Uncovering the fractional order phase transitions in AdS black holes

Based on : *M. Chabab and S. Iraoui, Front. in Phys. 8 (2021), 666*

Presented by : Samir IRAOUI

Ibn al-Haytham Laboratory, Regional Center For education And Training of Marrakech, Morocco
High Energy Physics and Astrophysics Laboratory (LPHEA), Cadi Ayyad University, Morocco

10 Mars 2022

Table of content

Introduction

Shawrzshild BH in flat space times	$\mapsto$	**No thermodynamic stability**
Shawrzshild in Anti-de Sitter	$\mapsto$	**thermodynamic stability**

Thermodynamics in Anti-de Sitter spacetime

- In the AdS black hole thermodynamics in extended phase space, the cosmological constant Λ has been related to the thermodynamical pressure by relation

$$P = -\frac{\Lambda}{8\pi}$$

- Λ has been treated as a thermodynamic variable included in the first law of BH thermodynamics

Introduction

Shawrzshild BH in flat space times	$\mapsto$	**No thermodynamic stability**
Shawrzshild in Anti-de Sitter	$\mapsto$	**thermodynamic stability**

Thermodynamics in Anti-de Sitter spacetime

- In the AdS black hole thermodynamics in extended phase space, the cosmological constant Λ has been related to the thermodynamical pressure by relation

$$P = -\frac{\Lambda}{8\pi}$$

- Λ has been treated as a thermodynamic variable included in the first law of BH thermodynamics

- Critical behavior appears in some types of black holes and in some theories of gravity :
  - **RN-AdS BH, Kerr AdS-BH, ...**
  - **Gauss Bonnet gravity, Massive gravity,...**
  - **Quintessence background,..**

Ehrenfest classification

Ehrenfest classification of PT (n the smallest integer ≥ 1)

$$\lim_{T \to T_c^+} \frac{\mathrm{d}^n G(T)}{\mathrm{d}T^n} = A^+ \neq A^- = \lim_{T \to T_c^-} \frac{\mathrm{d}^n G(T)}{\mathrm{d}T^n}$$

Ehrenfest classification

Ehrenfest classification of PT (n the smallest integer ≥ 1)

$$\lim_{T\to T_c^+} \frac{\mathrm{d}^n G(T)}{\mathrm{d}T^n} = A^+ \neq A^- = \lim_{T\to T_c^-} \frac{\mathrm{d}^n G(T)}{\mathrm{d}T^n}$$

2 types of phase transition are largely studies in the black hole system

- First order small/large black hole phase transition $n = 1$
- Second order small/large black hole phase transition $n = 2$

- **Hawking-Page phase transitions : 1st order $n = 1$**
- **Small/Large BH PT at the critical point : 2nd order $n = 2$**

Generalized Ehrenfest's classification

More general classification scheme

$$\lim_{T \to T_c^+} \frac{\mathrm{d}^\beta G(T)}{\mathrm{d}T^\beta} = A^+ \neq A^- = \lim_{T \to T_c^-} \frac{\mathrm{d}^\beta G(T)}{\mathrm{d}T^\beta}$$

β the smallest fractional > 0

Generalized Ehrenfest's classification

More general classification scheme

$$\lim_{T \to T_c^+} \frac{\mathrm{d}^\beta G(T)}{\mathrm{d}T^\beta} = A^+ \neq A^- = \lim_{T \to T_c^-} \frac{\mathrm{d}^\beta G(T)}{\mathrm{d}T^\beta}$$

β the smallest fractional > 0

- **Nagle[1] : Order/disorder transition of dipalmitoyl lecithin DPL-system is of 3/2 order.**
- **Ma[2] : RN-AdS black holes can undergo 4/3 order FPT.**

[1] Nagle. J. F. *Lipid bilayer phase transition : Density measurements and theory.* Proceedings of the National Academy of Sciences, 70(12) :3443–3444, 1973.

[2] M. S. Ma, *Fractional-order phase transition of charged AdS black holes*, Phys. Lett. B 795 (2019) 490.

Chosen black hole solutions

We analyze the FPT for several AdS black hole prototypes

1. Black hole surrounded by quintessence background (spherical solutions)
2. D dimensional RN-AdS BH (spherical solutions)
3. $5D$-Gauss-Bonnet gravity (spherical solutions)
4. Kerr black holes (axisymmetric solutions)

Caputo fractional derivative

- **Fractional derivatives can be defined in a variety of different ways.**
- **Several definitions exists in literature : Riemann–Liouville, Atangana-Baleanu, Caputo-Fabrizio, Riesz derivatives...**

Mathematical tool

- Caputo definition which enables easy use of conventional boundaries and initial conditions

$$D_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} \frac{\partial^n f(\tau)}{\partial \tau^n} \mathrm{d}\tau, \quad n-1 < \beta < n$$

β is the order of derivative and n an integer.

The Kiselev solution of four dimensional charged AdS black holes surrounded by quintessence

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2 - \frac{\alpha}{r^{3\omega_q+1}},$$

- ω is the quintessence parameter
- α a positive normalisation parameter ;
- M and Q are the mass and electric charge of the black hole

The Kiselev solution of four dimensional charged AdS black holes surrounded by quintessence

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2 - \frac{\alpha}{r^{3\omega_q+1}},$$

- ω is the quintessence parameter
- α a positive normalisation parameter ;
- M and Q are the mass and electric charge of the black hole

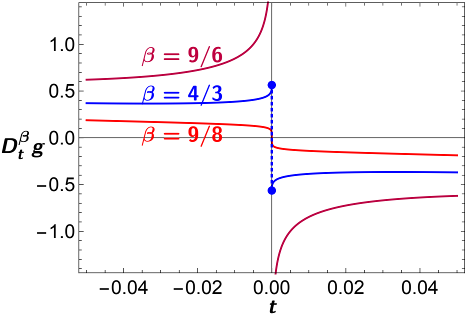
The rescaled Gibbs free energy $g = G/G_c$, near critical point

$$g(t,p) \approx \left[\sqrt{\frac{2}{3}} + \frac{p}{2\sqrt{6}} + \dots\right] - \left[\sqrt{\frac{2}{3}} - 3Q\alpha - \frac{\left(2-3\sqrt{6}Q\alpha\right)}{6^{1/6}}\left(p^{1/3} - \frac{19}{4\times 6^{2/3}}p^{2/3}\right) + \dots\right] t$$
$$- \left[\frac{2^{1/6}\left(2-6\sqrt{6}Q\alpha+27Q^2\alpha^2\right)}{9\times 3^{1/6}}\left(\frac{2^{5/3}}{p^{2/3}} - \frac{13}{9^{1/3}p^{1/3}}\right) + \dots\right] t^2 + \mathcal{O}[t^3],$$

where $t = T/T_c - 1$

Charged AdS black hole surrounded by quintessence

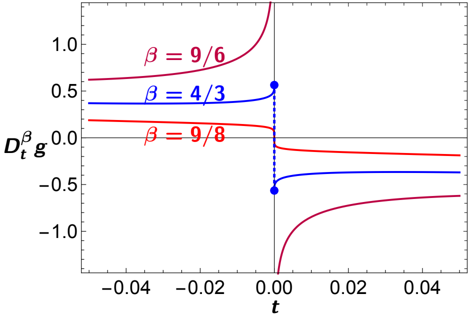
$$\lim_{t\to 0^{\pm}} D_t^{\beta} g(t,p) = \begin{cases} 0 & \text{for } \beta < 4/3, \\ \mp \frac{\left(2-3\sqrt{6}\alpha Q\right)^{4/3}}{\sqrt{6}\Gamma\left(\frac{2}{3}\right)} & \text{for } \beta = 4/3, \\ \mp\infty & \text{for } \beta > 4/3. \end{cases}$$



- The behavior of fractional derivatives $D_t^{\beta} g$ near the critical point of charged AdS black hole surrounded by quintessence.

Charged AdS black hole surrounded by quintessence

$$\lim_{t \to 0^{\pm}} D_t^{\beta} g(t,p) = \begin{cases} 0 & \text{for } \beta < 4/3, \\ \mp \frac{\left(2-3\sqrt{6}\alpha Q\right)^{4/3}}{\sqrt{6}\Gamma\left(\frac{2}{3}\right)} & \text{for } \beta = 4/3, \\ \mp\infty & \text{for } \beta > 4/3. \end{cases}$$



- The behavior of fractional derivatives $D_t^\beta g$ near the critical point of charged AdS black hole surrounded by quintessence.

- Spherical symmetric AdS black hole in quintessence background : the FPT stand at **4/3** order.
  $\Rightarrow$ **Not affected by the external quintessence field surrounding the charged AdS black hole.**

Charged AdS black holes in higher dimensions

The rescaled Gibbs free energy is

$$g(t,p)=\frac{(5-2D)^2(\nu+1)^{3-D}}{2[(D-2)(2D-5)]^{3/2}}+\frac{1}{2}\sqrt{\frac{2D-5}{D-2}}(\nu+1)^{D-3}-\sqrt{\frac{2D-5}{(D-2)^3}}\,\frac{(D-3)^2}{2(D-1)}(p+1)(\nu+1)^{D-1}$$

Dimension D	Order FPT	$\lim_{t\to 0^\pm} D_t^\beta g(t,p)$
$D=4$	$\beta=4/3$	$\mp\frac{2\,2^{5/6}}{3\sqrt{3}\,\Gamma\left(\frac{5}{3}\right)}$
$D=5$	$\beta=4/3$	$\mp\frac{8\sqrt[3]{2}}{\sqrt[6]{3}\,5^{5/6}\,\Gamma\left(\frac{5}{3}\right)}$
$D=6$	$\beta=4/3$	$\mp\frac{12\sqrt[3]{6}}{7^{5/6}\,\Gamma\left(\frac{5}{3}\right)}$
$D=7$	$\beta=4/3$	$\mp\frac{32\sqrt[3]{\frac{2}{3}}\sqrt{5}}{9\,\Gamma\left(\frac{5}{3}\right)}$
$D=8$	$\beta=4/3$	$\mp\frac{50\left(\frac{2}{11}\right)^{5/6}}{\sqrt[6]{3}\,\Gamma\left(\frac{5}{3}\right)}$
$D=9$	$\beta=4/3$	$\mp\frac{24\sqrt[3]{6}\sqrt{7}}{13^{5/6}\,\Gamma\left(\frac{5}{3}\right)}$
$D=10$	$\beta=4/3$	$\mp\frac{196\left(\frac{2}{5}\right)^{5/6}}{3\sqrt{3}\,\Gamma\left(\frac{5}{3}\right)}$

Whatever the dimension D of the RN-AdS black hole, the fractional order of phase transition near the critical point arises decidedly at $\beta=\frac{4}{3}$

5D Gauss-Bonnet-AdS black hole

- The third prototype : $D = 5$ Gauss-Bonnet-AdS black holes with solutions

$$f_{GB}(r) = 1 + \frac{r^2}{2\tilde{\alpha}}\left(1 - \sqrt{1 - \frac{16\pi P\tilde{\alpha}}{3} + \frac{32M\tilde{\alpha}}{3\pi r^4}}\right),$$

$\tilde{\alpha}$ is a Gauss-Bonnet coupling constant.

5D Gauss-Bonnet-AdS black hole

- The third prototype : $D = 5$ Gauss-Bonnet-AdS black holes with solutions

$$f_{GB}(r) = 1 + \frac{r^2}{2\tilde{\alpha}} \left(1 - \sqrt{1 - \frac{16\pi P \tilde{\alpha}}{3} + \frac{32 M \tilde{\alpha}}{3\pi r^4}} \right),$$

$\tilde{\alpha}$ is a Gauss-Bonnet coupling constant.

- The fractional derivative of rescaled free energy :

$$\lim_{t \to 0^{\pm}} D_t^{\beta} g(t,p) = \begin{cases} 0 & \text{for} \quad \beta < 4/3, \\ \mp \dfrac{2^{5/3}}{\Gamma\left(\frac{5}{3}\right)} \pi & \text{for} \quad \beta = 4/3, \\ \mp \infty & \text{for} \quad \beta > 4/3. \end{cases}$$

Again, the PT is fractional and happens at order 4/3

Axisymmetric BH : Kerr-AdS black holes

- **What about axisymmetric black hole solutions ? Is the 4/3 order of FPT universal ?**

- **What about axisymmetric black hole solutions ? Is the 4/3 order of FPT universal ?**
- The Kerr asymptotically AdS black hole solutions :

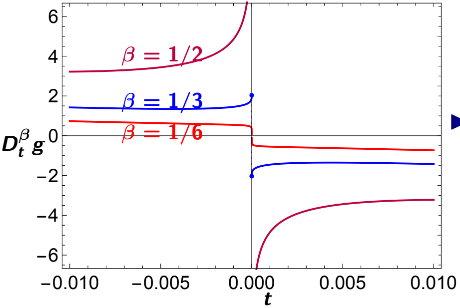
$$ds^2 = -\frac{\Delta}{\rho^2}\left[dt - \frac{a\sin^2\theta}{\Xi}d\varphi\right]^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{S}d\theta^2 + \frac{S\sin^2\theta}{\rho^2}\left[adt - \frac{r^2+a^2}{\Xi}d\varphi\right]^2$$

- t-expansion of the rescaled Gibbs free energy :

$$g(t,p) \approx C_1 + \left[C_2 + \frac{C_3}{p^{2/3}} + \frac{C_4}{p^{1/3}} + \dots\right]t + \left[C_5 + \frac{C_6}{p^{5/3}} + \frac{C_7}{p^{4/3}} + \frac{C_8}{p^{2/3}} + + \frac{C_9}{p^{1/3}} + \dots\right]t^2 + \mathcal{O}[t^3]$$

Axisymmetric BH : Kerr-AdS black holes

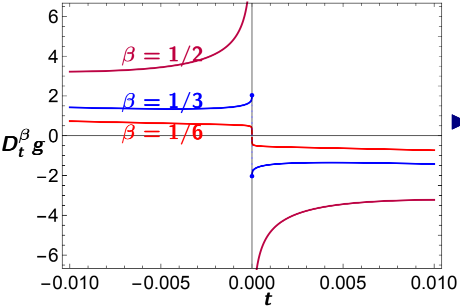
$$\lim_{t\to 0^\pm} D_t^\beta g(t,p) = \begin{cases} 0 & \text{for } \beta < 1/3, \\ \pm \dfrac{5\left(\frac{10}{3}\right)^{2/3}(2C_3 - C_6)}{9\Gamma\left(\frac{11}{3}\right)} & \text{for } \beta = 1/3, \\ \mp\infty & \text{for } \beta > 1/3, \end{cases}$$



- Near the critical point, fractional order of the phase transition is no longer $\beta = 4/3$.

Axisymmetric BH : Kerr-AdS black holes

$$\lim_{t\to 0^\pm} D_t^\beta g(t,p) = \begin{cases} 0 & \text{for } \beta < 1/3, \\ \pm\dfrac{5\left(\frac{10}{3}\right)^{2/3}(2C_3 - C_6)}{9\Gamma\left(\frac{11}{3}\right)} & \text{for } \beta = 1/3, \\ \mp\infty & \text{for } \beta > 1/3, \end{cases}$$



- Near the critical point, fractional order of the phase transition is no longer $\beta = 4/3$.

- **This may suggest 4/3 order FPT is not universal and only holds for static black holes with spherical symmetry**

Conclusion

- Continuous thermodynamic phase transitions of AdS black holes, according to the generalized Ehrenfest classification, has been studied

Results

Black hole solutions	Order of PT
$5D$ Gauss-Bonnet; AdS charged black hole surrounded by quintessence; RN-AdS_D black holes	$4/3$ order PT
AdS Kerr black hole	$1/3$ order PT

- **The phase transition of order 4/3 valid for static AdS black holes with spherical symmetry ;**
- **This order is not universal because it is not valid for non-spherical solutions.**

The End

Thanks for your attention.

samir.iraoui@ced.uca.com