observations reported in previous work find a clear explanation using the results of adiabatic theory.
- The analysis can be extended to the case in which nonlinear detuning with amplitude is present.
The Hamiltonian system

The system under consideration is

\[ H(p_x, p_y, x, y) = \frac{p_x^2 + p_y^2}{2} + \frac{1}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 + 2q x y \right) \]

- With \( q = -\sqrt{\beta_x/\beta_y} \hat{q} \)
- \( \hat{q} = \frac{1}{2B\rho} \left( \frac{\partial B_y}{\partial y} + \frac{\partial B_x}{\partial x} \right) \)

- The analyses can be carried out in
  - Physical space
  - Normal mode space
The co-ordinate systems

Physical space
- The frequencies are not those around the closed orbit (due to the presence of linear coupling).
- The resonance can be crossed

\[ \delta(\lambda) = \omega_x(\lambda) - \omega_y(\lambda) \quad \lambda = \epsilon t, \; \epsilon \ll 1 \]

Normal modes space
- The frequencies are given by

\[ \omega_{1,2}^2 = \frac{\omega_x^2 + \omega_y^2 \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 4q^2}}{2} \]

- And satisfy

\[ \omega_1^2 + \omega_2^2 = \omega_x^2 + \omega_y^2 \]

\[ \omega_1^2 - \omega_2^2 = \sqrt{(\omega_x^2 - \omega_y^2)^2 + 4q^2} \]

\[ \omega_1 \omega_2 = \sqrt{\omega_x^2 \omega_y^2 - q^2} \]

Resonance cannot be crossed!

Condition for stability of closed orbit
The co-ordinate systems

Normal modes space

- The eigenvectors
  \[ \mathbf{v}_1(\lambda) = c_1 \left( \frac{\delta_2(\lambda) + \sqrt{\delta_2^2(\lambda) + 4q^2}}{2}, q \right) \]
  \[ \mathbf{v}_2(\lambda) = c_2 \left( -q, \frac{\delta_2(\lambda) + \sqrt{\delta_2^2(\lambda) + 4q^2}}{2} \right) \]

- For small q
  - \( \delta_2(\lambda) > 0 \) then \( \mathbf{v}_1 \rightarrow \text{ex} \) and \( \mathbf{v}_2 \rightarrow \text{ey} \)
  - \( \delta_2(\lambda) = 0 \) then \( \mathbf{v}_1 \) is the bisectrix of first quadrant, \( \mathbf{v}_2 \) is the bisectrix of second quadrant
  - \( \delta_2(\lambda) < 0 \) then \( \mathbf{v}_1 \rightarrow \text{ey} \) and \( \mathbf{v}_2 \rightarrow -\text{ex} \)

The passage through the resonance implies a change of direction of the eigenvectors

\[ \delta_2(\lambda) = \omega_x^2(\lambda) - \omega_y^2(\lambda) = \delta(\lambda)(\omega_x(\lambda) + \omega_y(\lambda)) \]
The prototype Hamiltonian system

Physical space

• A cascade of transformations
  • Action-angle variables
  • Resonant variables, i.e. with a slow phase $\phi_a = \theta_x - \theta_y$
  • Averaging over the fast variables
  • Use a perturbative approach up to order $O(q^2) + O(\epsilon^2)$

$$H(\phi, J, \lambda) = \delta(\lambda) J + q \sqrt{(1 - J) J} \sin \phi$$

We observe the effect of the linear coupling, which distorts the invariant.

Normal modes space

$$H(\phi, J, \lambda) = \gamma(\lambda) J + \epsilon \sqrt{(1 - J) J} \sin \phi$$

We observe the effect of the adiabaticity, which breaks the invariance of the action.
A comment on the prototype Hamiltonian system

Physical space

- The Hamiltonian has the form
  \[ H(\phi, J, \lambda) = \delta(\lambda)J + q\sqrt{1 - J}J \sin \phi \]

- Leading to
  \[
  \frac{dJ}{dt} = -q \frac{\partial H_1}{\partial \phi} \\
  \frac{d\phi}{dt} = \epsilon t + q \frac{\partial H_1}{\partial J}
  \]

- That becomes
  \[
  \frac{dJ}{d\bar{t}} = -\frac{\partial H_1}{\partial \phi} \\
  \frac{d\phi}{d\bar{t}} = +\frac{\epsilon}{q^2} \bar{t} + \frac{\partial H_1}{\partial J}
  \]

  with \( \bar{t} = q t \) and \( \bar{\lambda} = \epsilon / q^2 \bar{t} \)

  This defines a new adiabatic parameter.
The phase-space topology

\[ H(\phi, J, \lambda) = \delta(\lambda)J + q\sqrt{1 - J}J\sin\phi \]

Physical space

- The dynamics can be studied in the following co-ordinate systems
  - Action-angle
  - In \[ \mathcal{X} = \sqrt{2J}\sin\phi \quad \mathcal{Y} = \sqrt{2J}\cos\phi \] that gives \[ H(\mathcal{X}, \mathcal{Y}, \lambda) = \frac{\delta(\lambda)}{2}(\mathcal{X}^2 + \mathcal{Y}^2) + \frac{q}{2}\sqrt{2 - (\mathcal{X}^2 + \mathcal{Y}^2)}\mathcal{X} \]

- On a sphere. In this case one needs two charts
  - One to describe the motion in the southern hemisphere (corresponding to \( J=0 \))
  - One to describe the motion in the northern hemisphere (corresponding to \( J=1 \))
The phase-space topology

$q = 1$ and $\delta = 1$ (top)

$\delta = 0$ (center)

$\delta = -1$ (bottom)
The phase-space topology

\[ H(\phi, J, \lambda) = \gamma(\lambda) J + \epsilon \sqrt{(1 - J)} J \sin \phi \]

Normal modes space

- Similar considerations as for the Hamiltonian in physical space allow to state that the dynamics is analytic when the resonance is crossed.
- The Hamiltonian can be written as
  \[ H(\phi, J, \lambda) = H(J, \lambda) + \epsilon H_1(\phi, J, \lambda) \]
- A theorem by A. Neishtadt states that for an analytic system like the previous one, the variation of the action during the resonance crossing is given by
  \[ \Delta J = O(\exp(-c/\epsilon)) \]
- If one considers the re-scaled adiabatic parameter, then
  \[ \Delta J = O(\exp(-c q^2/\epsilon)) \]
Impact of detuning with amplitude

Normal modes space

- Detuning terms can be included in our models. In this case, the final prototype Hamiltonian is

\[ H(\phi, J, \lambda) = \gamma(\lambda)J + \hat{\alpha}(\lambda)J^2 + \epsilon \sqrt{J(1 - J)} \sin \phi \]

- where the detuning terms are hidden in \( \gamma \) and \( \hat{\alpha} \)

- The additional term in the Hamiltonian has deep implications: there can be hyperbolic fixed points in phase space, which implies the existence of separatrices that are singularities for the dynamics. Hence, the theorem by A. Neishtadt does not apply anymore and the exponential estimate is lost.
Numerical simulations

The model

• A simple one-turn map

\[
\begin{pmatrix}
X \\
X' \\
Y \\
Y'
\end{pmatrix}
_{n+1}
=
\begin{pmatrix}
R(\omega_x) & 0 \\
0 & R(\omega_y)
\end{pmatrix}
\begin{pmatrix}
X \\
X' + qY \\
Y \\
Y' + qX
\end{pmatrix}
_n
\]

• The figure of merit

\[
P_{na} = 1 - \frac{\langle I_{x,f} \rangle - \langle I_{x,i} \rangle}{\langle I_{y,i} \rangle - \langle I_{x,i} \rangle}
\]
Results

$q = -0.002$  \quad exp. fit
$q = -0.003$  \quad exp. fit
$q = -0.005$  \quad exp. fit

$P_{na}$ vs $10^7/\epsilon$

$\epsilon = 2 \times 10^{-6}$  \quad \square
$\epsilon = 4 \times 10^{-6}$  \quad \bullet

$\epsilon = \text{const.} \times q^2$

$P_{na}$ vs $10^{-3} |q|$
Conclusions

- Hamiltonian theory has been used to build a number of prototype systems representing the crossing of the linear coupling resonance.

- Adiabatic theory has been used to study in detail the process.

- Two descriptions (physical space or normal modes space) have been used.

- The exact adiabatic condition has been worked out.

- The singularities (separatrices) of the dynamics are the culprit of the observed behaviour.

- Small detuning terms might not spoil the exponential behaviour of emittance exchange.

- Periodic crossing of the linear coupling term is not reversible.

- Next step: emittance exchange when crossing 2D nonlinear resonances.
Thank you very much for your attention