3-Dimensional BF Theory on Seifert Fibered Manifold

Tony Mbambu Kakona

ICTP, Trieste

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Outline (1/)

A bit of History

- **1988** Andreas Floer: $H_1(M, \mathbb{Z}) = 0$ twice Casson's invariant;
- **1988** Edward Witten:Jones polynomial, S^3 to any 3-manifold;
- **1991** Nicolai Reshetikhin and Vladimir Turaev;
- **1993** Matthias Blau and George Thompson: Abelianisation ;
- **1994** J.M.F.Labastida and M. Mariño: HOMFLY polynomial.
- **2005** Chris Beasley, Edward Witten: κ, $d\kappa = c_1(\mathfrak{L}_M)\,\pi^{\star}\omega$;
- **2006** M. Blau and G.Thompson: Abelianisation, S^1 over Σ;
- **2013** M. Blau and G.Thompson: Abelianisation, Seifert;
- **2017** When $c_1(\mathfrak{L}_M) = 0$, Mapping Torus, how to handle?

Chern-Simons Theory (1/)

Let $P \to M$ be a trivial principal G-bundle over M and a connection A on P taking value on $\mathfrak{g}$ of G; $\mathcal{A}$ the space of all connections on P.

$$\begin{aligned} A = A^a_\mu(x) T_a dx^\mu, \;\; A \in \Omega^1(M, (g)) \\ F_A = dA + A^2 \in \Omega^2(M, (g)) \end{aligned} \tag{1}$$

The Chern-Simons action is

$$S_{CS}[A] = \frac{k}{4\pi} \int_M Tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{2}$$

where Tr is in the fundamental representation of the Lie algebra $\mathfrak{g}$ for $SU(N)$: $\text{Tr}\, T_a T_b = 2\delta_{ab}$.

Chern-Simons Theory (2/)

Gauge Transformation: the action (2) is **not** gauge invariant!

- $S_{CS}[A^g] = S_{CS}[A] + \frac{1}{4\pi} \int_{\partial M} Tr\left(A \wedge g^{-1} dg\right) + 2\pi\omega(g)$

with $\omega(g) = \frac{1}{24\pi^2} \int_M \text{Tr}\left(g^{-1}dg\right)^3 \in \mathbb{Z}$
There for M such that $\partial M = 0$,

- $S_{CS}[A^g] = S_{CS}[A] + 2\pi n, \quad n \in \mathbb{Z}$

In QFT, we are interested in $\exp\left(ikS_{CS}[A]\right)$.

Fix $k \in \mathbb{Z}$

The corresponding Partition Function is gauge invariant,

$$Z\left[M, G, k\right] = \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A \exp\left(ikS_{CS}[A]\right) \tag{3}$$

Chern-Simons Theory (3/)

Gauge invariant **observables**: Wilson lines,

$$W[A, R, C] = \mathbf{T}r_R \left(P \exp \left(\int_C A \right) \right) \quad (4)$$

Equations of Motion and The Moduli Spaces of Flat Connections

Chern-Simons Theory (4/)

The equation of motion for A, tells us

- $F_A = 0$ The stationary points of the action are **flat connections**
- Under gauge transformation g, $F_{A^g} = g^{-1} F_A g$. Therefore, the space of solutions or the **moduli space** is
- $\mathcal{M}^G = \{A \in \mathcal{A} : F_A = 0\}/\mathcal{G}$ For each flat connection A and a loop $\gamma : s^1 \longrightarrow M$ one has an holonomy
- $Hol_\gamma(A) \equiv Pe^{\int_\gamma A}$ This determines a representation $\rho : \pi_1(M) \longrightarrow G$ and a map between the moduli space of flat connections and the set of homomorphisms of $\pi_1(M)$ into the gauge group G modulo gauge transformations. Hence, one has
- $M^G \equiv Hom(\pi_1(M), G)/G$

Path Integral Quantization

Consider **isolated** and **irreducible** flat connections so that there no ghost zero modes. Then for $A_k = A + \frac{1}{\sqrt{k}}\omega$ the path integral with observables is defined as

- $Z[\mathcal{O}_i, k] = \sum_a \int D\omega \, \exp\left(ikS_{CS}(A_a + \frac{1}{\sqrt{k}}\omega) + iS_{gf}\right) \prod_i \mathcal{O}_i(A_a + \frac{1}{\sqrt{k}}\omega)$;
- $S_{gf} = \mathrm{Tr} \int_M \left(b * d_{A_a} * \omega + \overline{c}\, d_{A_a} * d_{A_a + \frac{1}{\sqrt{k}}\omega}\, c\right)$;
- $Z[k] = \sum_a \exp\left(ikS_{CS}(A_a)\right). \frac{\mathrm{Det}\Delta_{A_a}|_{\Omega^0(M,\mathfrak{g})}}{\mathrm{Det}^{1/2} D_{A_a}|_{\Omega^1(M,\mathfrak{g}) \oplus \Omega^0(M,\mathfrak{g})}} \left(1 + O\left(\frac{1}{k}\right)\right)$

BF Theory and The TG group (1/)

BF limit of Chern-Simons

- $G \longrightarrow TG$
- TG is an example of the non-compact group $G \ltimes V$

- Let V be a G-module, i.e. a real vector space carrying a representation $\rho : G \to GL(V)$ of the group G.
- V is an Abelian group with respect to addition: $v, \omega \in V \to v + \omega \in V$
- In particular, the group structure on the G-module V allows us to equip the product (as a set) G with a semi-direct product group structure $G \ltimes V \equiv G \times_\rho V$, with $(g, v).(h, \omega) = (gh, \rho(g)\omega + v)$

BF Theory and The TG group (2/)

- The Abelian group (e, V) is a normal Abelian subgroup of $G \times_\rho V$.
- The Lie algebra takes the form Lie($G \times_\rho V$)$= \mathfrak{g} \otimes_\rho V_{ab}$ with $[(x, v), (y, \omega)] = ([x, y], \rho(x)\omega - \rho(y)v + v - \omega)$
- Example 1 :TG (or T*G), where $V = \mathfrak{g}$ the Lie algebra of G: $G \ltimes \mathfrak{g}$,
- Example 2 :$ISO(p, q) = SO(p, q) \ltimes \mathbb{R}^{p+q}$

BF Theory and The TG group (3/)

Invariants Scalar Products on The Lie Algebra $\mathfrak{tg}$

- Lie (TG) $\equiv \mathfrak{tg} \simeq \mathfrak{g} \otimes_{ad} \mathfrak{g}_{ab}$
- let l_a be a basis of $\mathfrak{g}$, with $[l_a, l_b] = f_{ab}^c l_c$
- Then $j_a = (l_a, 0)$ and $p_a = (0, l_a)$ are basis of $\mathfrak{tg}$
- Commutation relations are: $[j_a, j_b] = f_{ab}^c j_c$, $[j_a, p_b] = f_{ab}^c p_c$, $[p_a, p_b] = 0$
- For gsimple, the Cartan-Killing: $B(x, y) = \text{Tr}(ad_x \, ad_y)$
- In terms of the basis l_a, one has $g_{ab} = \text{Tr} \, ad_{l_a} \, ad_{l_b}$.
- The scalar product on $\mathfrak{g}$ is $< x, y >= \frac{1}{2} B(x, y)$ with $< l_a, l_b >= g_{ab}$

BF Theory and The TG group (4/)

Two invariant Scalar Products on $\mathfrak{tg}$

- 1. $< j_a, j_b >= 2g_{ab}$, $< j_a, p_b >=< p_a, p_b >= 0$
- 2. $<< j_a, j_b >>=<< p_a, p_b >>= 0$, $<< j_a, p_b >>= g_{ab}$

TG Gauge Theory: connections, gauge transformations and curvatures

- $\mathbb{A} = (A, B) = j_a A^a + p_a B^a$. Under gauge transformation, (g,v) it transforms as
- $\mathbb{A} \to \mathbb{A}^{(g,v)} = \left(g^{-1}Ag + g^{-1}dg, g^{-1}(B + d_A v)g\right)$
- The curvature is $\mathbb{F}_{\mathbb{A}} = (F_A, d_A B) = j_a F^a_A + p_a (d_A B)^a$
- G Chern-Simons Theory $(<,>)$:
  $S_{CS}[A] = \frac{1}{2}\int_M < \mathbb{A}, d\mathbb{A} + \frac{1}{3}[\mathbb{A}, \mathbb{A}] >= \int_M \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right)$.

BF Theory and The TG group (5/)

TG BF Theory

- 3-Dimensional BF theory arises from $(<<,>>)$:
  $S_{BF}[A,B] = \frac{1}{2}\int_M << \mathbb{A}, d\mathbb{A} + \frac{1}{3}[\mathbb{A},\mathbb{A}] >> = \int_M \operatorname{Tr} BF_A$
- B $\equiv$ Lagrange muliplier: $\delta B \Rightarrow F_A = 0$
- B satisfies the linearised flatness equation: $\delta A \Rightarrow d_A B = 0$.
- The space solutions is the tangent bundle of the moduli space of flat connections: TM^G.

On a 3-manifolds which have isolated flat connections, the BF path integral is formally a sum over the so-called Ray-Singer.

BF Theory and The TG group (6/)

Ray-Singer Torsion

- $$\frac{\mathrm{Det}\Delta_{A_a}|_{\Omega^0(M,\mathfrak{g})}}{\mathrm{Det}^{1/2}D_{A_a}|_{\Omega^1(M,\mathfrak{g})\oplus\Omega^0(M,\mathfrak{g})}} = e^{\frac{i}{2}\pi\eta} \cdot \left|\frac{\mathrm{Det}\Delta_{A_a}|_{\Omega^0(M,\mathfrak{g})}}{\mathrm{Det}^{1/2}D_{A_a}|_{\Omega^1\oplus\Omega^0}}\right|$$
- $$\left|\frac{\mathrm{Det}\Delta_{A_a}|_{\Omega^0(M,\mathfrak{g})}}{\mathrm{Det}^{1/2}D_{A_a}|_{\Omega^1(M,\mathfrak{g})\oplus\Omega^0(M,\mathfrak{g})}}\right| = \left|\frac{\mathrm{Det}^{3/4}\Delta_{A_a}|_{\Omega^0(M,\mathfrak{g})}}{\mathrm{Det}^{1/4}\Delta_{A_a}|_{\Omega^1(M,\mathfrak{g})}}\right| = \sqrt{\tau_M(A_a)};$$

As we noted BF theories pose a number of issues that still require resolution. Here is an incomplete list

BF and ISSUES

- 1. BF theory has a non-compact gauge group and such gauge theories have notorious issues with unitary and/ or renormalisability.
- 2. There are singularities in the moduli spaces that we end up on. These arise at reducible connections.
- 3. The space of classical solutions is non-compact in general. The path integral devolves to an integral over this space but with a measure that is not integrable.

Thank you!

References

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