Distribution amplitudes of pseudoscalar mesons

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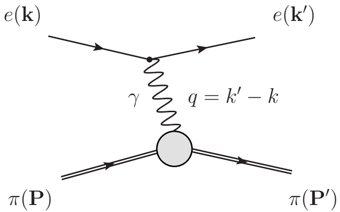
Introduction: Electromagnetic structure of hadrons

$\gamma^*\pi \to \pi$ elastic form factor

- Electron-hadron scattering is used to probe the (electromagnetic) structure of hadrons.
- The electron scatters from a quark by the electromagnetic interaction



$$\mathcal{M} = (-\mathrm{ie})\overline{u}(k')\gamma_\mu u(k)\frac{-i}{q^2}\langle\pi^+(P')|J_\mu|\pi^+(P)\rangle$$

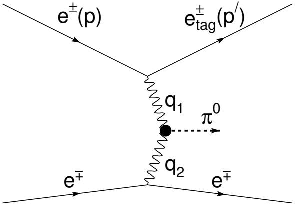
$$\langle\pi^+(P')|J_\mu|\pi^+(P)\rangle = (P'+P)_\mu F_\pi(Q^2)$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} \left|F_\pi(Q^2)\right|^2$$

- By measuring $\frac{d\sigma}{d\Omega} / \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}}$ we can obtain information on $F_\pi(Q^2)$.
- $F_\pi(Q^2)$ parametrises (our ignorance about) the distribution of charge inside the hadron.

$\gamma\gamma^* \to \pi^0$ transition form factor

- From C-symmetry the elastic FF of the π^0 vanishes.
- However, one can access the electromagnetic vertex behaviour of the π^0 in the reaction $\gamma\gamma^* \to \pi^0$ by studying the π^0 production process $e^+e^- \to e^+e^-\pi^0$



- The $\gamma\gamma^* \to \pi^0$ transition is described by the amplitude $T_{\mu\nu}$

$$T_{\mu\nu} \propto e^2 \epsilon_{\mu\nu\alpha\beta} \epsilon_1^\alpha \epsilon_2^\beta Q_1^\alpha Q_2^\beta G_{\pi\gamma\gamma^*}(Q_1^2, Q_2^2).$$

- $\frac{d\sigma}{dQ^2}$ for $e^+e^- \to e^+e^-\pi^0$ depends only on $G_{\pi\gamma\gamma^*}$.

- $G_{\pi\gamma\gamma^*}(Q^2, 0)$ has been measured at CELLO (1991), CLEO (1997) for $Q^2 = 0.7 - 2.2$ GeV^2 and $Q^2 = 1.6 - 8.0$ GeV^2, respectively.
- Recently it has been measured at BABAR (2009) and BELLE (2012) for $Q^2 = 4 - 40$ GeV^2.

Perturbative QCD predictions for FF

pQCD analysis of form factors (E-R,B-L, 1977-1979)

- pQCD yields predictions for pion elastic and transition form factors at asymptotically high energies. But how high?
- QCD factorisation property:

$$F_\pi(Q^2) = \int\int dx\, dy\, \phi_\pi(x,Q)\, T_B(x,y,Q)\phi_\pi(y,Q)$$

- The quark-gluon subprocesses are encoded in the hard-scattering amplitude $T_B(x,y,Q)$ and can be computed order-by-order in pQCD.
- The nonperturbative effects are absorbed into a universal pion distribution amplitude (PDA) $\phi_\pi(x,Q)$.

pQCD analysis of form factors (E-R,B-L, 1977-1979)

- pQCD yields predictions for pion elastic and transition form factors at asymptotically high energies. But how high?
- pQCD: Large Q^2. But how large?

$$
\begin{aligned}
T_B(x, y, Q) &= 16\pi C_F \frac{\alpha_s(Q^2)}{Q^2}\frac{1}{xy} \\
\phi_\pi(x) &= \phi_\pi^{asy}(x) = 6x(1-x)
\end{aligned}
$$

- pQCD prediction for $F_\pi(Q^2)$

$$
\begin{aligned}
Q^2 F_\pi(Q^2) &= 16\pi\alpha_s(Q^2) f_\pi^2 \left|\frac{1}{3}\int_0^1 \mathrm{d}x \frac{1}{x}\phi_\pi(x)\right|^2 \\
&= 16\pi\alpha_s(Q^2) f_\pi^2 .
\end{aligned}
$$

- The normalization of $F_\pi(Q^2)$ is controlled by $\left|\frac{1}{3}\int_0^1 \mathrm{d}x \frac{1}{x}\phi_\pi(x)\right|^2$.
- More about $\phi_\pi(x)$ later.

pQCD analysis of form factors (E-R,B-L, 1977-1979)

- QCD factorisation:

$$G_{\pi\gamma\gamma^*}(Q^2) = 4\pi^2 f_\pi \int_0^1 \mathrm{d}x\, T_H(x, Q^2, \alpha_s)\phi_\pi(x, Q)$$

- The quark-gluon subprocesses are encoded in the hard-scattering amplitude $T_H(x, y, Q)$ and can be computed order-by-order in pQCD.
- The nonperturbative effects are absorbed into the universal PDA $\phi_\pi(x, Q)$.
- pQCD:

$$T_H = \frac{e_u^2 - e_d^2}{xQ^2}.$$

$$\phi_\pi(x) = \phi_\pi^{asy}(x) = 6x(1-x).$$

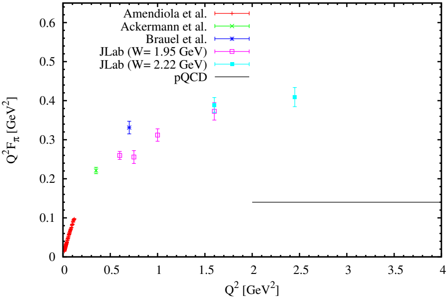
- pQCD prediction for $G_{\pi\gamma\gamma^*}(Q^2)$:

$$Q^2 G_{\pi\gamma\gamma^*}(Q^2) \to 4\pi^2 f_\pi.$$

- More about $\phi_\pi(x)$ later.

pQCD predictions for FF versus experimental data

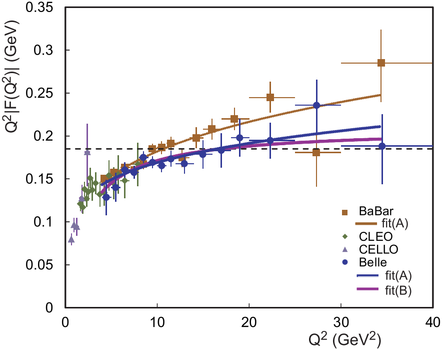
Comparision to experimental data



- There is a weak suggestion that $Q^2 F_\pi(Q^2) = \text{constant}$.
- Nonperturbative effects dominate; e.g. for $Q^2 = 2.45\text{GeV}^2$ $Q^2 F_\pi(Q^2) = 0.41\,\text{GeV}^2$; compare to $Q^2 F_\pi(Q^2) = 0.15$ using pQCD for $Q^2 = 4\text{GeV}^2$.
- Has $\phi_\pi(x)$ reached its asymptotic value ϕ_π^{asy} at present energies?
- We are awaiting for JLab 12 GeV upgrade experimental data.

Comparision to experimental data

- The pion TFF has also attracted a lot of attention since the publication of the BABAR data



- There is controversy in the BABAR data—strong scaling violation
- Computation of this quantity is of course very important (eg models of $\phi_\pi(x)$ can be tested using data on the transition FF)

Comparision to experimental data

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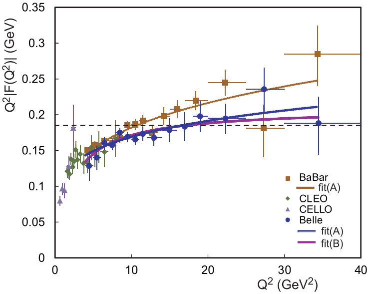

- Numerous attempts to explain BABAR, producing a TFF that exceeds the UV limit for $Q^2 > 10$ GeV^2
- Others argue that the BABAR data are not accurate measurements
- BELLE appears to be in general agreement with the asymptotic limit
- BELLE II (2018) may help to resolve these controversies.

Nonpertubative QCD, SDEs and all that

The rest of this presentation

- The PDA $\phi_\pi(x, Q^2)$ plays an important role in the theoretical description of many QCD processes (it is a universal function) (e.g. $\gamma^*\pi \to \pi$, $\gamma\gamma^* \to \pi^0$)

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  - Focus on the pion Distribution Amplitude (DA) $\phi_\pi(x)$
  - Present recent results for DA for heavy mesons and quarkonia
  - Present resulst for $\gamma^* + \pi \to \pi$ EFF and $\gamma\gamma^* \to \pi^0$ TFF (if there is time)

Quantum Chromodynamics—a reminder

- QCD is the fundamental theory of the strong interactions– quarks (spin 1/2 fermions), gluons (spin 1 gauge bosons), and their interactions
- It is a consistent QFT with a simple and elegant Lagrangian– based entirely on the invariance under the local non-Abelian $SU(3)$ colour gauge group and renormalisability
- QCD is a powerful tool in the description of large momentum transfer experiments (pQCD) due to its property of asymptotic freedom eg QCD backgrounds at the LHC
- Over the years QCD has become the accepted theory of the strong interactions at the fundamental level
- What about exclusive reactions?
- What about nonperturbative objects like the PDA?

Strong QCD

- The Lagrangian of QCD written on the blackboard does not by itself explain the data of strong interacting matter
- Furthermore, it is not clear how the plethora of the observed bound state objects, the hadrons, and their properties arise from the fundamental quarks and gluons of QCD.
- Emergent Phenomena (not seen in $\mathcal{L}_{QCD}$)
  - Confinement (means that) quarks and gluons cannot be removed from hadrons and studied in isolation.
  - Dynamical Chiral Symmetry Breaking (is) responsible for the existence of light pions and the generation of quark masses via interactions.
- Neither DCBS nor Confinement can be accounted for in pQCD, and are therefore genuine effects of strong QCD .
- Both can be studied in the functional approach to hadron physics, ie SDE-BSE, in particular the SDE for the quark propagator.

QCD generating functional (quarks, gluons, ghosts)

- Schwinger-Dyson equations (SDE) provide a nonperturbative formulation of QCD
- Their derivation makes no assumption about α_s

$$\mathcal{Z}[J,\xi,\xi^*,\overline{\eta},\eta] = \int \mathcal{D}A\mathcal{D}\overline{\eta}\mathcal{D}\eta\mathcal{D}\xi\mathcal{D}\xi^* \exp\left\{ \mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L}_{\text{eff}} + \text{Sources}\right)\right\},$$

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{G}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} + \mathcal{L}_{\text{F}},$$

$$\text{Sources} = A^a_\mu J^{a\mu} + \chi^{a*}\xi^a + \xi^{a*}\chi^a + \overline{\psi}\eta + \overline{\eta}\psi,$$

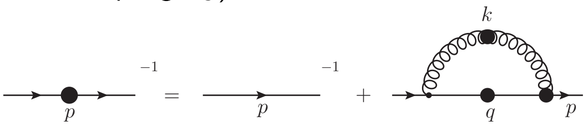
where

$$\mathcal{L}_{\text{G}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a, \qquad \mathcal{L}_{\text{GF}} = -\frac{1}{\xi}\left(\partial^\mu A^a_\mu\right)^2,$$

$$\mathcal{L}_{\text{FP}} = \left(\partial^\mu\chi^a\right)^* D^{ab}_\mu\chi^b, \qquad \mathcal{L}_{\text{F}} = \sum_{k=1}^{N_f}\overline{\psi}_k\left(\mathrm{i}\gamma_\mu D^\mu - m_k\right)\psi_k$$

Schwinger-Dyson Eqns—non pert QCD formulation

- The SDEs are the eqns for the Green functions of the theory.
- The SDEs provide a nonperturbative formulation of QCD in the continuum.
- The QCD quark propagator (a 2-point function) SDE eqn (valid for any value of the coupling α_S).



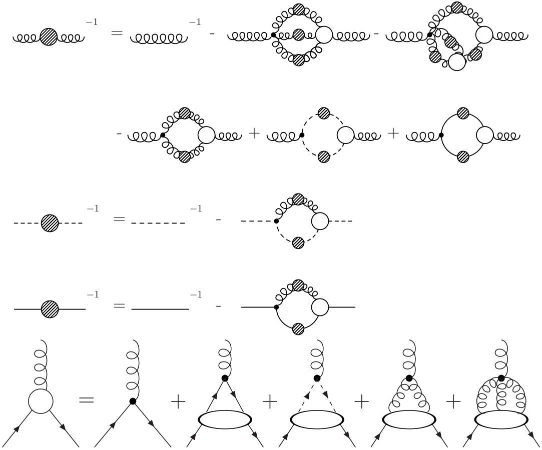
$$S_F{}^{-1}(p) = Z_2 \left[S_F^{\text{bare}}(p) \right]^{-1} - \Sigma(p)$$

$$\Sigma(p) = Z_{1F} \mathrm{i} g^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} D^{\mu\nu}(k) t^a \gamma_\mu S_F(q) \Gamma^a_\nu(p,q;k)$$

- $D_{\mu\nu}$ propagator, Γ^a_ν the quark-gluon vertex

SDEs form an infinite set of coupled equations

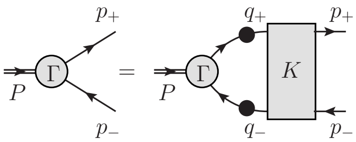
- The SDEs form an infinite tower of coupled nonlinear integral eqns.



- There is a need to introduce a truncation scheme.

Hadron physics—real-world QCD

- Quarks and gluons are not free so we must study bound state's properties to test our understanding of npQCD—and explain/understand hadron physics
- Mesons are bound states of a nonperturbative quark-antiquark pair (the BSE)



$$[\Gamma^H(p;P)]_{tu} = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} [K(p,q;P)]_{tu;rs} [S^a(q_+)\Gamma^H(q;P)S^b(q_-)]_{sr}$$

- The quark propagators are solns to the quark SDE.
- How do we determine $K(p,q;P)$?
- Need to determine the quark SDE kernel that appears in Σ first.

Hadron physics—Axial-Vector Ward-Takahashi identity

- How do we determine K? Look at the chiral symmetry breaking ppties of QCD.
- The AxWTI relates $K(p,k;P)$ in the BSE to $\Sigma(p)$ in the quark SDE:

$$[\Sigma(p_+)\gamma_5 + \gamma_5\Sigma(p_-)]_{tu} = \int \frac{d^4q}{(2\pi)^4} K^{rs}_{tu}(p,q;P)[\gamma_5 S(q_-) + S(q_+)\gamma_5]_{sr}$$

- A truncation in Σ implies a truncation in K.
- The chiral symmetry breaking pattern of QCD guarantees a massless pion in the chiral limit when DCSB occurs (Goldstone theorem!)
- When chiral symmetry is explicitly broken, it ensures a light pion

The Quark propagator—pQCD solution

- General form of the quark propagator:

$$S^{-1}(p) = \mathrm{i}\not{p}A(p^2,\mu^2) + B(p^2,\mu^2) = Z^{-1}(p^2,\mu^2)\left[\mathrm{i}\not{p} + M(p^2)\right],$$

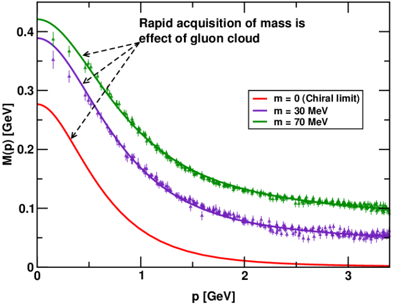
$$M(p^2) = \frac{B(p^2)}{A(p^2)};\ Z(p^2) = A^{-1}(p^2)$$

- pQCD prediction (i.e. perturbative truncation of the quark SDE)

$$M(p^2) = m\left[1 - \frac{\alpha}{\pi}\ln\left(\frac{p^2}{m^2}\right) + \cdots\right];\ \lim_{m\to 0} M(p^2) = 0$$

- It is always true that at any order in perturbation theory there is no dynamical mass generation, i.e. $M(p^2) = 0$.

The Quark propagator—Nonperturbative solution



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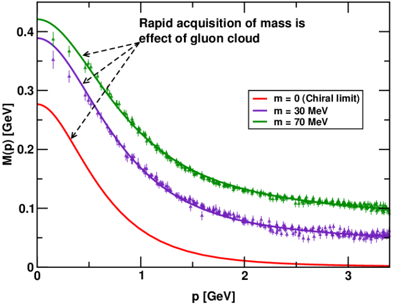

- DCSB IS realized in the nonperturbative solutions.

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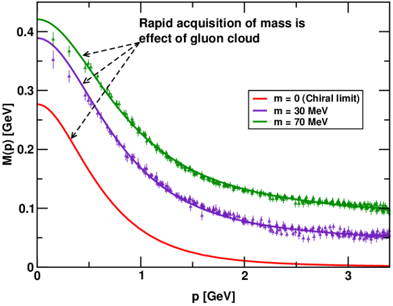

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- DCSB is more important for light quarks.

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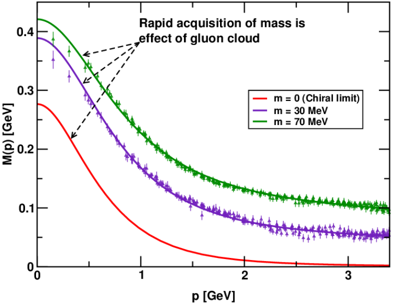

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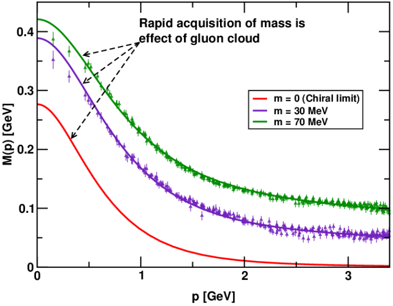
- The $M(p^2)$ connects the current-quark mass (UV) to a constituent-like quark mass (IR)—in agreement with pQCD

The Quark propagator—Nonperturbative solution



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- DCSB is more important for light quarks.
- The $M(p^2)$ connects the current-quark mass (UV) to a constituent-like quark mass (IR)—in agreement with pQCD
- DCSB is the most important mass generating mechanism for visible matter in the Universe (98% of the proton's mass).

Hadron physics from SDEs: PDAs and FFs

Hadron physics from SDEs (mass spectrum, decay constants, FFs, etc)

- A meson is characterized by its Dirac structure.
- In the $J^P = 0^-$ channel the lowest mass solution is the π

$$\Gamma_\pi(k;P) = \gamma_5[\mathrm{i}E_\pi(k;P) + \not{P}F_\pi(k;P)\not{k}(k\cdot P)G_\pi(k;P) + \sigma_{\mu\nu}k_\mu P_\nu H_\pi(k;P)$$

- By solving the BSE we obtain the meson mass spectrum (including excited states) m_π and its BS amplitude $\Gamma_\pi(k;P)$
- A simple observable: The pseudoscalar leptonic decay constant $(\pi^+ \to \mu^+ + \nu_\mu)$

$$f_\pi P_\mu = Z_2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \mathrm{Tr}\left[\gamma_5\gamma_\mu S^a(q_+)\Gamma_\pi(q;P)S^b(q_-)\right]$$

- $S(q)$ and $\Gamma_\pi(q;P)$ are solutions to the SDE and BSE, respectively.

Hadron physics from SDEs

- Recall pion DA $\phi_\pi(x)$ that appears in $F_\pi(Q^2)$?

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- The leading twist PDA:

$$f_\pi \phi_\pi(x) = Z_2 N_c \int \frac{d^4k}{(2\pi)^4} \delta(n\cdot k - x n\cdot P) \mathrm{Tr}\left[\gamma_5 \not{n} S(k) \Gamma_\pi(k;P) S(k-P)\right]$$

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- Calculate its moments $\langle x^m \rangle = \int_0^1 \mathrm{d}x x^m \phi_\pi(x)$:

$$f_\pi \langle x^m \rangle = \frac{Z_2 N_c}{(n\cdot P)^{m+1}} \int \frac{d^4k}{(2\pi)^4} (n\cdot k)^m \,\mathrm{Tr}\left[\gamma_5 \not{n} S(k) \Gamma_\pi(k;P) S(k-P)\right]$$

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- We can compute arbitrarily many of them [PRL 110, 132001 (2013)]

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- We can compute arbitrarily many of them [PRL 110, 132001 (2013)]
- $\phi_\pi(x)$ is then reconstructed from its moments $\langle x^m \rangle$–see PRL 110, 132001 (2013) for details

Hadron physics: $\phi_\pi(x)$ reconstruction at 2 GeV

- $\phi_\pi(x)$ is then reconstructed from its moments $\langle x^m \rangle$
- $\phi_\pi(x)$ can be parametrised as

$$\phi_\pi(x,\mu) = 6x(1-x)[1 + \sum_{n=2,4,\dots} a_n(\mu) C_n^{3/2}(2x-1)]$$

- It is more efficient to fit the moments with order α Gegenbauer pols

$$\phi_\pi(x,\mu) = N(\alpha)\,[x(1-x)]^{\alpha-1/2}\,[1 + \sum_{n=2,4,\dots} a_n^\alpha(\mu) C_n^\alpha(2x-1)]$$

- For our (RL) model $\alpha_{\mathrm{RL}} = 0.79$, $a_2^{\mathrm{RL}} = 0.0029$—This is sufficient to get a very accurate fit.
- We then project onto order $\alpha = 3/2$ Gegenbauers: $a_2^{3/2} = 0.23, \dots,$ $a_{14}^{3/2} = 0.022$—This underscores the merit of reconstruction via Gegenbauer-α pols (many $a_n^{3/2}$ are needed!).

Comparison to Lattice QCD: $\phi_\pi(x)$ at 2 GeV

- Lattice QCD can calculate only one nontrivial moment:

$$\begin{aligned} \langle (2x-1)^2 \rangle &= 0.27 \pm 0.04 \ \text{(Braun et al, 2006)} \\ \langle (2x-1)^2 \rangle &= 0.24 \pm 0.01 \ \text{(Braun et al, 2015)} \end{aligned}$$

- Our RL and DB results give

$$\langle (2x-1)^2 \rangle^{\text{RL}} = 0.28, \ \ \langle (2x-1)^2 \rangle^{\text{DB}} = 0.25$$

- While the asymptotic PDA $[6x(1-x)]$ gives

$$\langle (2x-1)^2 \rangle^{\text{Asy}} = 0.2$$

Hadron physics: $\phi_\pi(x, Q^2)$ scale evolution

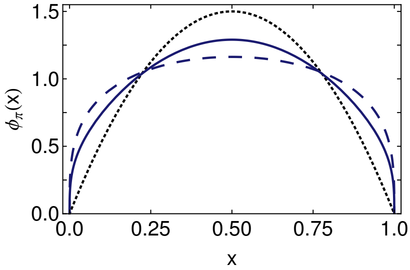
- The Q^2 scale evolution can be computed from pQCD (similar to DGLAP equation for PDFs).
- At leading order, the expansion coefficients evolve according to

$$a_n^{3/2}(\mu) = a_n^{3/2}(\mu_1)\left[\frac{\alpha_s(\mu_1)}{\alpha_s(\mu)}\right]^{\gamma_n^{(0)}/\beta_0}, \quad \alpha_s(Q^2) = (4\pi/\beta_0)\ln^{-1}\left(Q^2/\Lambda^2_{\mathrm{QCD}}\right),$$

$$\beta_0 = 11 - (2/3)N_f, \quad \gamma_n^{(0)} = \frac{4}{3}\left[3 + \frac{2}{(n+1)(n+2)} - 4\sum_{k=1}^{n+1}\frac{1}{k}\right]$$

- Recall $a_2^{3/2}(2\,\mathrm{GeV}) = 0.23, \ldots, a_{14}^{3/2}(2\,\mathrm{GeV}) = 0.022$.
- It is necessary to evolve to $\mu = 100\,\mathrm{GeV}$ before $a_2^{3/2}(2\,\mathrm{GeV})$ falls to 50 % of its value!($N_f = 4$, and $\Lambda_{\mathrm{QCD}} = 0.234\,\mathrm{GeV}$)
- The asymptotic limit $\phi_\pi^{asy}(x) = 6x(1-x)$ is a poor approximation to the PDA at currently accessible energies.

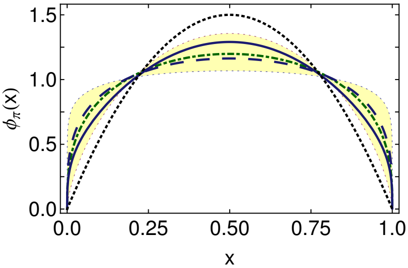
Hadron physics: $\phi_\pi(x)$ [PRL 110, 132001 (2013)]



- RL result (dashed curve)
- DB result (solid curve)
- Asymptotic PDA (dotted curve), $\phi_\pi^{asy}(x) = 6x(1-x)$

- Need to evolve to $\mu = 100\,\text{GeV}$ before $a_2^{3/2}(2\,\text{GeV})$ reduces 50 %.
- The asymptotic domain (ϕ^{asy}) lies at very very large momenta.
- The dilation is an expression of dynamical chiral symmetry breaking on the light front.

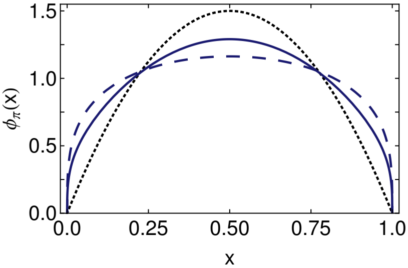
Comparison with Lattice QCD [PRL 111, 092001 (2013)]



- RL result (dashed curve); DB result (solid curve); Asymptotic PDA (dotted curve)
- Lattice QCD (dot-dashed curve)

- The shaded region indicates the extremes allowed by the errors $\langle(2x-1)^2\rangle$ from Lattice QCD
- The Lattice results are linearly extrapolated from large (20-50 times the empirical pion mass) pion masses
- The Lattice result favors the SDE results
- It is anticipated that improved lattice simulations will produce a PDA in better agreement with the DB results

Back to the perturbative analysis of $F_\pi(Q^2)$



- RL result (dashed curve)
- DB result (solid curve)
- Asymptotic PDA (dotted curve), $\phi_\pi^{asy}(x) = 6x(1-x)$

- In evaluating $Q^2 F_\pi$, ϕ_π has not yet reached its asymptotic value:

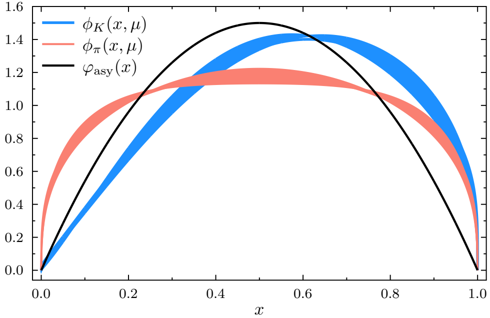
$$\left| \frac{1}{3} \int_0^1 \mathrm{d}x \frac{1}{x} \phi_\pi^{\mathrm{asy}}(x) \right|^2 = 1, \qquad \left| \frac{1}{3} \int_0^1 \mathrm{d}x \frac{1}{x} \phi_\pi(x, 2\,\mathrm{GeV}) \right|^2 = 3.2$$

- The pQCD result has to be multiplied by 3.2 (at the scale of 2 GeV) and the asymptotic analysis of various models have to be compared to this new result—the normalization of F_π runs with Q^2.

Distribution amplitudes of heavy mesons and quarkonia

(Eur.Phys.J.C 80 (2020) 10, 955, arXiv: 2008.09619 [hep-ph])

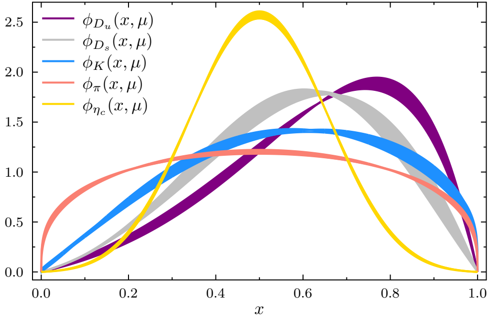
Distribution amplitudes of heavy mesons and quarkonia



- Asymptotic PDA (black curve), $\phi_{asy}(x) = 6x(1-x)$
- Pion PDA (red curve), $\phi_\pi(x)$
- Kaon PDA (blue curve), $\phi_K(x)$

- The Kaon PDA $\phi_K(x)$ is concave as is $\phi_\pi(x)$
- $\phi_K(x)$ is an asymmetric function in x: $\phi_K(x) \neq \phi_K(1-x)$–it peaks at $x \simeq 0.61$
- This is a clear signal of dynamical SU(3) symmetry breaking, where the heaviest valence quark in the the kaon carries a greater amount of the meson momentum

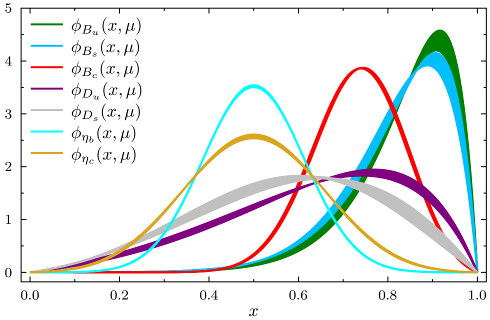
Distribution amplitudes of heavy mesons and quarkonia



- Pion PDA (red curve), $\phi_\pi(x)$
- Kaon PDA (blue curve), $\phi_K(x)$
- D_u PDA (purple curve), $\phi_{D_u}(x)$
- D_s PDA (gray curve), $\phi_{D_s}(x)$
- η_c PDA (yellow curve), $\phi_{\eta_c}(x)$

- D_u and D_s PDAs are not anymore concave–they are convex-concave
- The charm quark carries most of the fractions of the meson's momentum in both the D_u and D_s
- $\phi_{D_u}(x)$ is slightly more asymmetric and peaks at higher x than $\phi_{D_s}(x)$z–$x = 0.76$ and $x = 0.63$, becayse $m_c - m_u < m_c - m_s$
- This stands in constrast to the PDA of the η_c, which is symmetric about at $x = 0.5$, though much more sharply peak than ϕ_{asy}

Distribution amplitudes of heavy mesons and quarkonia



- B_u PDA (green curve), $\phi_{B_u}(x)$
- B_s PDA (blue curve), $\phi_{B_s}(x)$
- B_c PDA (red curve), $\phi_{B_c}(x)$
- η_b PDA (cyan curve), $\phi_{\eta_b}(x)$

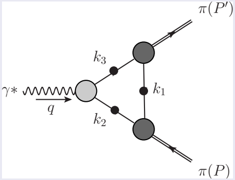
- B_u and B_s PDAs are extremely assymetric–the heavy valence quark inside the B_u and the B_s carries almost all of the meson's momentum
- $\phi_{B_u}(x)$ and $\phi_{B_s}(x)$ peak at $x = 0.92$ and $x = 0.90$
- $\phi_{B_c}(x)$ is less assymetric than B_u and B_s PDAs–it peaks at $x = 0.74$
- $\phi_{\eta_b}(x)$ is, as expected, narrower than $\phi_{\eta_c}(x)$

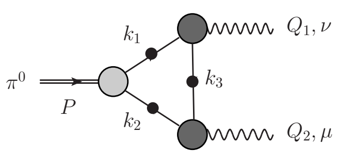
Elastic and transition form factors

Hadron physics—real-world QCD

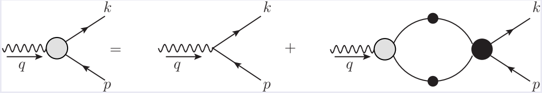
EM interactions of hadrons: the nonperturbative quark-photon vertex

- Once we have $S_{u/d}(k)$, m_π, $\Gamma_\pi(k;P)$ we can calculate anything that involves these objects, e.g. $F_\pi(Q^2)$, $G_{\pi\gamma\gamma^*}(Q^2)$:





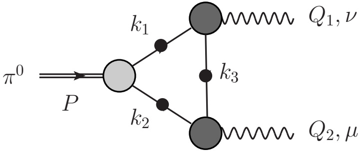
- Quark electromagnetic interaction: quark-photon vertex SDE



- The quark-photon vertex $\Gamma_\mu(k,p;Q)$ is constrained by the WTI of QED.

The pion transition form factor [PRD 93, 074017 (2016)]

- Once we have $S_{u/d}(k)$, m_π, $\Gamma_\pi(k;P)$ we can calculate anything that involves these objects.

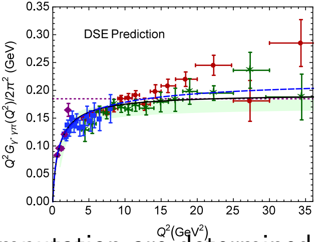


$$T_{\mu\nu}(Q_1, Q_2) = \frac{e^2}{4\pi^2}\epsilon_{\mu\nu\alpha\beta}\epsilon_1^\alpha\epsilon_2^\beta Q_1^\alpha Q_2^\beta G_{\pi\gamma\gamma^*}(Q_1^2, Q_2^2)$$
$$= \mathrm{Tr}\int [S(q_2)\Gamma^\pi(q_2, q_1)S(q_1)i\Gamma_\mu(q_1, k)S(k)i\Gamma_\nu(k, q_2)]$$

- $S(q)$ and $\Gamma_\pi(q;P)$ are solutions to the SDE and BSE, respectively.

The pion transition form factor [PRD 93, 074017 (2016)]

- SDE prediction [PRD 93, 074017 (2016)]



- All elements in the computation are determined by solutions to QCD SDE.
- The SDE prediction is a member of a class of studies (shaded band) which are consistent with all non-BABAR data and confirm the standard QCD factorisation result.
- The BaBar data cannot be reconciled with QCD factorisation.
- New measurements (Belle II) may help resolve the controversies around the BaBar data.

The pion form factor

- Once we have $S_{u/d}(k)$, m_π, $\Gamma_\pi(k;P)$ we can calculate anything that involves these objects.

$$\Lambda^\pi_\mu(P,P';q) \equiv \langle \pi(P')|J_\mu|\pi(P)\rangle = (P+P')_\mu F_\pi(Q^2),$$

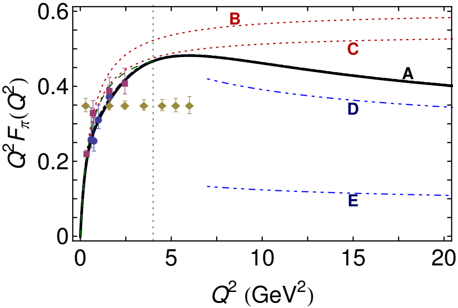
- Flavor decompostion

$$\Lambda^\pi_\mu(P,P';q) = \hat{Q}^u \Lambda^{\pi,u}_\mu(P,P';q) + \hat{Q}^{\bar{d}} \Lambda^{\pi,\bar{d}}_\mu(P,P';q)$$

$$\Lambda^{\pi,\bar{d}}_\mu(P,P';q) = N_c \int d\ell \, \mathrm{Tr} \Big[S^u(k_1)\Gamma_\pi(k_1,k_2;P)S^d(k_2) \\ i\Gamma^d_\mu(k_2,k_3;q)S^d(k_3)\overline{\Gamma}_\pi(k_3,k_1;-P') \Big]$$

The pion form factor [PRL 111, 141802 (2013)]

- A: New computation of F_π (any Q^2).
- B: Monopole parametrisation; C: Old NBF computation (up to 4 GeV).



- E: Perturbative normalization.
- D: Nonperturbative normalization.

Summary and outlook

- The SDE-BSEs are well suited to the study of hadrons as composites of dressed quarks and gluons:
  - Static properties (light and heavy mesons): masses, decay constants.
  - Electromagnetic and Transition Form Factors (light and heavy mesons).
  - Parton distribution functions (PDFs).
  - Parton distribution amplitudes (PDAs).
  - Generalized parton distrubution functions (GPDs), etc.
  - QCD at finite temperature and chemical potential (QCD phase diagram).
- Due to their infinite-set nature a truncation must be introduced.
- There is an interconnection between truncation schemes and symmetries.
- Phenomenology and Lattice QCD provide valuable information to design a robust truncation scheme.