Relativistic kinematics

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Outline of the course

- Monday – introduction
  - the need for relativity; Lorentz transforms; basic consequences; four vectors; proper time;
- Tuesday – kinematics and decays
  - kinematics; Fermi Golden rule; Lorentz invariant phase space; two-body decays
- Wednesday – more decays and cross sections
  - three-body decay; Dalitz plots; cross section calculations; pseudorapidity
- Thursday - tutorial

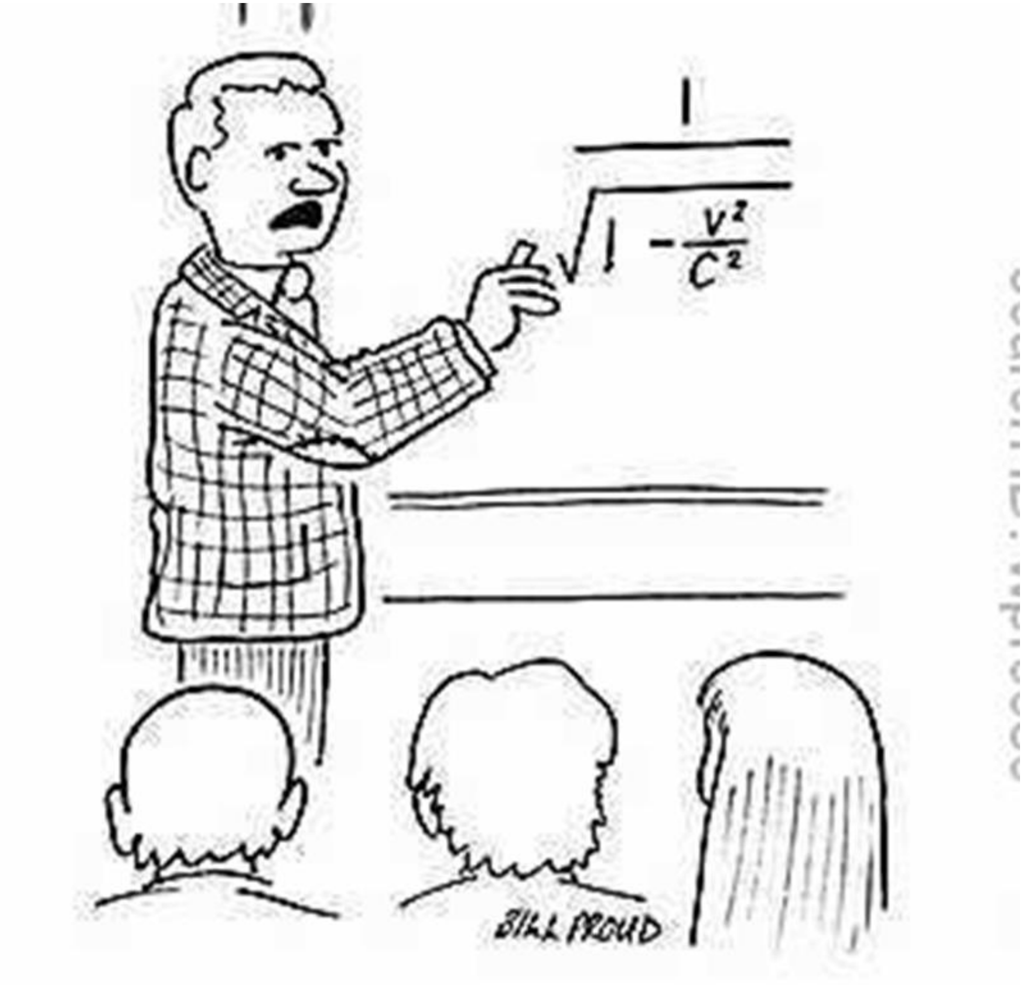
Additional resources

- Books
  - A.P. French – Special Relativity (Taylor & Francis)
  - D. Griffiths – Introduction to Elementary Particles (Wiley)
  - M. Thomson – Modern Particle Physics (Cambridge)
- Lecture courses
  - Relativity – M. Tegmark
    - https://ocw.mit.edu/courses/physics/8-033-relativity-fall-2006/
  - Relativistic kinematics – K. Mazumdar – XIth SERC School on EHEP
    - https://www.niser.ac.in/sercehep2017/
  - Quantum Field Theory – S. Coleman
    - https://arxiv.org/abs/1110.5013

An apology

Normally I would like to give this type of course as chalk'n'talk but given the large amount of material and the virtual setting I am using slides.

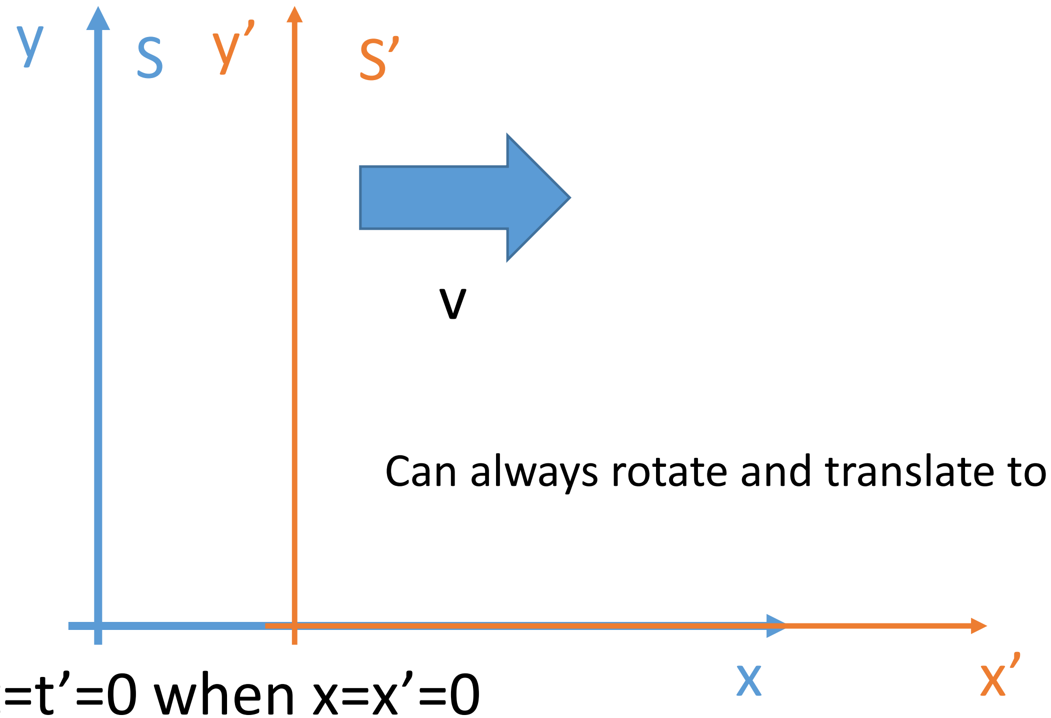
I will try to slow myself down. A good way to do that is ask questions, **please stop me any time that something is not clear.**



If v is the number of qualified physics teachers, and c is the number of unqualified science teachers, this factor reduces to zero

A bit of history

- Relativity is not new
- “The fundamental laws of physics are the same in all frames of reference moving with constant velocity with respect to one another”
  - Galileo Galilei 1632 AD



$$\vec{r}' = \vec{r} - \vec{v}t$$

$$t' = t$$

Can always rotate and translate to this scenario

t=t’=0 when x=x’=0

Classical physics

- Newtonian physics is unchanged e.g.

$$F_x' = m\frac{d^2x'}{dt'^2} = m\frac{d^2(x - v_x t)}{dt^2} = m\frac{d^2x}{dt^2} = F_x$$

- But classical electrodynamics is not
- Maxwell's equations in a vacuum lead to

$$\frac{\partial^2 \vec{E}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \Rightarrow \vec{E}(x,t) = A\vec{f}(x - ct) + B\vec{g}(x + ct)$$

$$\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 \vec{E}}{\partial x'^2} + 2\frac{v}{c^2}\frac{\partial^2 \vec{E}}{\partial x' \partial t'} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t'^2} = 0 \Rightarrow \vec{E}'(x',t') = \vec{f}'(x' - [c \pm v]t') + \vec{g}'(x' + [c \pm v]t')$$

Tutorial problem 1

Einstein’s postulate

Finding evidence for the medium ‘aether’ that the waves travelled through was not forthcoming c.f. Michelson-Morley experiment

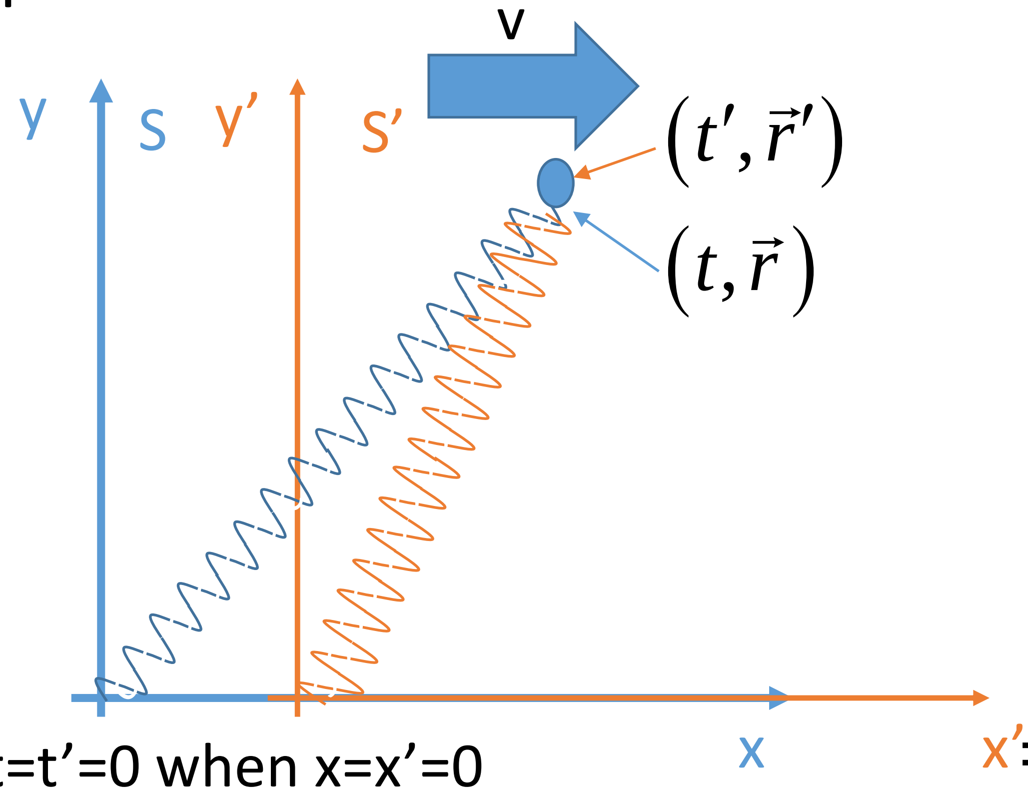
So Einstein dispensed with it and amended Galilean relativity with

1) “The fundamental laws of physics are the same in all frames of reference moving with constant velocity with respect to one another (inertial)”

2) **“The speed of light is the same in all inertial frames”**

Toward the Lorentz transformations

- Light pulse at t=t'=0



t=t'=0 when x=x'=0

With Einstein's postulate this leads to two ways to define the distance travelled by light in each frame that is equal

$$(ct)^2 = |\vec{r}|^2$$

$$(ct')^2 = |\vec{r}'|^2$$

$$\Rightarrow (ct)^2 - |\vec{r}|^2 = (ct')^2 - \left|\overrightarrow{r'}\right|^2$$

Lorentz transformation ensures this relationship

Lorentz transformation

- The transform between inertial frames

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma ct - \gamma\beta x \\ -\gamma\beta ct + \gamma x \\ y \\ z \end{bmatrix} \quad \text{where } \beta = \frac{v}{c} \text{ and } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

- Time now frame dependent
- When v << c, $\beta \to 0$ and $\gamma \to 1$, and Lorentz $\to$ Galilean transformation
- Derivation in back up

Reminder of the basic consequences

Inverse transform: S moves with velocity –v in the x' direction in S' i.e. $\beta \to -\beta$

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \Lambda^{-1} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma ct' + \gamma\beta x' \\ \gamma\beta ct' + \gamma x' \\ y \\ z \end{bmatrix}$$

Time dilation: time interval observed in S for a clock at fixed position x' = 0 is

$$ct_2 - ct_1 = \gamma\,(ct'_2 - ct'_1) \Rightarrow \Delta t = \gamma \Delta t'$$

$\gamma > 1$ therefore 'a moving clock runs slow' i.e. cosmic ray muons

Basic consequence II

At time t what length x_1 to x_2 is measured in S for a stick of length l' on x' axis that is at rest in S' with ends at x_1' and x_2'

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma ct - \gamma\beta x \\ -\gamma\beta ct + \gamma x \\ y \\ z \end{bmatrix}$$

Length contraction:

$$x'_2 - x'_1 = \gamma\,(x_2 - x_1) \Rightarrow l' = \gamma l$$

$\gamma > 1$ so the stick appears shorter

There is much fun to be had with these, e.g. twin paradox, but not the thrust of these lectures so we will move on to the language of relativity

Natural units

As you are aware in particle physics we dispense with [kg, m, s] and use $[\hbar, c, GeV]$ and we go further to just use GeV by setting $\hbar = c = 1$

So I am getting bored of writing *c* so I will drop it unless I am making a specific point in the lectures

Table 2.1 Relationship between S.I. and natural units.

Quantity	[kg, m, s]	$[\hbar, c, \text{GeV}]$	$\hbar = c = 1$
Energy	$\text{kg m}^2\,\text{s}^{-2}$	GeV	GeV
Momentum	kg m s^{-1}	GeV/c	GeV
Mass	kg	GeV/c^2	GeV
Time	s	$(\text{GeV}/\hbar)^{-1}$	GeV^{-1}
Length	m	$(\text{GeV}/\hbar c)^{-1}$	GeV^{-1}
Area	m^2	$(\text{GeV}/\hbar c)^{-2}$	GeV^{-2}

From Mark Thomson. Modern Particle Physics

Four vectors

So far we have seen that we must treat time differently to classical physics and it has become relative in a similar way to space coordinates

We have a way of transforming coordinates between any two inertial frames via the LT

Matrix multiplication using the Einstein summation convention

$$x^{\mu} = (t, x, y, z) \equiv (x^0, x^1, x^2, x^3)$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \quad \left(\Lambda^{\mu}{}_{\nu} \equiv \Lambda_{i\,j} \text{ in LT derivation}\right)$$

A contravariant four vector is one that transforms from one inertial frame to another following LT c.f. a three-vector is defined via its behaviour under rotations

….but it doesn't have to be (t,x,y,z)

Invariant

We go back to our master Eq. for SR $\Rightarrow t^2 - |\vec{r}|^2 = t'^2 - \left|\vec{r'}\right|^2$

This motivates another definition – covariant four-vector

$$x_\mu = (t, -x, -y, -z)$$

$$x^\mu x_\mu = t^2 - x^2 - y^2 - z^2$$

$$= t'^2 - x'^2 - y'^2 - z'^2$$

$$= x'^\nu x'_\nu$$

This is equivalent to the invariance of $|\vec{r}|^2$ under rotations in Euclidean 3D

The metric and inverse

This leads to the definition of the metric

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$g_{\mu\nu} x^{\mu} x^{\nu} = g_{\alpha\beta} x'^{\alpha} x'^{\beta} = g_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} x^{\mu} \Lambda^{\beta}{}_{\nu} x^{\nu}$$

$$\therefore g_{\mu\nu} = g_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} = \Lambda^{\alpha}{}_{\mu} \Lambda_{\alpha\ \nu}$$

$$\therefore g_{\mu\nu} g^{\nu\delta} = \Lambda^{\alpha}{}_{\mu} \Lambda_{\alpha\ \nu} g^{\nu\delta} = \Lambda^{\alpha}{}_{\mu} \Lambda_{\alpha}{}^{\delta}$$

$$\Rightarrow \delta^{\delta}_{\mu} = \Lambda^{\alpha}{}_{\mu} \Lambda_{\alpha}{}^{\delta}$$

$$\Rightarrow \delta^{\delta}_{\mu} = \left(\Lambda^{-1}\right)^{\delta}{}_{\alpha} \Lambda^{\alpha}{}_{\mu}$$

$$\text{where } \left(\Lambda^{-1}\right)^{\delta}{}_{\alpha} \equiv \Lambda_{\alpha}{}^{\delta} = g_{\alpha\beta} \Lambda^{\beta}{}_{\nu} g^{\nu\delta}$$

Important to be comfortable navigating this notation, as it appears many places, but I will not be doing a lot of index manipulation in this course

Four derivative

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \Lambda^{-1} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma ct' + \gamma\beta x' \\ \gamma\beta ct' + \gamma x' \\ y \\ z \end{bmatrix}$$

Consider the derivatives w.r.t. x' and t'

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = \gamma\frac{\partial}{\partial x} + \gamma\beta\frac{\partial}{\partial t} \Rightarrow -\frac{\partial}{\partial x'} = \gamma\left(-\frac{\partial}{\partial x}\right) - \gamma\beta\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = \gamma\beta\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial t} \Rightarrow \frac{\partial}{\partial t'} = -\gamma\beta\left(-\frac{\partial}{\partial x}\right) + \gamma\frac{\partial}{\partial t}$$

$$\therefore \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$$

$$\Rightarrow \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box \qquad \text{(d'Alembertian)}$$

Wave eq in EM is
is an invariant!

EM Lorentz invariant

Problem set Q2

Symmetry of Lorentz Transforms

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\cosh^2\eta - \sinh^2\eta = \gamma^2 - \gamma^2\beta^2 = \frac{1-\beta^2}{1-\beta^2} = 1$$

$$\eta = \tanh^{-1}(-\beta) \equiv \text{rapidity}$$

More abstract a rotation by -iη in the (ct,x) plane

But this is a useful way to write the transformation for practical reasons (lecture 3) and to understand the symmetry of Lorentz transformation

Conservation laws and infinitesimal transformations

Invariance of a system under a continuous transformation leads to a conserved quantity – Noether's theorem – so there are associated quantities with LT, but they are not much used.

(see Sidney Coleman's QFT lectures (6 October) for more detail about this)

However, thinking about the infinitesimal Lorentz transformations elucidates another important connection with symmetry groups

We define infinitesimal transformation as (Problem 3)

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu\nu} x_{\nu} \delta\eta$$

Four vectors in general

- In general a four vector a^μ when combined with another b^μ

$$a^\mu b_\mu = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 = \text{invariant}$$

- Further four vectors transform according to Lorentz transformations between two inertial frames
- So far we have met space-time four vectors (and we have alluded to some in electromagnetism) but we don't have what we really need the energy and momentum that form a four vector
- The first thing to consider is 'proper time'

Proper time

A non-accelerating particle will have an inertial frame of reference associated with it where it is at rest.

The ‘clock’ in this frame will have a time agreed upon by observers in all other inertial frame

This is referred to as the proper time τ c.f. the lifetime of a particle

Can we use this information to find the energy and momentum

We know that if all the laws of physics are invariant then let us use Lagrangian formalism for this

$$\text{Action} = S \propto \int d\tau$$

Derivation of energy and momentum four vector

Recall dimensions of action are

$$[\text{Energy}][\text{t}] \equiv [\text{GeV}][\text{GeV}]^{-1} \equiv \text{dimensionless}$$

The only other invariant quantity we have that has dimension energy is the mass M of the particle so we multiply by –M

$$S = -M \int d\tau \; = -M \int \frac{dt}{\gamma}$$

$$L = -M\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \Rightarrow p_x = \frac{M\dot{x}}{\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}} = M\gamma\dot{x} \text{ (conserved quantity)}$$

$$\vec{p} = M\gamma\vec{v}$$

Energy and four-momentum

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = M\gamma(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{M}{\gamma} = M\gamma\left(1 - \frac{1}{\gamma^2} + \frac{1}{\gamma^2}\right) = M\gamma$$

$$p^\mu = (M\gamma, M\gamma\vec{v}) = (E, \vec{p})$$

$$\Rightarrow p^\mu p_\mu = M^2\gamma^2\left(1 - |\vec{v}|^2\right) = M^2\gamma^2\frac{1}{\gamma^2} = M^2$$

$$\Rightarrow E^2 - |\vec{p}|^2 = M^2$$

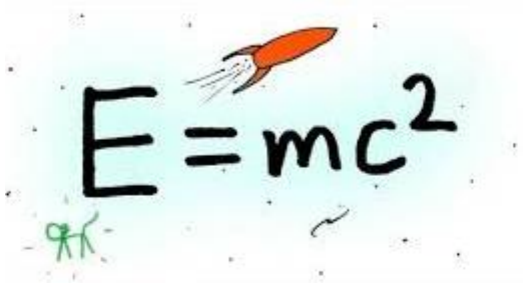
You can just differentiate x^μ by τ to get proper velocity and multiple by M to get the four-momenta

Recap of yesterday and plan for today

- Yesterday
  - the need for relativity
  - Lorentz transforms
  - four vectors
  - proper time and $p^{\mu} = (M\gamma, M\gamma\vec{v}) = (E, \vec{p})$
- Today
  - Using the four-momentum: two-body decay kinematics, centre-of-mass and threshold
  - Fermi Golden rule and Lorentz invariant phase space
  - two body decay rate

What about classical physics

E=M when v=0 or as it should appear in a course on relativity



minutephysics

Therefore kinetic energy is

$$
\begin{aligned}
T &= E - mc^2 \\
&= mc^2(\gamma - 1) \\
&= mc^2\left((1-\beta^2)^{-\frac{1}{2}} - 1\right) \\
&\approx mc^2\left(\frac{1}{2}\beta^2\right) \quad \text{when } \beta^2 \ll 1 \\
&\approx \frac{1}{2}mv^2
\end{aligned}
$$

Four-momenta and massless particles

So we have shown two ways – based upon proper time – that

$$p^{\mu} = (E, \vec{p})$$

is the representation of energy and momentum relativistically.

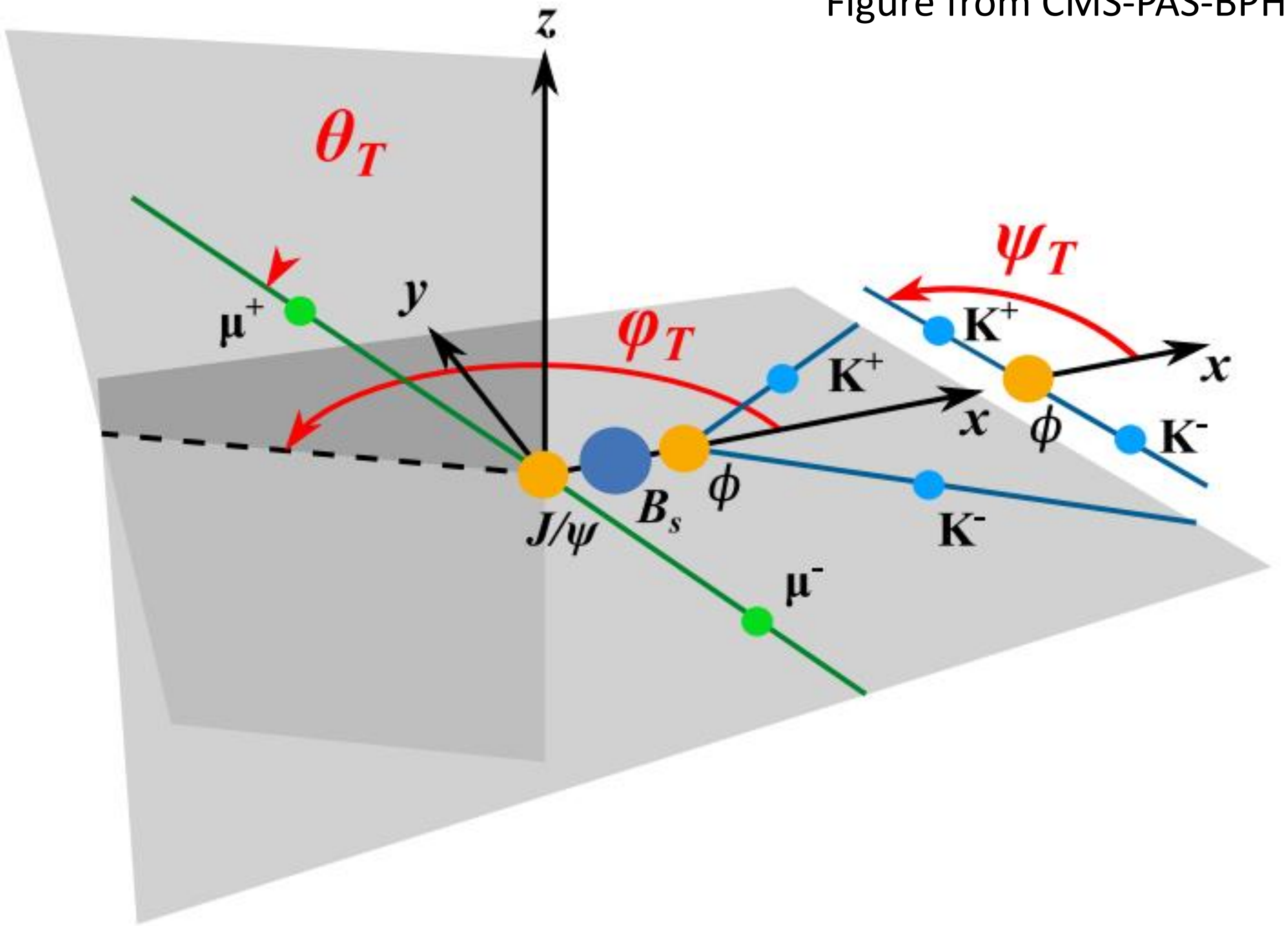
Special case m=0

$$E^2 - |\vec{p}|^2 = m^2 \Rightarrow E = |\vec{p}| \text{ when } m = 0 \Rightarrow \frac{|\vec{p}|}{E} = 1 = \beta$$

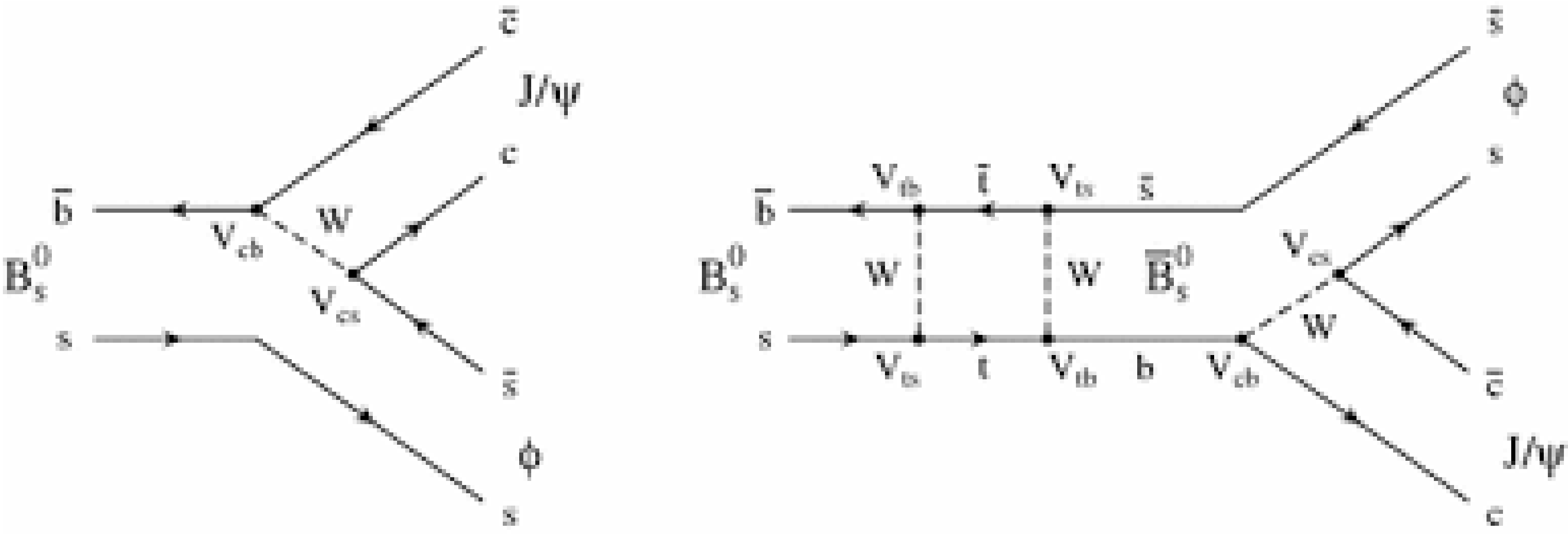
Not so special case at LHC unless particle masses at EW scale – W, Z, H and t – mass makes little difference in calculations so assuming m=0 hence E=p often chosen

Example: two-body decay, opening angle (and some B physics)

Figure from CMS-PAS-BPH-20-001



Example: two-body decay, opening angle (and some B physics)



From: T. Kuhr, CP-Violation in Mixing and the Interference of Mixing and Decay, in Flavor Physics at the Tevatron, Springer Tracts in Modern Physics (2013)

What is the ϕ momentum in the B rest frame?

$$p_B = p_\phi + p_{J/\psi}$$

$$\Rightarrow \left(p_B - p_\phi \right)^2 = p_{J/\psi}^2$$

$$\Rightarrow p_B^2 + p_\phi^2 - 2 p_B p_\phi = m_{J/\psi}^2$$

$$\Rightarrow 2 p_B p_\phi = m_B^2 + m_\phi^2 - m_{J/\psi}^2$$

$$\Rightarrow E_\phi = \frac{m_B^2 + m_\phi^2 - m_{J/\psi}^2}{2m_B}$$

What is the ϕ momentum in the B rest frame?

$$4m_B^2 E_\phi^2 = \left(m_B^2 + m_\phi^2 - m_{J/\psi}^2\right)^2$$

$$\Rightarrow 4m_B^2\left(\left|\vec{p}_\phi\right|^2 + m_\phi^2\right) = m_B^4 + 2m_B^2\left(m_\phi^2 - m_{J/\psi}^2\right) + \left(m_\phi^2 - m_{J/\psi}^2\right)^2$$

$$\Rightarrow 4m_B^2\left|\vec{p}_\phi\right|^2 = m_B^4 - 2m_B^2\left(m_\phi^2 + m_{J/\psi}^2\right) + \left(m_\phi - m_{J/\psi}\right)^2\left(m_\phi + m_{J/\psi}\right)^2$$

$$= m_B^4 - 2m_B^2\frac{1}{2}\left[\left(m_\phi + m_{J/\psi}\right)^2 + \left(m_\phi - m_{J/\psi}\right)^2\right] + \left(m_\phi - m_{J/\psi}\right)^2\left(m_\phi + m_{J/\psi}\right)^2$$

$$\Rightarrow \left|\vec{p}_\phi\right| = \frac{1}{2m_B}\sqrt{\left(m_B^2 - \left(m_\phi + m_{J/\psi}\right)^2\right)\left(m_B^2 - \left(m_\phi - m_{J/\psi}\right)^2\right)}$$

A important formula for any 1→2+3 process

1) $$\left|\vec{p}_2\right| = \frac{1}{2m_1}\sqrt{\left(m_1^2 - \left(m_2 + m_3\right)^2\right)\left(m_1^2 - \left(m_2 - m_3\right)^2\right)} = \left|\vec{p}_3\right| \quad (2 \leftrightarrow 3)$$

2) $$\left|\vec{p}_2\right| = \frac{1}{2}\sqrt{\left(m_1^2 - 4m_2^2\right)} = \frac{m_1}{2}\sqrt{\left(1 - \frac{4m_2^2}{m_1^2}\right)} \text{ if } m_2 = m_3 \Rightarrow \beta = \frac{\left|\vec{p}_2\right|}{E} = \sqrt{\left(1 - \frac{4m_2^2}{m_1^2}\right)}$$

3) $$\left|\vec{p}_2\right| = \frac{m_1^2 - m_2^2}{2m_1} \text{ if } m_3 = 0 \Rightarrow \beta = \frac{\left|\vec{p}_2\right|}{E} = \frac{m_1^2 - m_2^2}{m_1^2 + m_2^2}$$

Centre of mass frame

How to find the boost to the centre-of-mass (CM) frame?

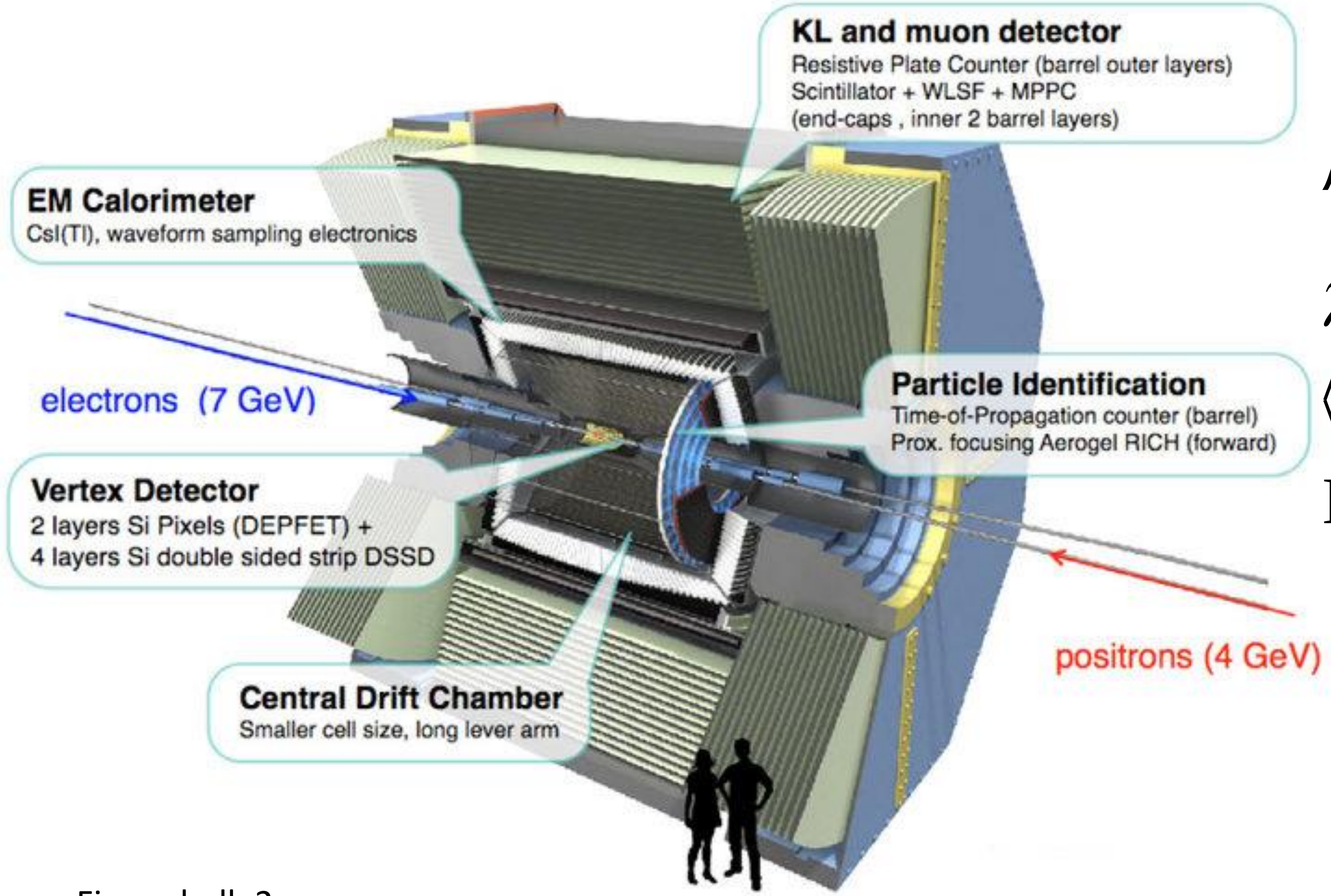
In general $\sum_{i=1}^{N} \vec{p}_i^{CM} = 0$

In the original frame $\sum_{i=1}^{N} \vec{p}_i = \vec{p}^{\text{total}} = \left| p^{\text{total}} \right| \hat{e}_{\parallel}$

So we can resolve all original frame momenta into $\perp$ and $\parallel$, then

look for a boost to make $\sum_{i=1}^{N} p_{\parallel,i}^{CM} = 0 = \sum_{i=1}^{N} \gamma\left(p_{\parallel,i} - \beta E_i \right) \Rightarrow \gamma \sum_{i=1}^{N} \vec{p}_{\parallel,i} = \beta\gamma \sum_{i=1}^{N} E_i$

$$\Rightarrow \beta = \frac{\sum_{i=1}^{N} p_{\parallel,i}}{\sum_{i=1}^{N} E_i} \Rightarrow \vec{\beta} = \frac{\sum_{i=1}^{N} \vec{p}_{\parallel,i}}{\sum_{i=1}^{N} E_i} = \frac{\sum_{i=1}^{N}\left(\vec{p}_{\parallel,i} + \vec{p}_{\perp,i} \right)}{\sum_{i=1}^{N} E_i} = \frac{\sum_{i=1}^{N}\left(\vec{p}_i \right)}{\sum_{i=1}^{N} E_i} = \frac{\vec{p}^{\text{total}}}{E^{\text{total}}}$$



$$\beta = \frac{7-4}{7+4} = \frac{3}{11} \approx 0.27$$

$$\gamma = 1.04$$

$$\langle l_B \rangle = \gamma\beta \times c\tau_B \sim 130\ \mu\text{m}$$

Here $\tau_B = 1.5\ \text{ps}$

Figure belle2.org

Threshold production

Bevatron was a fixed target (one proton at rest) p+p experiment with the goal of inducing

$$p + p \to p + p + p + \bar{p}$$

What is the energy of the beam at threshold?

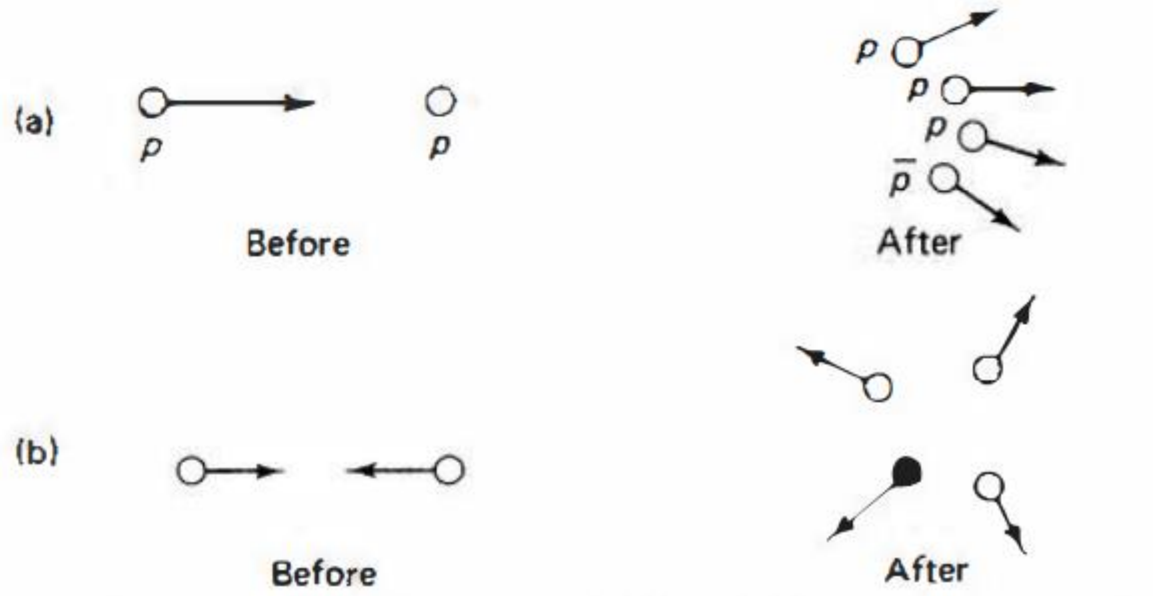


Fig. 3.6 $p + p \to p + p + p + \bar{p}$. (a) In the lab frame; (b) in the CM frame.

Griffiths, Introduction to Elementary Particles

In lab frame before collision

$$p^{\mu}_{\text{Total}} = (E_{\text{beam}} + m_p, \vec{p}_{\text{beam}}) \Rightarrow p^{\mu}_{\text{Total}} p_{\mu,\text{Total}} = (E_{\text{beam}} + m_p)^2 - |\vec{p}_{\text{beam}}|^2 = E^2_{\text{beam}} - |\vec{p}_{\text{beam}}|^2 + m_p^2 + 2m_p E_{\text{beam}}$$

$$\Rightarrow s = 2m_p^2 + 2E_{\text{beam}} m_p$$

In CM frame after collision at threshold (all particles at rest)

$$\Rightarrow p^{*\mu}_{\text{Total}} = (4m_p, 0) \Rightarrow s = 16m_p^2$$

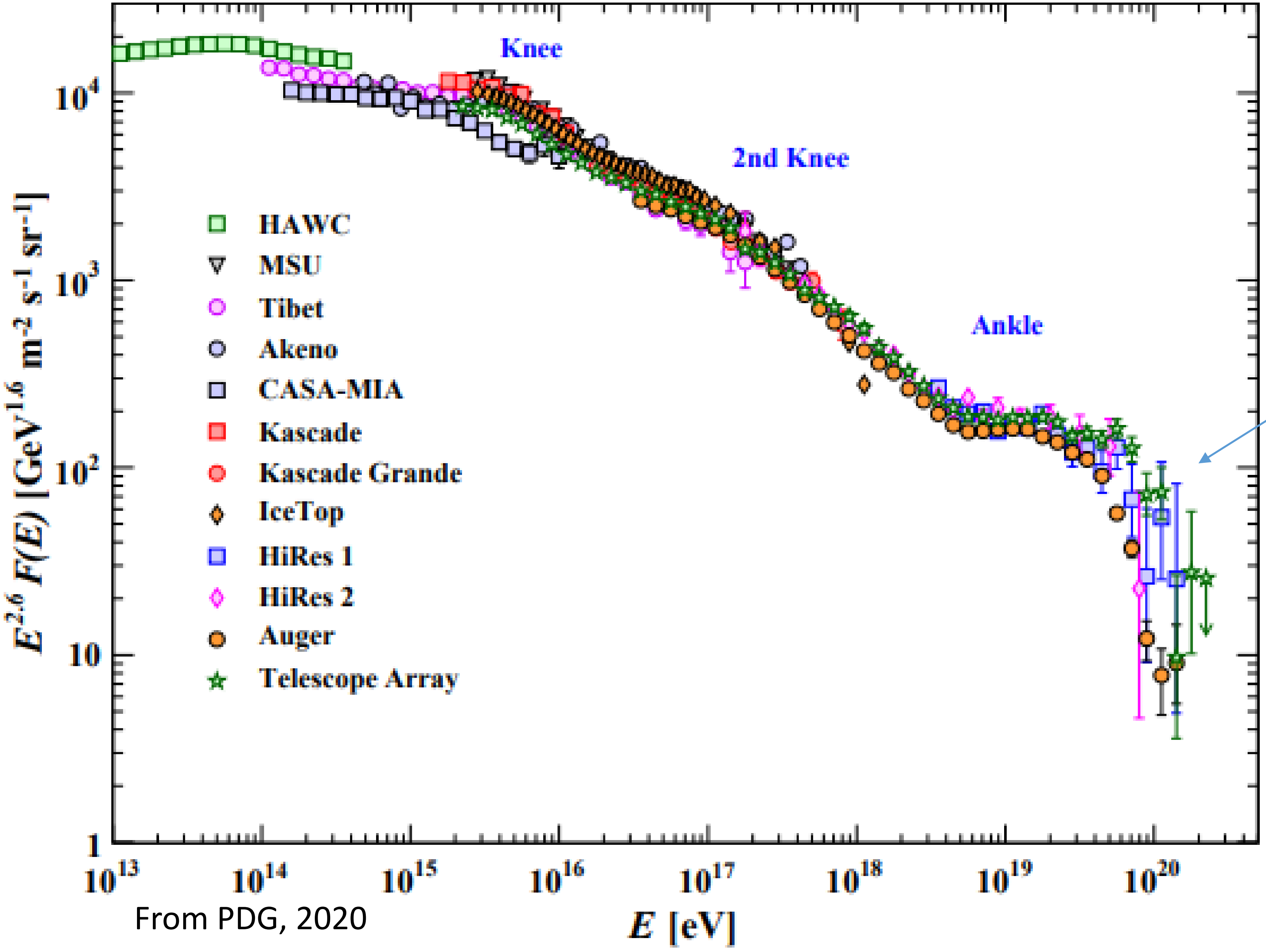
Equating s

$$\Rightarrow E_{\text{beam}} = 7m_p$$

If colliding beams CM and lab equivalent

$$\Longrightarrow E^*_{\text{beam}} = 2m_p$$

Greisen–Zatsepin–Kuzmin



GZK cut-off problem set

From PDG, 2020

Griffiths' suggestions

1) To get the energy of a particle, when you know its momentum (or vice versa) use the invariant $E^2 - \overrightarrow{|p|}^2 = m^2$
2) If you know the energy and momentum of a particle, and you want to determine its velocity, use $\vec{\beta} = \vec{p}/E$
3) Use four-vector notation, and exploit the invariant dot product $p^2 = m^2$
4) If the problem seems cumbersome in the lab frame try analysing it in the CM system

Fermi's Golden Rule (number 2)

- We are now in a position to start thinking about calculations of the most important quantities in HEP: Γ and σ
- Fermi Golden rule is the key: Sec. 2.3 Thomson derivation

Transition rate → $$W = \frac{2\pi}{\hbar}\left|m_{if}\right|^2 \rho(E)$$ ← density of states available for energy E (phase space factor)

Matrix element of transition $i \to f$ $\left(\left|m_{if}\right|^2 = \left|\langle \psi_f \left| V_{if} \right| \psi_i \rangle\right|^2\right)$

- $|m_{if}|^2$ maybe unknown
  - extreme case it is a constant so the kinematics of the final state is purely governed by $\rho(E)$
- Therefore, we need to calculate $\rho(E)$ to understand the dynamics of the matrix element

Density of states

- State of motion of a single particle with a momentum between 0 to p confined to volume V is specified by a point in 6-D phase space (x,y,z,p_x,p_y,p_z)
- Limit to which a momentum and spatial coordinate can be specified is h from the uncertainty principle
    - Elemental volume of phase space is h^3
- Therefore, the number of states available to an individual particle, N_i, is:

$$N_i = \frac{\text{total phase space volume}}{\text{elementary volume}} = \frac{1}{(2\pi\hbar)^3}\int dx\,dy\,dz\,dp_x\,dp_y\,dp_z = \frac{V}{(2\pi\hbar)^3}\int d^3\mathbf{p}$$

- For a system of n particles the number of available final states, N_n, is the product of the individual particles:

$$N_n = \left(\frac{V}{(2\pi)^3}\right)^n \int \prod_{i=1}^{n} d^3\mathbf{p}_i \quad (\hbar = 1)$$

Phase space

- The phase space factor is defined as the number of states per unit energy interval per unit volume (V=1)

$$\rho(E) = \frac{dN_n}{dE} = \frac{1}{(2\pi)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n} d^3\mathbf{p}_i$$

- However, not all momenta are independent because of momentum conservation so there is the constraint:

$$\left(\sum_{i=1}^{n} \mathbf{p}_i \right) - \mathbf{P} = 0 \quad \text{where } \mathbf{P} \text{ is the total momentum}$$

- Can be accommodated by integrating over $n-1$ particles

$$\rho(E) = \frac{1}{(2\pi)^{3(n-1)}} \frac{d}{dE} \int \prod_{i=1}^{n-1} d^3\mathbf{p}_i$$

Phase space continued

- This can be re-expressed more usefully using Dirac δ functions to take care of the momentum conservation

Write the momentum conservation as:

$$\mathbf{p_n} - \left(\mathbf{P} - \sum_{i=1}^{n-1} \mathbf{p}_i \right) = 0 \quad \text{so} \int d^3\mathbf{p}_n \delta\left[\mathbf{p_n} - \left(\mathbf{P} - \sum_{i=1}^{n-1} \mathbf{p}_i \right) \right] = 1$$

$$\therefore \rho(E) = \frac{1}{(2\pi)^{3(n-1)}} \frac{d}{dE} \int \prod_{i=1}^{n-1} d^3\mathbf{p}_i = \frac{1}{(2\pi)^{3(n-1)}} \frac{d}{dE} \int \prod_{i=1}^{n} d^3\mathbf{p}_i \delta\left[\mathbf{p_n} - \left(\mathbf{P} - \sum_{i=1}^{n-1} \mathbf{p}_i \right) \right]$$

$$= \frac{1}{(2\pi)^{3(n-1)}} \frac{d}{dE} \int \prod_{i=1}^{n} d^3\mathbf{p}_i \delta\left[\mathbf{P} - \sum_{i=1}^{n} \mathbf{p}_i \right]$$

Phase space continued

- This can be re-expressed more usefully using Dirac δ functions to take care of the momentum conservation

$$\text{Energy conservation gives } \sum_{i=1}^{n} E_i - E = 0 \text{ so } \int dE\,\delta\left(\sum_{i=1}^{n} E_i - E\right) = 1$$

$$\therefore \rho(E) = \frac{1}{(2\pi)^{3(n-1)}} \frac{d}{dE} \int \prod_{i=1}^{n} d^3\mathbf{p}_i\, dE\,\delta\left[\mathbf{P} - \sum_{i=1}^{n} \mathbf{p}_i\right] \delta\left(\sum_{i=1}^{n} E_i - E\right)$$

$$= \frac{1}{(2\pi)^{3(n-1)}} \int \prod_{i=1}^{n} d^3\mathbf{p}_i\, \delta\left[\mathbf{P} - \sum_{i=1}^{n} \mathbf{p}_i\right] \delta\left(\sum_{i=1}^{n} E_i - E\right) \text{ as } \frac{d}{dE} \int f(E)\,dE = f(E)$$

Only problem this is not Lorentz invariant

Ensuring Lorentz invariance

- Fermi's golden rule: $$W = 2\pi \left| m_{if} \right|^2 \rho\left(E\right)$$
- If $\rho(E)$ is not Lorentz invariant then neither is $|m_{if}|^2$
- Consider a single massive particle moving with energy E in a volume V which is described by a wavefunction ψ normalised to $\int |\psi|^2 dV = 1$
- This normalisation implies that the particle density is $1/V$ for a stationary observer
- However, if the particle speed is relativistic then there will be a contraction by a factor $1/\gamma$ in the direction of motion so the particle density appears to be γ/V
- Normalising the wavefunctions to $\psi' \to \sqrt{\gamma}\psi$ ensures the particle density becomes invariant

Ensuring Lorentz invariance

Factor 2 later

For the transition rate we can redefine the matrix element to be:

$$\left|M_{if}\right|^2 = \left|m_{if}\right|^2 \prod_{j=1}^{n} 2m_j\gamma_j c^2 \prod_{i=1}^{n} 2m_i\gamma_i c^2 = \left|m_{if}\right|^2 \prod_{j=1}^{n} 2E_j \prod_{i=1}^{n} 2E_i$$

where j represents particles in the initial state so the transition rate to a single final state becomes

$$dW = 2\pi \frac{\left|M_{if}\right|^2}{\prod_{j=1}^{n} 2E_j} \frac{1}{(2\pi)^{3(n-1)}} \left(\prod_{i=1}^{n} \frac{d^3\mathbf{p}_i}{2E_i} \delta\left(\sum_{i=1}^{n} \mathbf{p}_i - \mathbf{P}\right) \delta\left(\sum_{i=1}^{n} E_i - E\right) \right)$$

Integrate over all final states to get:

$$\Rightarrow W = 2\pi \frac{\left|M_{if}\right|^2}{\prod_{j=1}^{n} 2E_j} \frac{1}{(2\pi)^{3(n-1)}} \int \left(\prod_{i=1}^{n} \frac{d^3\mathbf{p}_i}{2E_i} \delta\left(\sum_{i=1}^{n} \mathbf{p}_i - \mathbf{P}\right) \delta\left(\sum_{i=1}^{n} E_i - E\right) \right) = 2\pi \frac{\left|M_{if}\right|^2}{\prod_{j=1}^{n} 2E_j} \Phi_n(E)$$

Lorentz invariant phase space

$$\Phi_n(E) = \frac{1}{(2\pi)^{3(n-1)}} \int \prod_{i=1}^{n} \frac{d^3\mathbf{p}_i}{2E_i} \delta\left(\sum_{i=1}^{n} \mathbf{p}_i - \mathbf{P}\right) \delta\left(\sum_{i=1}^{n} E_i - E\right)$$

Recap and plan for today

- Monday and yesterday
  - the need for relativity, Lorentz transforms and four vectors
  - proper time and $p^{\mu} = (M\gamma, M\gamma\vec{v}) = (E, \vec{p})$
  - Using the four-momentum: two-body decay kinematics, centre-of-mass and threshold
  - Fermi Golden rule and Lorentz invariant phase space
- Today
  - two body decay rate
  - Dalitz plot
  - Cross section
  - Pseudorapidity

Ensuring Lorentz invariance

Factor 2 later

For the transition rate we can redefine the matrix element to be:

$$\left|M_{if}\right|^2 = \left|m_{if}\right|^2 \prod_{j=1}^{n} 2m_j\gamma_j c^2 \prod_{i=1}^{n} 2m_i\gamma_i c^2 = \left|m_{if}\right|^2 \prod_{j=1}^{n} 2E_j \prod_{i=1}^{n} 2E_i$$

where j represents particles in the initial state so the transition rate to a single final state becomes

$$dW = 2\pi \frac{\left|M_{if}\right|^2}{\prod_{j=1}^{n} 2E_j} \frac{1}{(2\pi)^{3(n-1)}} \left(\prod_{i=1}^{n} \frac{d^3\mathbf{p}_i}{2E_i} \delta\left(\sum_{i=1}^{n} \mathbf{p}_i - \mathbf{P}\right) \delta\left(\sum_{i=1}^{n} E_i - E\right) \right)$$

Integrate over all final states to get:

$$\Rightarrow W = 2\pi \frac{\left|M_{if}\right|^2}{\prod_{j=1}^{n} 2E_j} \frac{1}{(2\pi)^{3(n-1)}} \int \left(\prod_{i=1}^{n} \frac{d^3\mathbf{p}_i}{2E_i} \delta\left(\sum_{i=1}^{n} \mathbf{p}_i - \mathbf{P}\right) \delta\left(\sum_{i=1}^{n} E_i - E\right) \right) = 2\pi \frac{\left|M_{if}\right|^2}{\prod_{j=1}^{n} 2E_j} \Phi_n(E)$$

Lorentz invariant phase space

$$\Phi_n(E) = \frac{1}{(2\pi)^{3(n-1)}} \int \prod_{i=1}^{n} \frac{d^3\mathbf{p}_i}{2E_i} \delta\left(\sum_{i=1}^{n} \mathbf{p}_i - \mathbf{P}\right) \delta\left(\sum_{i=1}^{n} E_i - E\right)$$

Showing that it is invariant

To show that this Lorentz invariant consider the Lorentz transformations

for boost is in z direction:

$$p'_x = p_x \quad p'_y = p_y \quad p'_z = \gamma\left(p_z - \beta E\right) \quad E' = \gamma\left(E - \beta p_z\right)$$

$$\frac{dp'_z}{dp_z} = \gamma\left(1 - \beta\frac{dE}{dp_z}\right) = \gamma\left(1 - \beta\frac{p_z}{E}\right)$$

as $$\frac{dE}{dp_z} = \frac{d}{dp_z}\left(\sum_{i=xyz} p_i^2 + m^2\right)^{\frac{1}{2}} = p_z\left(\sum_{i=xyz} p_i^2 + m^2\right)^{-\frac{1}{2}} = \frac{p_z}{E}$$

$$\frac{dp'_z}{dp_z} = \gamma\left(1 - \beta\frac{p_z}{E}\right) = \frac{\gamma\left(E - \beta p_z\right)}{E} = \frac{E'}{E}$$

$$\Rightarrow \frac{dp'_z}{E'} = \frac{dp_z}{E} \therefore \frac{d^3\mathbf{p}'}{E'} = \frac{d^3\mathbf{p}}{E}$$

2 body phase space

$$\Phi_2(E) = \frac{1}{(2\pi)^3} \int \prod_{i=1}^{2} \frac{d^3\mathbf{p}_i}{2E_i} \delta\left(\sum_{i=1}^{2} \mathbf{p}_i - \mathbf{P}\right) \delta\left(\sum_{i=1}^{2} E_i - E\right)$$

$$= \frac{1}{(2\pi)^3} \iint \frac{d^3\mathbf{p}_1}{2E_1} \frac{d^3\mathbf{p}_2}{2E_2} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}) \delta(E_1 + E_2 - E)$$

$$= \frac{1}{(2\pi)^3} \iint \frac{d^3\mathbf{p}_1}{2E_1} \frac{d^3\mathbf{p}_2}{2E_2} \delta(\mathbf{p}_1 + \mathbf{p}_2) \delta(E_1 + E_2 - E) \quad \text{in centre of mass frame}$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}_1}{4E_1E_2} \delta(E_1 + E_2 - E) \quad \text{integrate over } \mathbf{p}_2$$

$$= \frac{1}{(2\pi)^3} \int \frac{4\pi |\mathbf{p}_1|^2 \, d|\mathbf{p}_1|}{4E_1E_2} \delta(E_1 + E_2 - E)$$

$$= \frac{1}{8\pi^2} \int \frac{|\mathbf{p}_1| \, dE_1}{E_2} \delta(E_1 + E_2 - E) \quad \text{as } |\mathbf{p}_1| \, d|\mathbf{p}_1| = E_1 dE_1 \text{ from } E_1^2 - p_1^2 = m_1^2$$

2 body phase space

To do the integral we need to write E_2 in terms of E_1, m_1 and m_2. In the centre of mass frame $\therefore$

$$\mathbf{p}_1^2 = \mathbf{p}_2^2 \Rightarrow E_1^2 - m_1^2 = E_2^2 - m_2^2 \Rightarrow E_2 = \left(E_1^2 - m_1^2 + m_2^2\right)^{\frac{1}{2}}$$

$$\Phi_2\left(E\right) = \frac{1}{8\pi^2}\int\frac{\left|\mathbf{p}_1\right| dE_1}{E_2}\delta\left(E_1 + \left(E_1^2 - m_1^2 + m_2^2\right)^{\frac{1}{2}} - E\right) = \frac{1}{8\pi^2}\int\frac{\left|\mathbf{p}_1\right| dE_1}{E_2}\delta\left(g\left(E_1\right)\right)$$

To integrate over E_1 we use the relation $\int dE_1 \delta\left(g(E_1)\right) = \left|\frac{dg}{dE_1}\right|^{-1}$

with $g(E_1) = E_1 + \left(E_1^2 - m_1^2 + m_2^2\right)^{\frac{1}{2}} - E$

$$\frac{dg}{dE_1} = 1 + E_1\left(E_1^2 - m_1^2 + m_2^2\right)^{-\frac{1}{2}} = \frac{E_2 + E_1}{E_2} = \frac{E}{E_2} \Rightarrow \left|\frac{dg}{dE_1}\right|^{-1}_{g(E_1)=0} = \frac{E_2}{E}$$

Two-body Lorentz invariant phase space is $\Phi_2\left(E\right) = \dfrac{1}{8\pi^2}\dfrac{\left|\mathbf{p}_1\right|}{E}$

Two body decay rate a$\rightarrow$1+2

Let's consider two-body decay of particle a mass m_a, so $E = m_a$ in CM frame

Two-body Lorentz invariant phase space is $\Phi_2\left(E\right) = \frac{1}{8\pi^2}\frac{\left|\mathbf{p}_1\right|}{E}$

$\left|\mathbf{p}_1\right| \equiv \left|\mathbf{p}*\right|$ is the momentum of the decay products of the rest frame a

Also, if $|\mathrm{M}_{if}|^2$ depends on the relative angle of the final state particles to the spin of the initial state

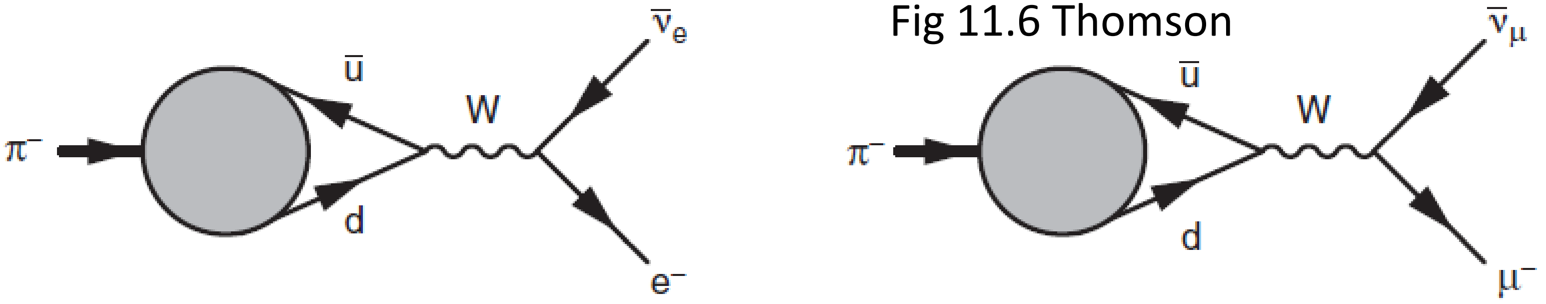
$$d\Phi_2\left(m_a,\Omega\right) = \Phi_2\left(m_a,\Omega\right)\frac{\mathrm{d}\Omega}{4\pi} = \frac{1}{32\pi^3}\frac{\left|\mathbf{p}*\right|}{m_a}\mathrm{d}\Omega$$

$$\therefore W = \Gamma = 2\pi\int\frac{\left|M_{if}\right|^2}{2E}d\Phi_2\left(M,\Omega\right) = 2\pi\ \frac{1}{2m_a}\frac{1}{32\pi^3}\frac{\left|\mathbf{p}*\right|}{m_a}\int|\mathrm{M}_{if}|^2\mathrm{d}\Omega = \frac{1}{32\pi^2}\frac{\left|\mathbf{p}*\right|}{m_a{}^2}\int|\mathrm{M}_{if}|^2\mathrm{d}\Omega$$

and $\quad \left|\mathbf{p}*\right| = \frac{1}{2m_a}\sqrt{\left(m_a^2-\left(m_1+m_2\right)^2\right)\left(m_a^2-\left(m_1-m_2\right)^2\right)}$

Pion decay

Fig 11.6 Thomson



Two of the three main decay modes for the π^-. The decay $\pi^- \to \mu^-\overline{\nu}_\mu\gamma$ (not shown) has a comparable branching ratio to that for $\pi^- \to e^-\overline{\nu}_e$.

So Feynman rules will lead to a well defined weak current for the lepton and quark part, but describing the strong non-perturbative binding in the initial state is impossible analytically

$$
\begin{aligned}
M_{fi} &\propto j_{\mu,\text{quarks}} j^{\mu}_{\text{leptons}} \\
&\propto F_{\mu} j^{\mu}_{\text{leptons}} \\
&\propto f_{\pi} p_{\mu} j^{\mu}_{\text{leptons}} \quad (f_{\pi} \text{ is pion decay constant})
\end{aligned}
$$

Lorentz invariance gives us the answer

Pion decay

$$M_{fi} = \frac{g_W^2 f_\pi}{8m_W^2} p_\mu \left[\bar{u}(p_2)\gamma^\mu (1-\gamma^5) v(p_1) \right]$$

$$\Rightarrow \left|\bar{M}_{fi}\right|^2 = \left(\frac{g_W^2 f_\pi}{8m_W^2} \right)^2 p_\mu p_\nu \mathrm{Tr}\left[\gamma^\mu (1-\gamma^5) \not{p}_1 \gamma^\mu (1-\gamma^5) \left(\not{p}_2 + m_l \right) \right]$$

$$= \frac{1}{8} \left(\frac{g_W^2 f_\pi}{m_W^2} \right)^2 \left[2(p.p_1)(p.p_2) - p^2 (p_1.p_2) \right]$$

Now we can use the four-momentum conservation to write

$$p = p_1 + p_2 \Rightarrow p.p_1 = p_1^2 + p_1.p_2 = p_1.p_2 \text{ and similarly } p.p_2 = m_l^2 + p_1.p_2$$

$$\text{Also, } p^2 = \left(p_1 + p_2\right)^2 \Rightarrow m_\pi^2 = m_l^2 + 2p_1.p_2 \Rightarrow p_1.p_2 = \frac{m_\pi^2 - m_l^2}{2}$$

$$\left|\bar{M}_{fi}\right|^2 \propto \left[2(p_1.p_2)(m_l^2 + p_1.p_2) - m_\pi^2 (p_1.p_2) \right]$$

$$\propto \left(\frac{m_\pi^2 - m_l^2}{2} \right) \left[2\left(m_l^2 + \frac{m_\pi^2 - m_l^2}{2} \right) - m_\pi^2 \right]$$

$$\propto m_l^2 \left(\frac{m_\pi^2 - m_l^2}{2} \right)$$

If m_l=0 the pion would never decay

Pion decay

$$\Gamma = \frac{1}{32\pi^2}\frac{|\mathbf{p}^*|}{m_a{}^2}\int |\mathrm{M}_{if}|^2 \mathrm{d}\Omega$$

$$= \frac{1}{32\pi^2}\frac{1}{m_\pi{}^2}\frac{(m_\pi^2 - m_l^2)}{2m_\pi}4\pi\frac{1}{8}\left(\frac{g_W^2 f_\pi}{m_W^2}\right)^2 m_l^2\left(\frac{m_\pi^2 - m_l^2}{2}\right)$$

$$= \frac{f_\pi^2}{\pi m_\pi{}^2}\left(\frac{g_W}{4m_W}\right)^4 m_l^2(m_\pi^2 - m_l^2)^2 \Rightarrow \frac{\Gamma(\pi \to e\nu)}{\Gamma(\pi \to \mu\nu)} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2}$$

Dalitz plot

Considering a scalar or pseudoscalar decaying into a three-body final state how many variables are required to describe it?

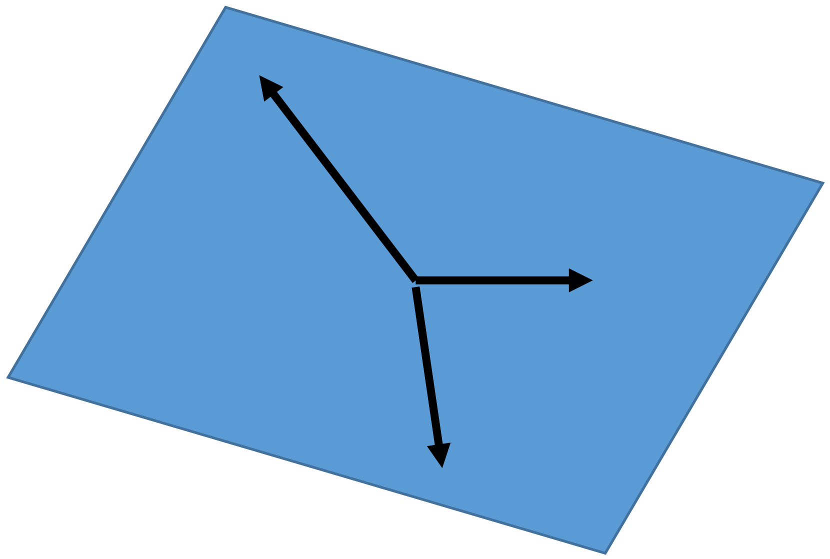
3 four-momenta = 12 variables

Constraints:

Energy-momentum conservation = 4

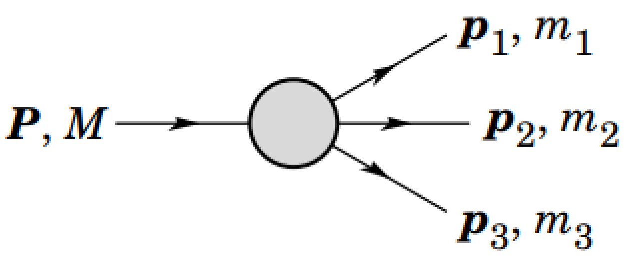
Particle masses = 3

Orientation of decay plane choice = 3



12-10 = two-independent variables

Dalitz plot



- Following (and figures) from PDG kinematics review general form with just the kinematic constraints - energies in rest frame of M and α, β and γ are Euler angles to define the orientation

$$d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M} |\mathscr{M}|^2 \, dE_1 \, dE_3 \, d\alpha \, d(\cos\beta) \, d\gamma$$

$$d\Gamma = \frac{1}{(2\pi)^3} \frac{1}{8M} \overline{|\mathscr{M}|^2} \, dE_1 \, dE_3$$

$$= \frac{1}{(2\pi)^3} \frac{1}{32M^3} \overline{|\mathscr{M}|^2} \, dm_{12}^2 \, dm_{23}^2$$

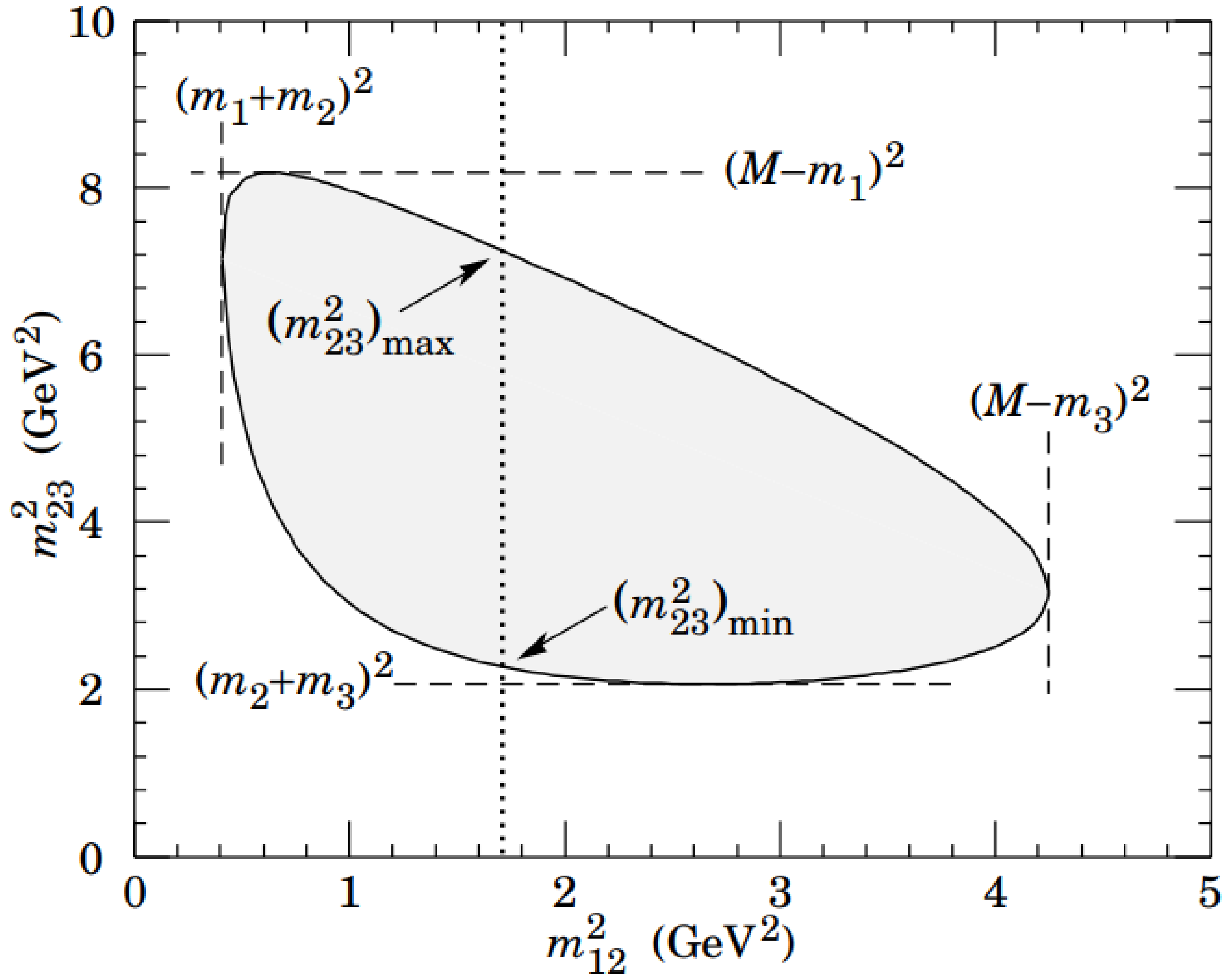
$$(p_i + p_j)^2 = m_{ij}^2$$

$$P = p_1 + p_2 + p_3$$

$$\Rightarrow (p_i + p_j)^2 = (P - p_k)^2 = M^2 + m_k^2 - 2P \cdot p_k$$

$$\therefore m_{ij}^2 = M^2 + m_k^2 - 2ME_k$$

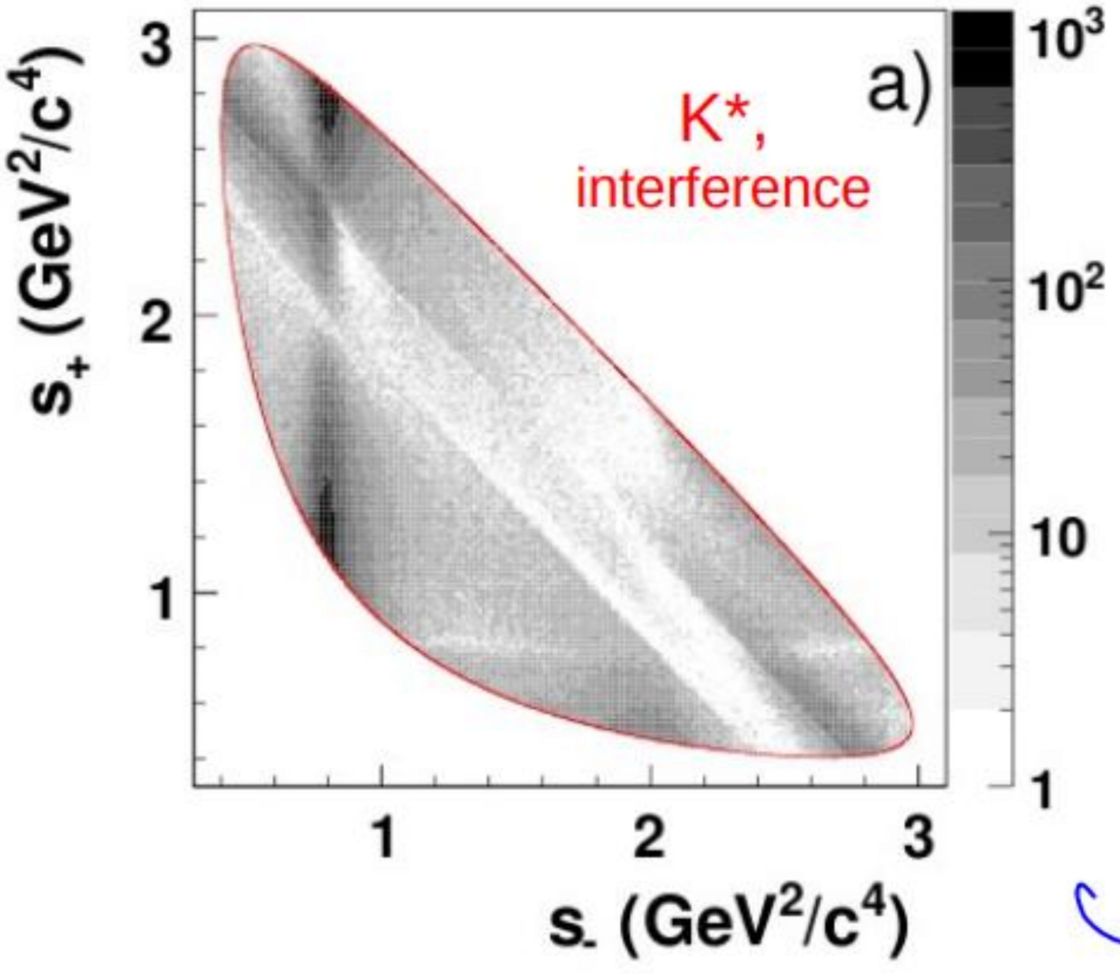
Dalitz plot



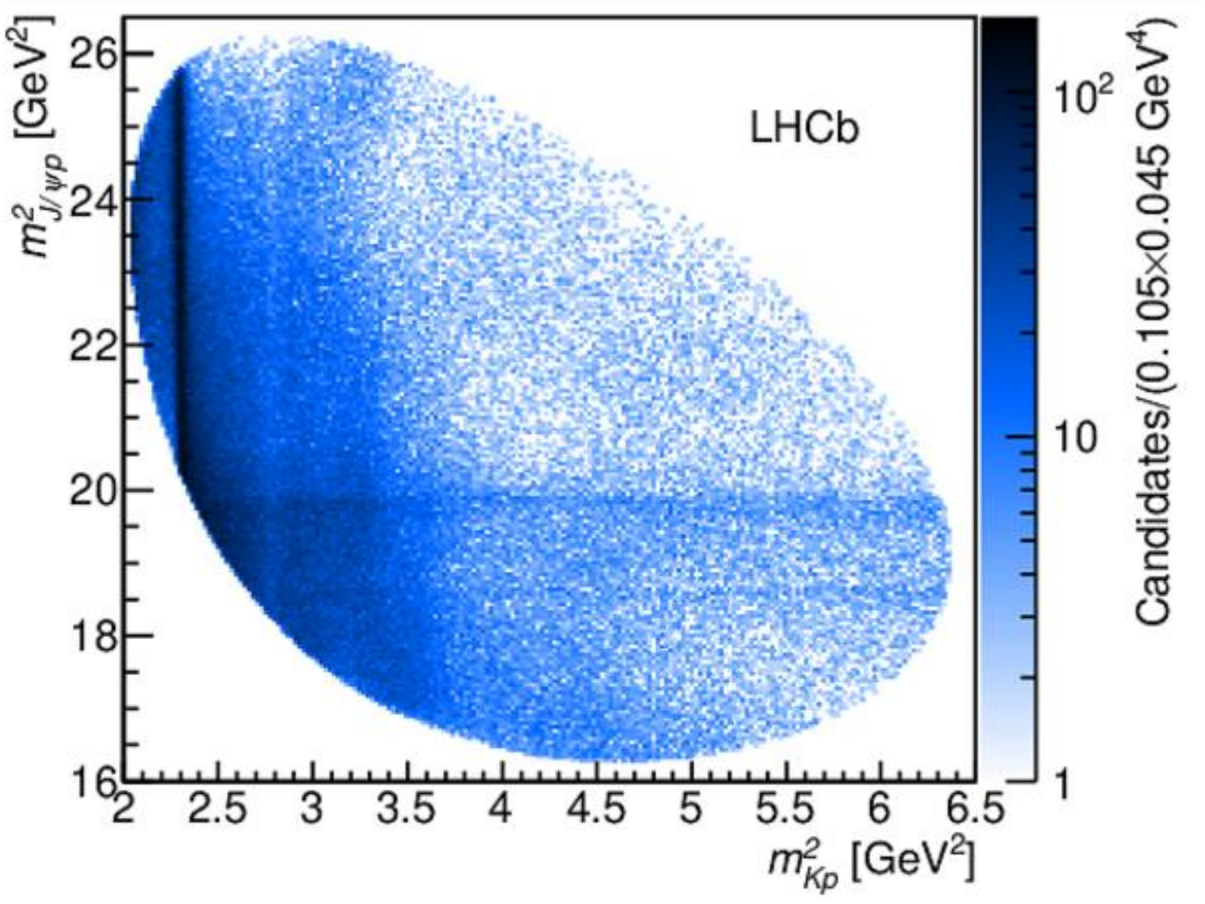
10
8
6
4
2
0
0
1
2
3
4
5
$(m_1+m_2)^2$
$(M-m_1)^2$
$(m^2_{23})_{max}$
$(M-m_3)^2$
$(m^2_{23})_{min}$
$(m_2+m_3)^2$
m^2_{23} (GeV^2)
m^2_{12} (GeV^2)

$D^0 \to K_s \pi^+ \pi^-$

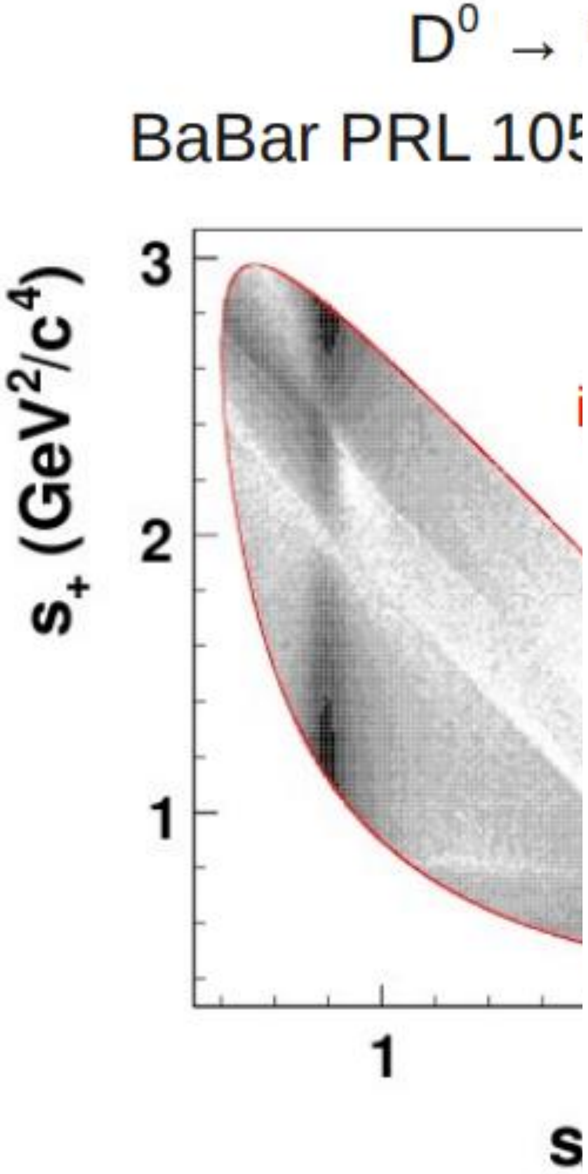
BaBar PRL 105 (2010) 081803

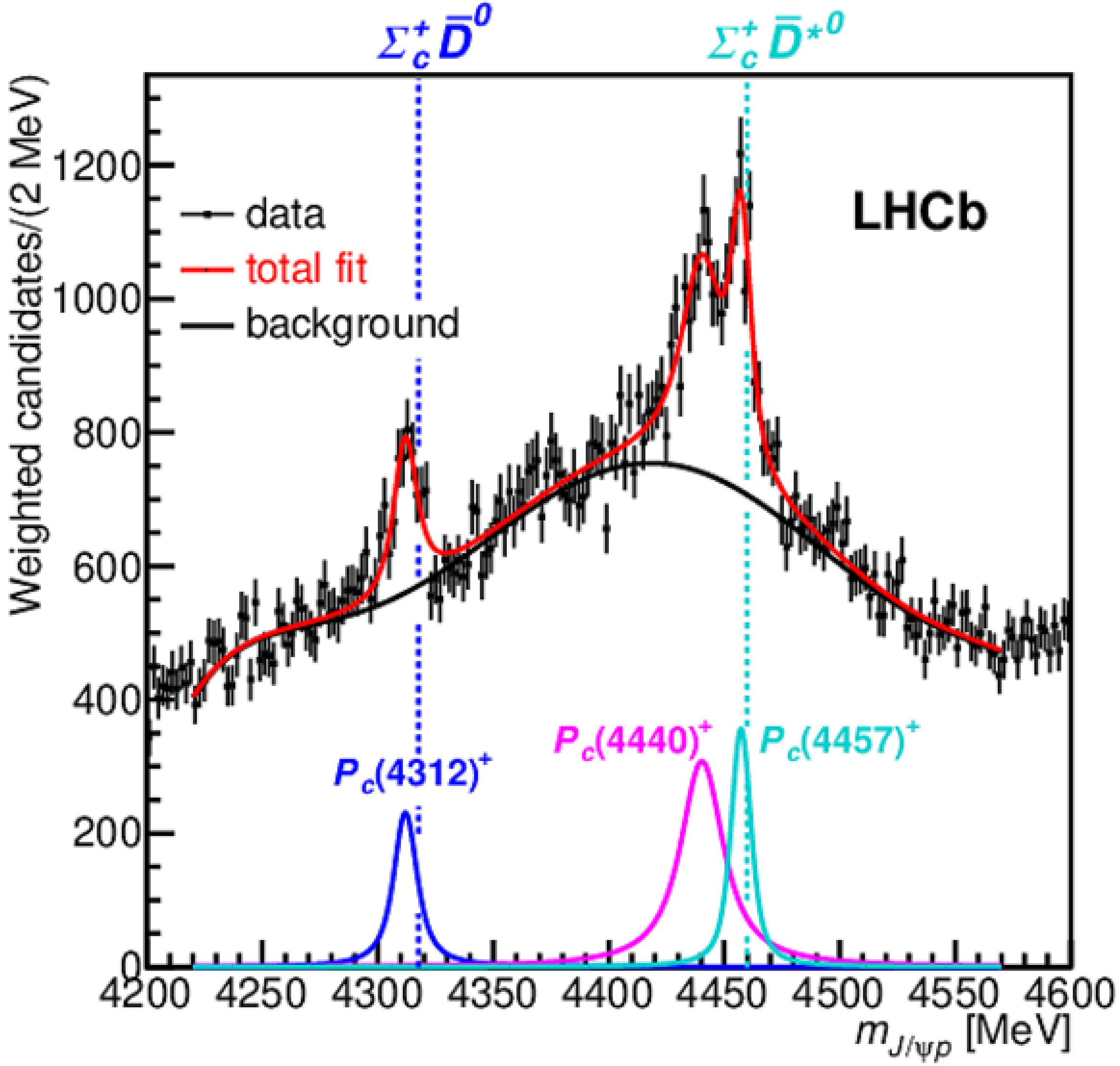


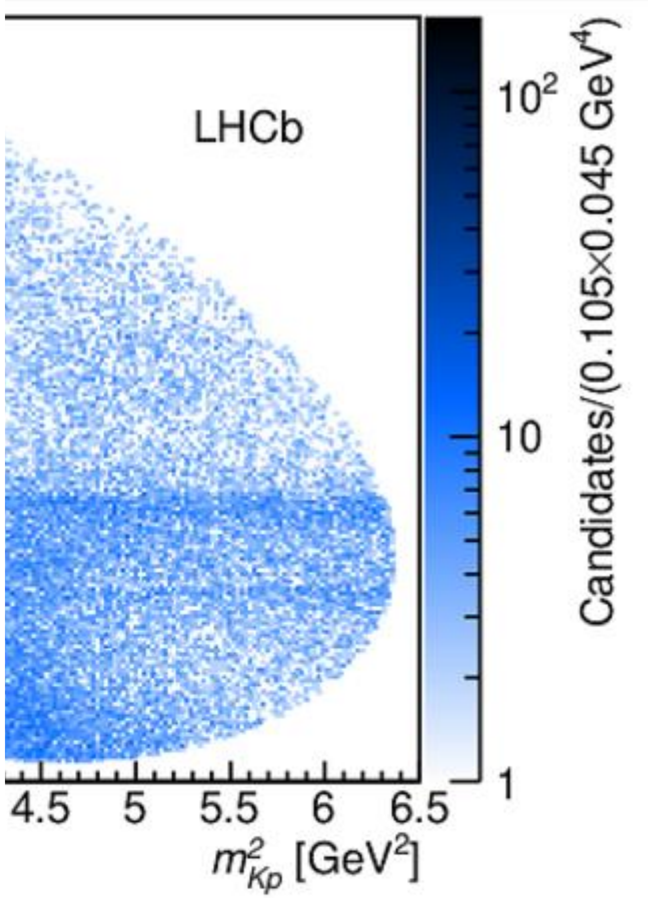
Observation of a narrow pentaquark state, Pc(4312)+, and of two-peak structure of the Pc(4450)+, **PHYS. REV. LETT. 122 (2019) 222001**



Observation of a narrow pentaquark state, Pc(4312)+, and of two-peak structure of the Pc(4450)+, **PHYS. REV. LETT. 122**

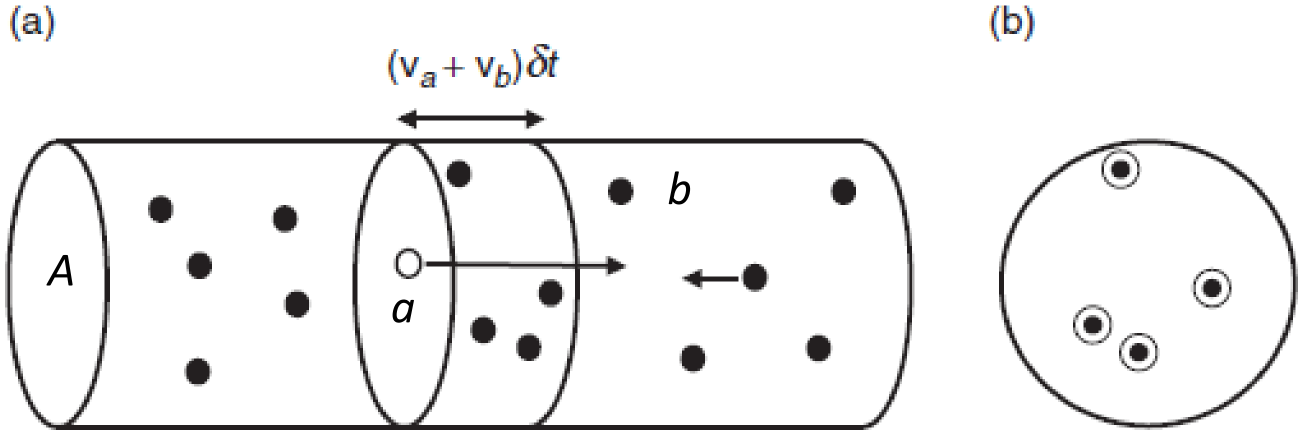






Thomson Fig. 3.5.

Cross section definition



The left-hand plot (a) shows a single incident particle of type *a* traversing a region containing particles of type *b*. The right-hand plot (b) shows the projected view of the region traversed by the incident particle in time δt.

Interaction probability

$$\delta P = \frac{\delta N \sigma}{A} = \frac{n_b (v_a + v_b) A \sigma \delta t}{A} = n_b v \sigma \delta t$$

$v_a + v_b$

Cross section definition

Interaction rate per particle is $r_a = \frac{dP}{dt} = n_b v \sigma$

flux

Total rate in volume V = $\Gamma = r_a n_a V = (n_a v)(n_b V)\sigma = \phi N_b \sigma$

No. of targets

σ=Number of interactions per unit time per target particle/incident flux

Now let us consider the scattering process *a+b→1+2* with 1 particle per unit volume normalization we have

$$\sigma = \frac{\Gamma}{v_a + v_b}$$

Recalling the Golden rule

Golden rule gives

$$\sigma = \frac{1}{(2\pi)^2 (v_a + v_b)} \int \left| m_{if} \right|^2 d^3\mathbf{p}_1 d^3\mathbf{p}_2 \delta\left(\mathbf{p}_a + \mathbf{p}_b - \mathbf{p}_1 - \mathbf{p}_2\right) \delta\left(E_a + E_b - E_1 - E_2\right)$$

$$= \frac{(2\pi)^{-2}}{4E_a E_b (v_a + v_b)} \int \left| M_{if} \right|^2 \frac{d^3\mathbf{p}_1}{2E_1} \frac{d^3\mathbf{p}_2}{2E_2} \delta\left(\mathbf{p}_a + \mathbf{p}_b - \mathbf{p}_1 - \mathbf{p}_2\right) \delta\left(E_a + E_b - E_1 - E_2\right)$$

$$= \frac{(2\pi)^{-2}}{F} \int \left| M_{if} \right|^2 \frac{d^3\mathbf{p}_1}{2E_1} \frac{d^3\mathbf{p}_2}{2E_2} \delta\left(\mathbf{p}_a + \mathbf{p}_b - \mathbf{p}_1 - \mathbf{p}_2\right) \delta\left(E_a + E_b - E_1 - E_2\right)$$

Where we define the ‘Lorentz invariant’ flux *F*

Lorentz invariant flux: check

$$F = 4E_a E_b (v_a + v_b) = 4E_a E_b \left(\frac{|\mathbf{p}_a|}{E_a} + \frac{|\mathbf{p}_b|}{E_b} \right) = 4\left(|\mathbf{p}_a| E_b + |\mathbf{p}_b| E_a \right)$$

$$\Rightarrow F^2 = 16\left(E_a^2 |\mathbf{p}_b|^2 + E_b^2 |\mathbf{p}_a|^2 + 2E_a E_b |\mathbf{p}_a||\mathbf{p}_b| \right)$$

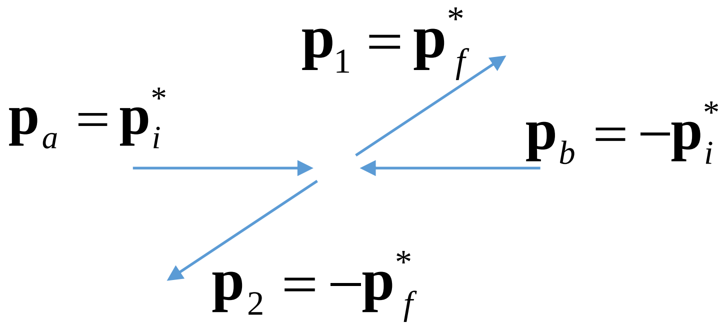
Note that when $\mathbf{p}_a$ and $\mathbf{p}_b$ are collinear in opposite directions

$$\Rightarrow \left(p_a \cdot p_b \right)^2 = \left(E_a E_b + |\mathbf{p}_a||\mathbf{p}_b| \right)^2 = E_a^2 E_b^2 + |\mathbf{p}_a|^2 |\mathbf{p}_b|^2 + 2E_a E_b |\mathbf{p}_a||\mathbf{p}_b|$$

$$\therefore F^2 = 16\left(\left(p_a \cdot p_b \right)^2 - \left(E_a^2 - |\mathbf{p}_a|^2 \right)\left(E_b^2 - |\mathbf{p}_b|^2 \right) \right)$$

$$\Rightarrow F = 4\sqrt{\left(p_a \cdot p_b \right)^2 - m_a^2 m_b^2}$$

Flux and cross section in CM



$$E_a^* + E_b^* = E_1^* + E_2^* = \sqrt{s}$$

Now we have a simple setup to analyse the problem

First we work out the flux:

$$F = 4E_a^* E_b^* \left(|\mathbf{v}_a^*| + |\mathbf{v}_b^*| \right) = 4E_a^* E_b^* \left(\frac{|\mathbf{p}_a^*|}{E_a^*} + \frac{|\mathbf{p}_b^*|}{E_b^*} \right) = 4|\mathbf{p}_i^*| \left(E_a^* + E_b^* \right) = 4|\mathbf{p}_i^*|\sqrt{s}$$

Now we can write the cross section calculation

$$\sigma = \frac{(2\pi)^{-2}}{4|\mathbf{p}_i^*|\sqrt{s}} \int |M_{if}|^2 \frac{d^3\mathbf{p}_1^*}{2E_1^*} \frac{d^3\mathbf{p}_2^*}{2E_2^*} \delta\left(\mathbf{p}_1^* + \mathbf{p}_2^*\right) \delta\left(\sqrt{s} - E_1^* - E_2^*\right)$$

$$= \frac{1}{16\pi^2 |\mathbf{p}_i^*| \sqrt{s}} \times \frac{|\mathbf{p}_f^*|}{4\sqrt{s}} \int |M_{if}|^2 \, d\Omega^* = \frac{|\mathbf{p}_f^*|}{64\pi^2 |\mathbf{p}_i^*| s} \int |M_{if}|^2 \, d\Omega^*$$

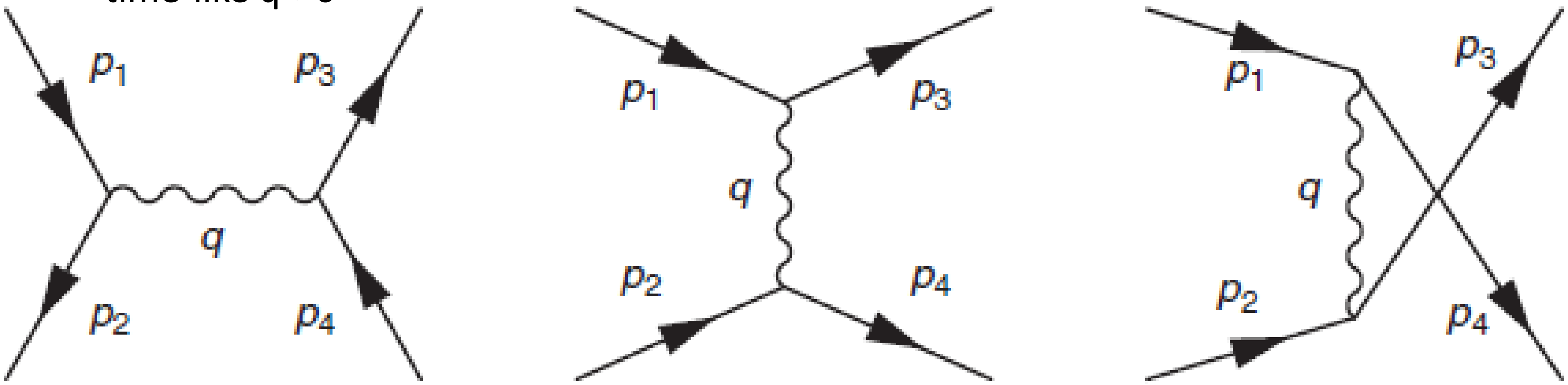
Using the a $\rightarrow$1 + 2 dLIPS with $m_a \rightarrow \sqrt{s}$

Mandelstam variables

Fig. 2.2 Thomson

time-like $q^2>0$

space-like $q^2<0$



Identical 3 and 4
cf Moller Scattering

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

Equivalent to q^2 of the propagator

In CM $s = (p_1 + p_2)^2 = (E_1^* + E_2^*)^2 - (\mathbf{p}_i^* - \mathbf{p}_i^*)^2 = (E_1^* + E_2^*)^2$ = (total energy in CM)^2

Mandelstam variables: a couple of useful relations

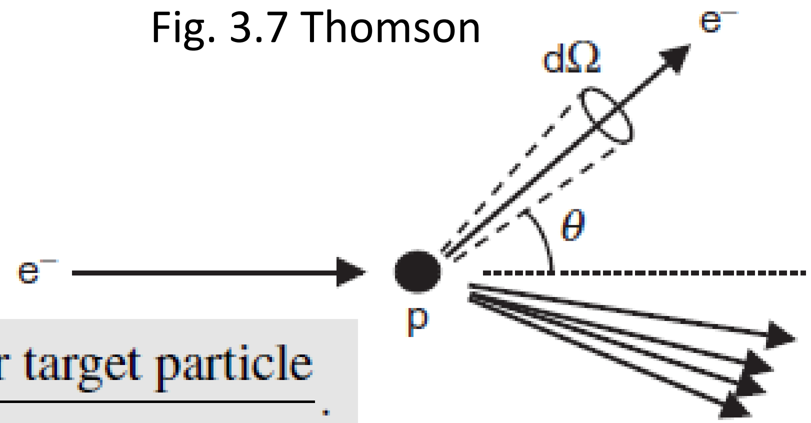
$$
\begin{aligned}
s+t+u &= \left(p_1+p_2\right)^2+\left(p_1-p_3\right)^2+\left(p_1-p_4\right)^2 \\
&= 3m_1^2+m_2^2+m_3^2+m_4^2+2\left(p_1\cdot p_2-p_1\cdot p_3-p_1\cdot p_4\right) \\
&= 3m_1^2+m_2^2+m_3^2+m_4^2+2p_1\cdot\left(p_2-p_3-p_4\right) \\
&= 3m_1^2+m_2^2+m_3^2+m_4^2+2p_1\cdot\left(-p_1\right) \quad \because\ p_1+p_2=p_3+p_4 \\
&= m_1^2+m_2^2+m_3^2+m_4^2
\end{aligned}
$$

When all particles in the relativistic limit i.e. $m_i^2 \approx 0 \Rightarrow$

$$
\begin{aligned}
s &\approx p_1\cdot p_2 \approx p_3\cdot p_4 \\
t &\approx -p_1\cdot p_3 \approx -p_2\cdot p_4 \\
u &\approx -p_1\cdot p_4 \approx -p_2\cdot p_3
\end{aligned}
$$

Differential cross sections

Fig. 3.7 Thomson



$$\frac{d\sigma}{d\Omega} = \frac{\text{number of particles scattered into } d\Omega \text{ per unit time per target particle}}{\text{incident flux}}.$$

Differential distributions contain a lot more information than the integrated total cross section - can be w.r.t. to other variables than solid angle too

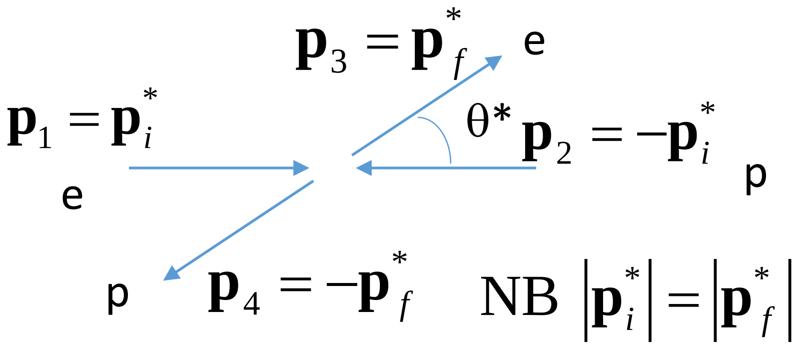
In CM

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s} \frac{\left|\mathbf{p}_f^*\right|}{\left|\mathbf{p}_i^*\right|} \left|M_{if}\right|^2$$

Lorentz invariant differential cross section

Elastic $e^- p \to e^- p$ in a fixed target experiment: lab frame $\neq$ CM frame

Can we find a invariant formulation of the differential cross section



NB $|\mathbf{p}_i^*| = |\mathbf{p}_f^*|$

$$t = \left(p_1^* - p_3^*\right)^2 = p_1^{*2} + p_3^{*2} - 2p_1^* \cdot p_3^*$$

$$= m_1^2 + m_3^2 - 2\left[E_1^* E_3^* - \mathbf{p}_1^* \cdot \mathbf{p}_3^*\right]$$

$$= m_1^2 + m_3^2 - 2\left[E_1^* E_3^* - |\mathbf{p}_1^*||\mathbf{p}_3^*|\cos\theta^*\right]$$

Only independent variable in elastic scat.

$$d\Omega^* = 2\pi d\left(\cos\theta^*\right) = \frac{2\pi}{2|\mathbf{p}_1||\mathbf{p}_3|} dt = \frac{\pi}{|\mathbf{p}_i^*||\mathbf{p}_f^*|} dt$$

$$\Rightarrow \frac{d\sigma}{dt} = \frac{1}{64\pi s|\mathbf{p}_i^*|^2} |M_{if}|^2$$

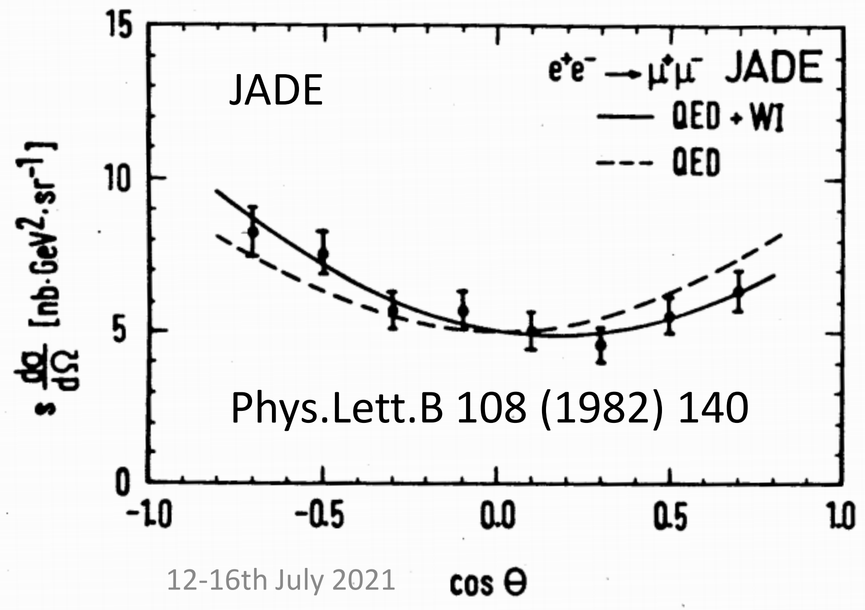
Recall $|\vec{\mathbf{p}}_i^*| = |\vec{\mathbf{p}}_f^*| = \frac{1}{2\sqrt{s}}\sqrt{\left(s - (m_1 + m_2)^2\right)\left(s - (m_1 - m_2)^2\right)}$

so this is a Lorentz invariant formulation in terms of s and t

Example: $e^+e^- \to f\bar{f}$

The spin-averaged matrix element for $e^+e^- \to \mu^+\mu^-$ is $\overline{\left|M_{fi}\right|^2} = e^4\left(1+\cos^2\theta^*\right)$ in QED when $s \gg m_\mu^2$

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\frac{\left|\mathbf{p}_f^*\right|}{\left|\mathbf{p}_i^*\right|}\overline{\left|M_{if}\right|^2} \Rightarrow \frac{d\sigma}{d\cos\theta^*} = \frac{1}{32\pi s}e^4\left(1+\cos^2\theta^*\right) = \frac{\pi\alpha^2}{2s}\left(1+\cos^2\theta^*\right)$$



$$e^+e^- \to \gamma \to \mu^+\mu^-$$

$$+$$

$$e^+e^- \to Z^0 \to \mu^+\mu^-$$

$$\sigma = \frac{\pi\alpha^2}{2s}\int_{-1}^{1}\left(1+\cos^2\theta^*\right)d(\cos\theta^*) = \frac{4\pi\alpha^2}{3s}$$

Rapidity

Recall
$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\eta \equiv y = \tanh^{-1}(\beta) \equiv \text{rapidity}$$

$$\Rightarrow \tanh\ y = \beta \Rightarrow \frac{e^{2y}-1}{e^{2y}+1} = \beta \Rightarrow e^{2y} = \frac{1+\beta}{1-\beta}$$

$$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$$

Rapidity at the LHC

- In pp collisions the there is only a fraction of each protons momentum associated with the partons in each collision x_1 and x_2 such that there is a boost in beam direction $(x_1-x_2)E_{proton}$ ($E_{proton}>>m_p$)
- If we define the z direction as that of the beams we can quote the rapidity of each particle or jet in the final state

$$y = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) = \frac{1}{2}\ln\left(\frac{1+p_z/E}{1-p_z/E}\right) = \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right)$$

- What is y' in an inertial frame moving in the beam direction i.e. the CM frame?

Rapidity gaps

$$y' = \frac{1}{2}\ln\left(\frac{E' + p'_z}{E' - p'_z}\right) = \frac{1}{2}\ln\left(\frac{\gamma(E - \beta p_z) + \gamma(p_z - \beta E)}{\gamma(E - \beta p_z) - \gamma(p_z - \beta E)}\right)$$

$$= \frac{1}{2}\ln\left(\frac{(1-\beta)(E + p_z)}{(1+\beta)(E - p_z)}\right) = \frac{1}{2}\ln\left(\frac{E + p_z}{E - p_z}\right) + \frac{1}{2}\ln\left(\frac{1-\beta}{1+\beta}\right)$$

$$= y + \frac{1}{2}\ln\left(\frac{1-\beta}{1+\beta}\right)$$

Therefore, $\Delta y' = \Delta y$, differences in rapidity are invariant.

Rapidity differences independent of the unknown boost in the z direction

Pseudorapidity

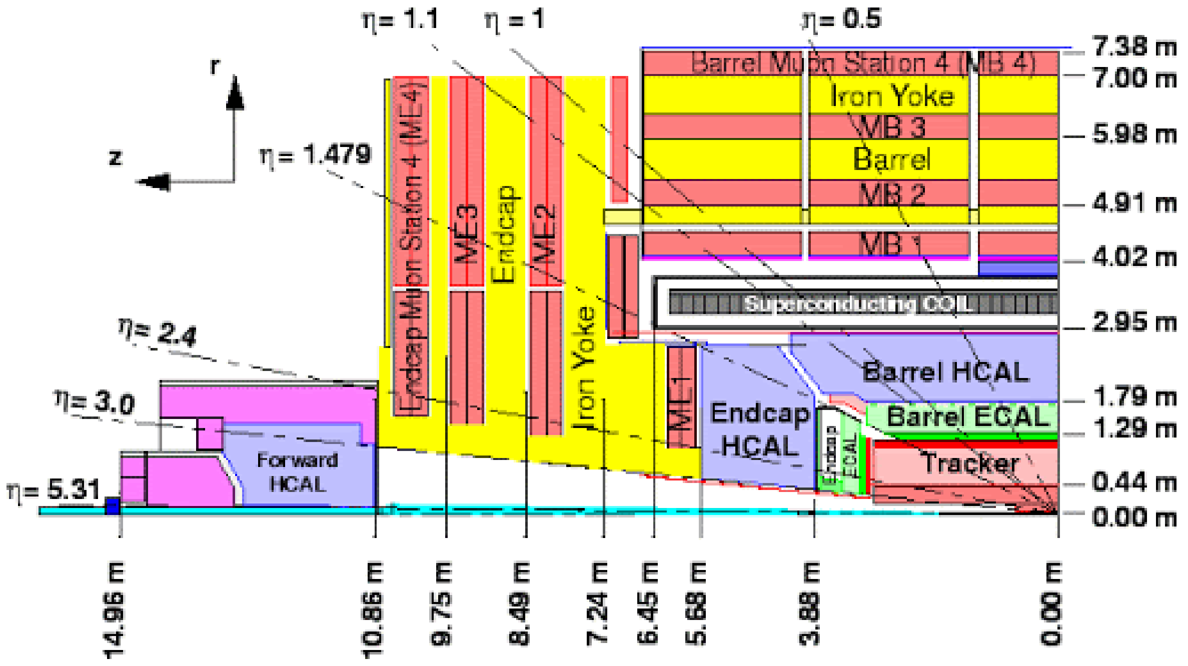
As already noted for quarks that fragment to jets and leptons produced the m_{jet} , $m_l << E_p$ so they can be treated as massless

So

$$y = \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right) \approx \frac{1}{2}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$$

$$\Rightarrow \eta = \frac{1}{2}\ln\left(\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right) = -\ln\left(\tan^2\frac{\theta}{2}\right)$$

$$= \text{pseudorapidity}$$



$$p_T = \sqrt{p_x^2 + p_y^2} \text{ (complementary variable invariant under } z \text{ boosts)}$$

Example: Drell Yan production

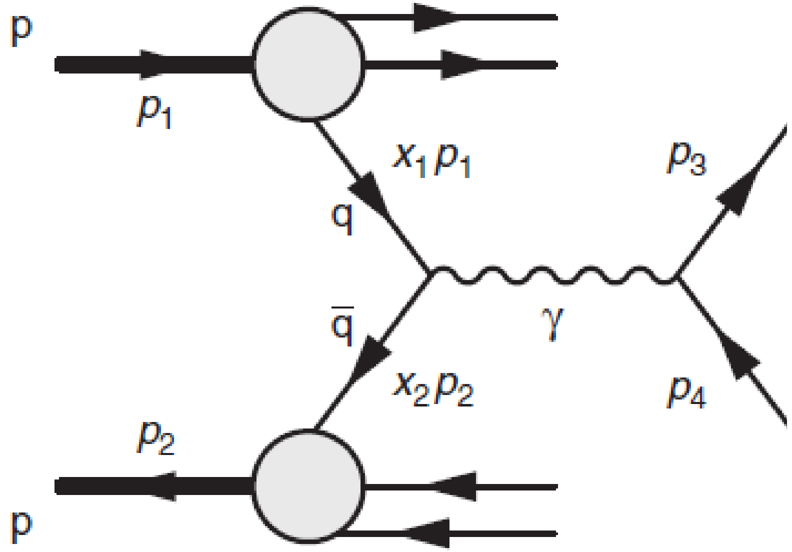
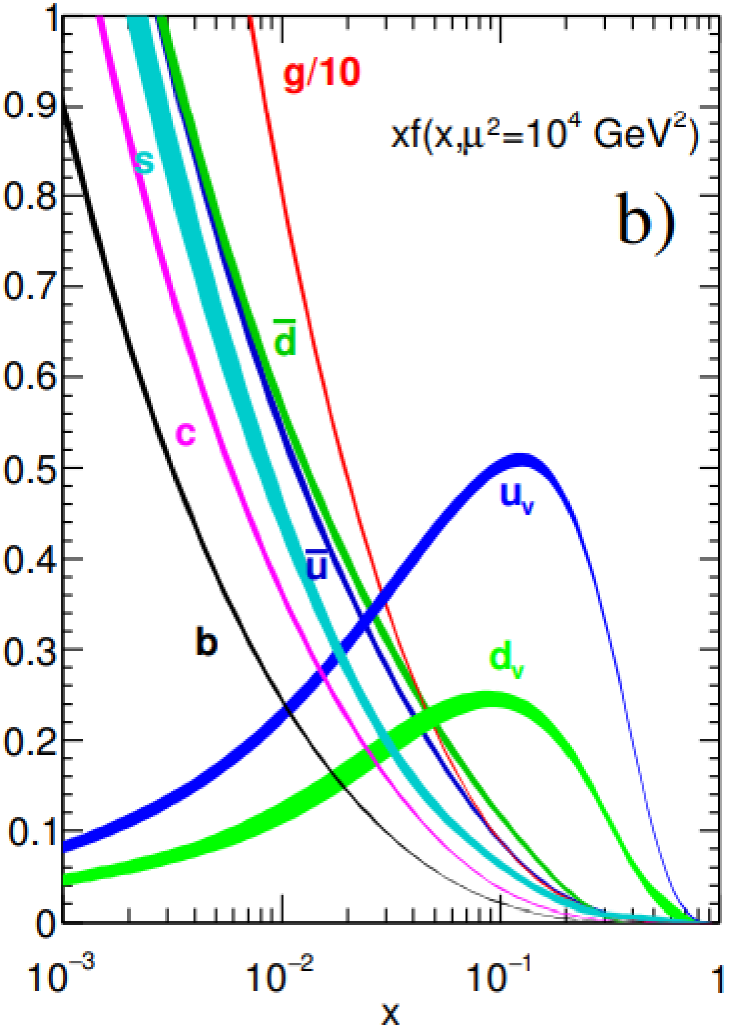


Fig. 10.28, Thomson



$(\hat{s})$

$2x_1x_2 p_1 \cdot p_2$

$$)\bar{u}(x_1)) + \frac{1}{9}\left(d(x_1)\bar{d}(x_2) + d(x_2)\bar{d}(x_1)\right)\bigg) dx_1 dx_2$$

Fig. 18.4 PDG

In terms of observables: y and M of muons

$$y_{\mu^+\mu^-} = y = \frac{1}{2}\ln\left(\frac{E_3 + E_4 + p_{3z} + p_{4z}}{E_3 + E_4 - p_{3z} - p_{4z}}\right) = \frac{1}{2}\ln\left(\frac{E_q + E_{\bar{q}} + p_{qz} + p_{\bar{q}z}}{E_q + E_{\bar{q}} - p_{qz} - p_{\bar{q}z}}\right) \text{ and}$$

$$p_q = \frac{\sqrt{s}}{2}(x_1, 0, 0, x_1) \quad p_{\bar{q}} = \frac{\sqrt{s}}{2}(x_2, 0, 0, -x_2)$$

$$y = \frac{1}{2}\ln\left(\frac{x_1 + x_2 + x_1 - x_2}{x_1 + x_2 - x_1 + x_2}\right) = \frac{1}{2}\ln\frac{x_1}{x_2}$$

$$M^2_{\mu^+\mu^-} = M^2 = \hat{s} = x_1 x_2 s$$

$$\Rightarrow M = \sqrt{x_1 x_2 s}$$

$$e^y = \sqrt{\frac{x_1}{x_2}} \Rightarrow x_1 = \frac{M}{\sqrt{s}}e^y \text{ and } x_2 = \frac{M}{\sqrt{s}}e^{-y}$$

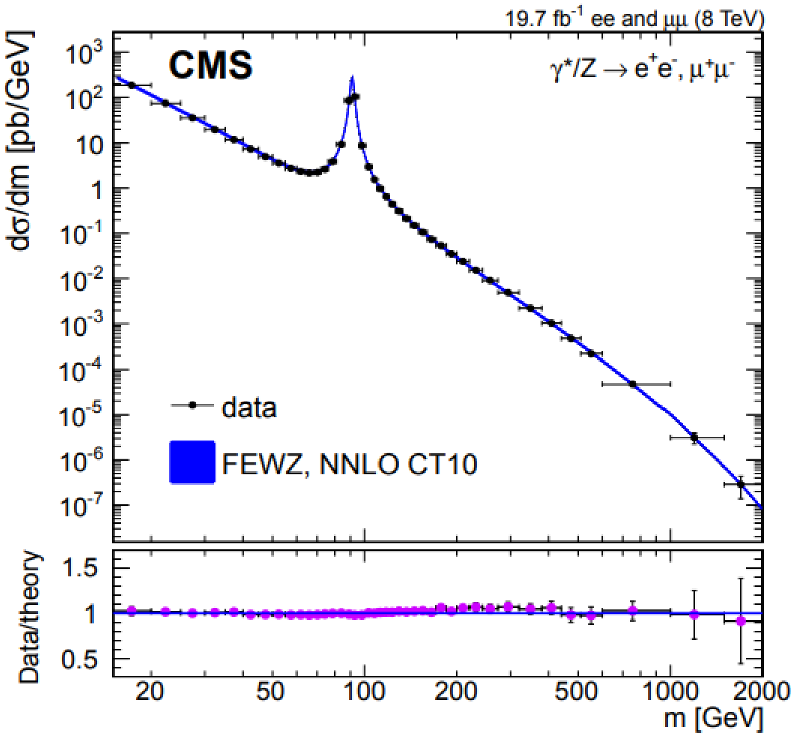
$$dydM = \frac{\partial(y, M)}{\partial(x_1, x_2)}dx_1 dx_2 = \begin{vmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} \\ \frac{\partial M}{\partial x_1} & \frac{\partial M}{\partial x_2} \end{vmatrix} dx_1 dx_2 = \frac{s}{2M}dx_1 dx_2$$

$$d^2\sigma = \frac{4\pi\alpha^2}{9x_1x_2s}\left(\frac{4}{9}\left(u(x_1)\bar{u}(x_2)+u(x_2)\bar{u}(x_1)\right)+\frac{1}{9}\left(d(x_1)\bar{d}(x_2)+d(x_2)\bar{d}(x_1)\right)\right)dx_1dx_2$$

$$= \frac{4\pi\alpha^2}{9x_1x_2s} f(x_1,x_2)dx_1dx_2$$

$$= \frac{4\pi\alpha^2}{9M^2} f\left(\tfrac{M}{\sqrt{s}}e^{y},\tfrac{M}{\sqrt{s}}e^{-y}\right)\frac{2M}{s}dydM$$

$$\Rightarrow \frac{d^2\sigma}{dydM} = \frac{8\pi\alpha^2}{9Ms} f\left(\tfrac{M}{\sqrt{s}}e^{y},\tfrac{M}{\sqrt{s}}e^{-y}\right)$$



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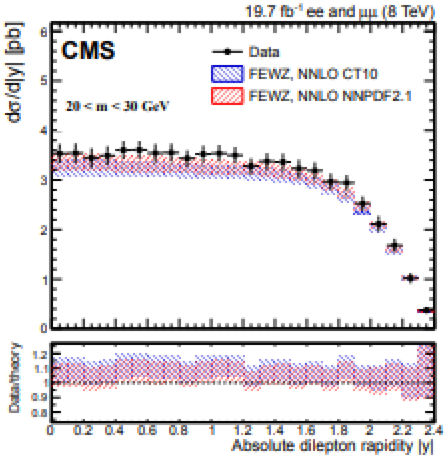
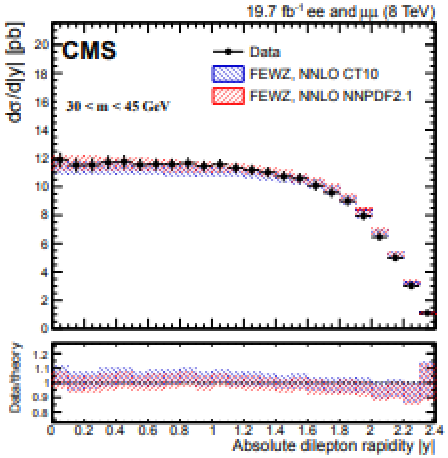
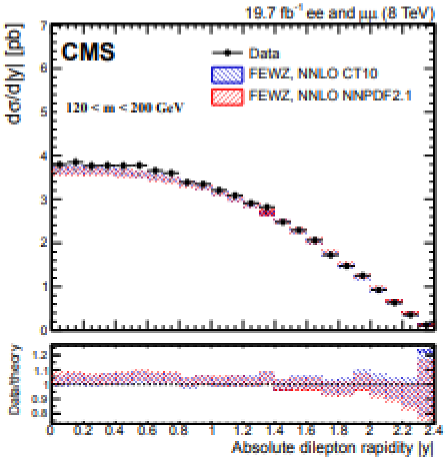
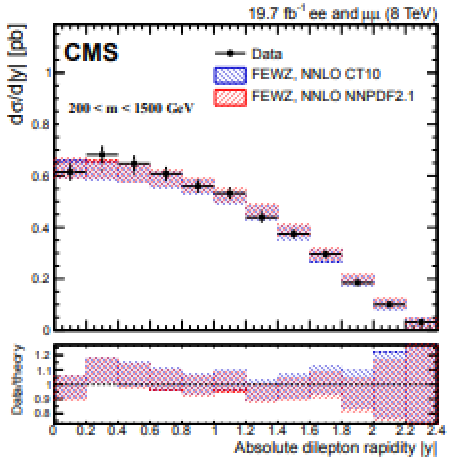








Additional slides

LT derivation

Muon decay

Recursive phase space

Lorentz transformations

- Why are they linear? c.f. rotations and translations
- So assume

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \Lambda\left(\vec{v}\right)\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44} \end{bmatrix}\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

- Without any loss of generality we can rotate so that Cartesian axes are aligned (i.e. $\Lambda(0)=I_4$) and that **v** is in the x (x') direction
- Latter means that as our transformation is invariant under rotations about the x-axis x' and t' cannot depend on y and z i.e.

$$\Lambda_{13} = \Lambda_{14} = \Lambda_{23} = \Lambda_{24} = 0$$

Lorentz transformation

By construction x and x' axes coincide so

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 & 0 \\ \Lambda_{21} & \Lambda_{22} & 0 & 0 \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Lambda_{12} \\ \Lambda_{22} \\ \Lambda_{32} \\ \Lambda_{42} \end{bmatrix} \Rightarrow \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 & 0 \\ \Lambda_{21} & \Lambda_{22} & 0 & 0 \\ \Lambda_{31} & 0 & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & 0 & \Lambda_{43} & \Lambda_{44} \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

Consider an events with x=t=0

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 & 0 \\ \Lambda_{21} & \Lambda_{22} & 0 & 0 \\ \Lambda_{31} & 0 & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & 0 & \Lambda_{43} & \Lambda_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \Lambda_{33} y + \Lambda_{34} z \\ \Lambda_{43} y + \Lambda_{44} z \end{bmatrix}$$

All events in (y',z') plane are simultaneous (t=t'=0) trivial comparison separation in this plane with y=y' and z=z' so we get

Lorentz transformation

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 & 0 \\ \Lambda_{21} & \Lambda_{22} & 0 & 0 \\ \Lambda_{31} & 0 & 1 & 0 \\ \Lambda_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

Now consider an object with x=vt in S i.e. x'=0

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 & 0 \\ \Lambda_{21} & \Lambda_{22} & 0 & 0 \\ \Lambda_{31} & 0 & 1 & 0 \\ \Lambda_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ vt \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Lambda_{11}ct + \Lambda_{12}vt \\ \Lambda_{21}ct + \Lambda_{22}vt \\ \Lambda_{31}ct \\ \Lambda_{41}ct \end{bmatrix} = \begin{bmatrix} ct' \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \Lambda_{31} = \Lambda_{41} = 0 \text{ and } \Lambda_{21} = -\frac{v}{c}\Lambda_{22} = -\beta\Lambda_{22}$$

Lorentz transformation

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 & 0 \\ -\beta\Lambda_{22} & \Lambda_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \Lambda_{11}ct + \Lambda_{12}x \\ -\beta\Lambda_{22}ct + \Lambda_{22}x \\ y \\ z \end{bmatrix}$$

Note we can get the Galilean transformation from this analysis because yet to assume constancy of c in S and S'. So with t=t' and $|\boldsymbol{r}_2 - \boldsymbol{r}_1| = |\boldsymbol{r}'_2 - \boldsymbol{r}'_1| \Rightarrow$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ct \\ -vt + x \\ y \\ z \end{bmatrix}$$

Lorentz transformation

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 & 0 \\ -\beta\Lambda_{22} & \Lambda_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \Lambda_{11}ct + \Lambda_{12}x \\ -\beta\Lambda_{22}ct + \Lambda_{22}x \\ y \\ z \end{bmatrix}$$

So recall c constant gave $(ct)^2 - |\vec{r}|^2 = (ct')^2 - \left|\overrightarrow{r'}\right|^2$

$$\Rightarrow c^2t^2 - x^2 - y^2 - z^2 = \left(\Lambda_{11}ct + \Lambda_{12}x\right)^2 - \left(-\beta ct + x\right)^2 \Lambda_{22}^2 - y^2 - z^2$$

$$\Rightarrow c^2t^2 - x^2 = \left(\Lambda_{11}^2 - \beta^2\Lambda_{22}^2\right)c^2t^2 - \left(\Lambda_{22}^2 - \Lambda_{12}^2\right)x^2 + 2\left(\Lambda_{11}\Lambda_{12} + \beta\Lambda_{22}^2\right)xct$$

$$\Rightarrow (i)\ \Lambda_{11}^2 - \beta^2\Lambda_{22}^2 = 1, \left(ii\right)\ \Lambda_{22}^2 - \Lambda_{12}^2 = 1, \left(iii\right)\ \Lambda_{11}\Lambda_{12} + \beta\Lambda_{22}^2 = 0$$

Lorentz transformation

$$\Rightarrow (i)\ \Lambda_{11}^2 - \beta^2\Lambda_{22}^2 = 1, \left(ii\right)\ \Lambda_{22}^2 - \Lambda_{12}^2 = 1, \left(iii\right)\ \Lambda_{11}\Lambda_{12} + \beta\Lambda_{22}^2 = 0$$

$$(iii) \Rightarrow \Lambda_{11}^2\Lambda_{12}^2 - \beta^2\Lambda_{22}^4 = 0 \quad (iv)$$

$$(i), (ii) \text{ and } (iv) \Rightarrow (1+\beta^2\Lambda_{22}^2)(1-\Lambda_{22}^2) + \beta^2\Lambda_{22}^4 = 0$$

Other solution?

$$\Rightarrow \Lambda_{22} = \gamma = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \Lambda_{11} = \gamma \Rightarrow \Lambda_{11} = -\beta\gamma$$

We are done

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma ct - \gamma\beta x \\ -\gamma\beta ct + \gamma x \\ y \\ z \end{bmatrix}$$

Muon decay: three-body phase space

$$\overline{|M|^2} = 2\left(\frac{g_W}{M_W}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4)$$

Working in the rest frame of the μ i.e. $p_1 = (m_\mu, 0)$

$p_1 \cdot p_2 = m_\mu E_2$ and

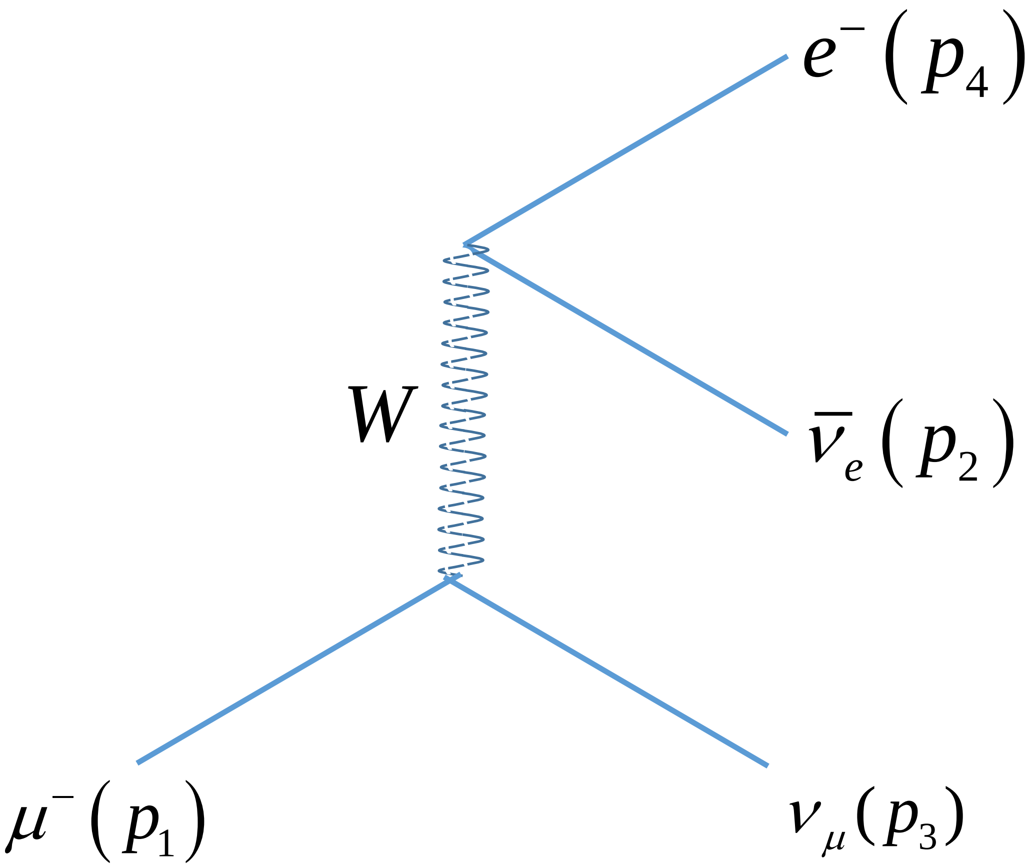
$$p_1 = p_2 + p_3 + p_4$$

$$\Rightarrow (p_3 + p_4)^2 = (p_1 - p_2)^2$$

$$\Rightarrow 2p_3 \cdot p_4 + m_e^2 = m_\mu^2 - 2p_1 \cdot p_2$$

$$\Rightarrow p_3 \cdot p_4 \approx \frac{m_\mu^2}{2} - m_\mu E_2 = \frac{m_\mu}{2}\left(m_\mu - 2E_2\right)$$

$$\therefore \overline{|M|^2} = \left(\frac{g_W}{M_W}\right)^4 m_\mu^2 E_2\left(m_\mu - 2E_2\right) = \left(\frac{g_W}{M_W}\right)^4 m_\mu^2 |\mathbf{p}_2|\left(m_\mu - 2|\mathbf{p}_2|\right)$$



$$dW = 2\pi \frac{\overline{|M|^2}}{\prod_{j=1}^{n} 2E_j} \frac{1}{(2\pi)^{3(n-1)}} \left(\prod_{i=1}^{n} \frac{d^3\mathbf{p}_i}{2E_i} \delta\left(\sum_{i=1}^{n} \mathbf{p}_i - \mathbf{P} \right) \delta\left(\sum_{i=1}^{n} E_i - E \right) \right)$$

$$\Rightarrow d\Gamma = \frac{\overline{|M|^2}}{2m_\mu} \prod_{i=2}^{4} \frac{d^3\mathbf{p}_i}{(2\pi)^3 2|\mathbf{p}_i|} (2\pi)^4 \delta(\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \delta(E_2 + E_3 + E_4 - m_\mu)$$

Integrating over $\mathbf{p}_3$ exploiting the delta function

$$d\Gamma = \frac{\overline{|M(|\mathbf{p}_2|)|^2}}{16(2\pi)^5 m_\mu} \frac{d^3\mathbf{p}_2 d^3\mathbf{p}_4}{|\mathbf{p}_2||\mathbf{p}_2 + \mathbf{p}_4||\mathbf{p}_4|} \delta(|\mathbf{p}_2| + |\mathbf{p}_2 + \mathbf{p}_4| + |\mathbf{p}_4| - m_\mu)$$

Now for $\mathbf{p}_2$ defining a polar angle θ w.r.t. the outgoing electron direction $\mathbf{p}_4$

Change of variables

$$d^3\mathbf{p}_2 = 2\pi\left|\mathbf{p}_2\right|^2 \sin\theta d\left|\mathbf{p}_2\right| d\theta \text{ and}$$

$$u^2 = \left|\mathbf{p}_2 + \mathbf{p}_4\right|^2 = \left|\mathbf{p}_2\right|^2 + \left|\mathbf{p}_4\right|^2 + 2\left|\mathbf{p}_2\right|\left|\mathbf{p}_4\right|\cos\theta$$

$$\Rightarrow 2udu = -2\left|\mathbf{p}_2\right|\left|\mathbf{p}_4\right|\sin\theta d\theta$$

$$\therefore d^3\mathbf{p}_2 = 2\pi\frac{\left|\mathbf{p}_2\right|}{\left|\mathbf{p}_4\right|}ud\left|\mathbf{p}_2\right|du$$

$$d\Gamma = \frac{\overline{\left|M(|\mathbf{p}_2|)\right|^2}}{16(2\pi)^4 m_\mu} \frac{d|\mathbf{p}_2| d^3\mathbf{p}_4}{|\mathbf{p}_4|^2} \int_{u_-}^{u_+} du\,\delta\left(|\mathbf{p}_2| + u + |\mathbf{p}_4| - m_\mu\right)$$

where $u_\pm = \sqrt{|\mathbf{p}_2|^2 + |\mathbf{p}_2|^2 \pm 2|\mathbf{p}_2||\mathbf{p}_4|} = \left||\mathbf{p}_2| \pm |\mathbf{p}_4|\right|$

Now the integral will be 1 if $u_- < m_\mu - |\mathbf{p}_2| - |\mathbf{p}_4| < u_+$

If $|\mathbf{p}_2| > |\mathbf{p}_4|$ then $|\mathbf{p}_2| - |\mathbf{p}_4| < m_\mu - |\mathbf{p}_2| - |\mathbf{p}_4| \Rightarrow |\mathbf{p}_2| < m_\mu / 2$

Similarly, $|\mathbf{p}_4| > |\mathbf{p}_2|$

$$d\Gamma = \frac{d^3\mathbf{p}_4}{16(2\pi)^4 m_\mu |\mathbf{p}_4|^2} \int_{m_\mu/2-|\mathbf{p}_4|}^{m_\mu/2} d|\mathbf{p}_2| \overline{|M(|\mathbf{p}_2|)|^2} = \frac{d^3\mathbf{p}_4}{16(2\pi)^4 m_\mu |\mathbf{p}_4|^2} \int_{m_\mu/2-|\mathbf{p}_4|}^{m_\mu/2} d|\mathbf{p}_2| \left(\frac{g_W}{M_W}\right)^4 m_\mu^2 |\mathbf{p}_2| \left(m_\mu - 2|\mathbf{p}_2|\right)$$

$$\Rightarrow \Gamma = \int_0^{m_\mu/2} \frac{4\pi d\mathbf{p}_4}{16(2\pi)^4 m_\mu} \int_{m_\mu/2-|\mathbf{p}_4|}^{m_\mu/2} d|\mathbf{p}_2| \left(\frac{g_W}{M_W}\right)^4 m_\mu^2 |\mathbf{p}_2| \left(m_\mu - 2|\mathbf{p}_2|\right)$$

You find

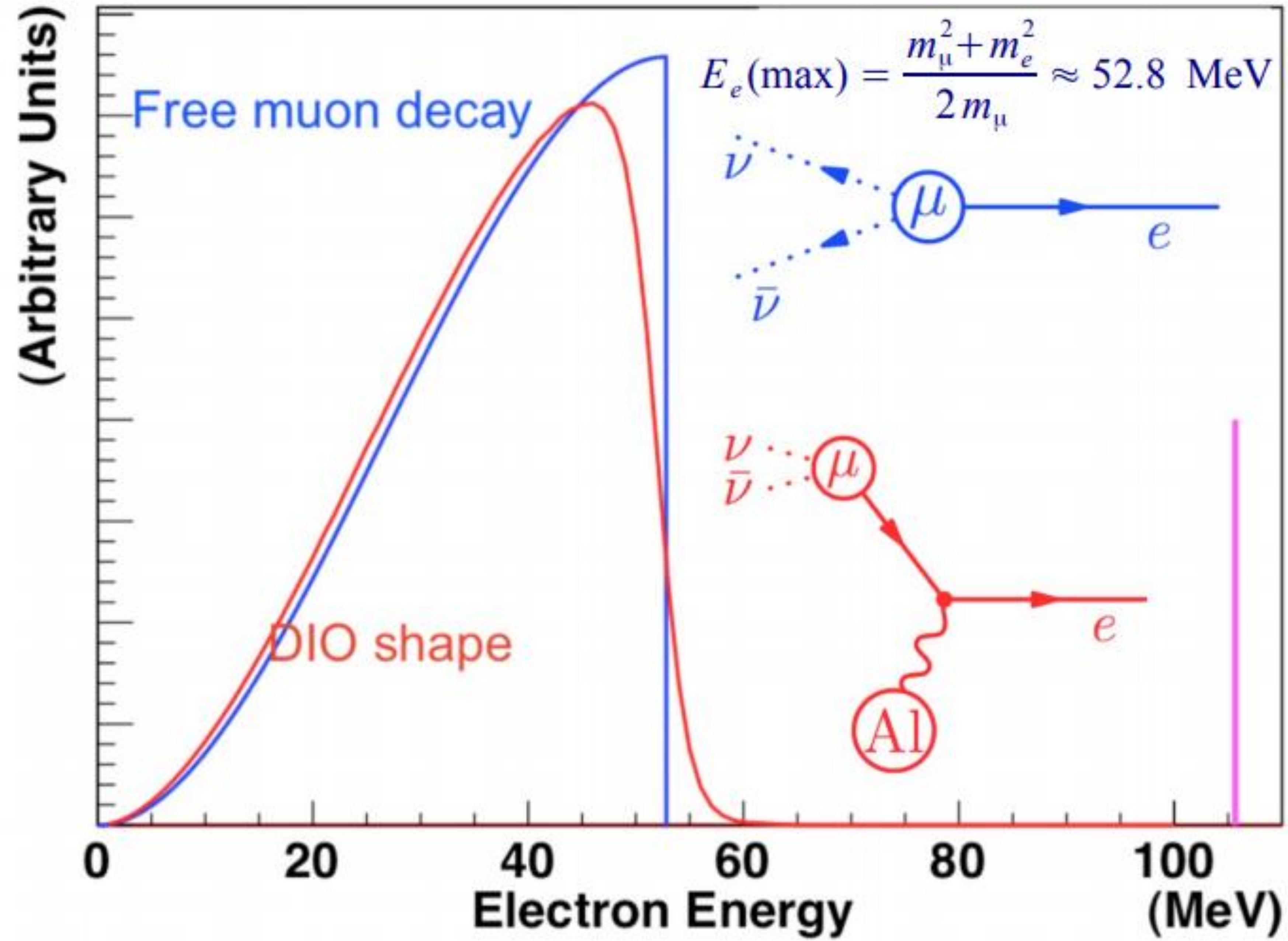
$$\frac{d\Gamma}{d|\mathbf{p}_4|} = \left(\frac{g_W}{M_W}\right)^4 \frac{m_\mu^2 |\mathbf{p}_4|^2}{2(4\pi)^3} \left(1 - \frac{4|\mathbf{p}_4|}{3m_\mu}\right) \text{ and } \Gamma = \left(\frac{m_\mu g_W}{M_W}\right)^4 \frac{m_\mu}{12(8\pi)^3}$$

$$d\Gamma = - \quad \ldots \quad n_\mu - 2|\mathbf{p}_2|)$$
$$1$$

$$\Rightarrow \Gamma =$$

You fi

$$\frac{d\Gamma}{d|\mathbf{p}_4|} =$$



https://mu2e.fnal.gov/public/index.shtml

Finding n-body phase space recursively

We can rewrite n-body phase space in the centre of mass frame

$$R_n(E) = \frac{1}{(2\pi)^{3(n-1)}} \int \prod_{i=1}^{n} \frac{d^3 p_i}{2E_i} \delta\left(\sum_{i=1}^{n} \mathbf{p}_i \right) \delta\left(\sum_{i=1}^{n} E_i - E \right) \quad \text{as}$$

$$R_n(E) = \frac{1}{(2\pi)^{3(n-1)}} \int \frac{d^3 p_n}{2E_n} \int \prod_{i=1}^{n-1} \frac{d^3 p_i}{2E_i} \delta\left(\sum_{i=1}^{n} \mathbf{p}_i - (-\mathbf{p_n}) \right) \delta\left(\sum_{i=1}^{n} E_i - (E - E_n) \right)$$

The second integral is the phase space integral for n-1 particles with total momentum $-\mathbf{p}_n$ and total energy $(E - E_n)$

Lorentz invariance allows this to be rewritten in terms of a system of zero total momentum and energy $\varepsilon^2 = (E - E_n)^2 - p_n^2$

As an example we can go to 3-body phase space from 2-body

$$R_3(E) = \frac{\pi}{(2\pi)^6} \int \frac{d^3 p_3}{2E_3} \frac{p_1(\varepsilon(E_3))}{\varepsilon} \quad \text{where } \varepsilon^2 = (E - E_3)^2 - p_3^2$$