Basic Statistics

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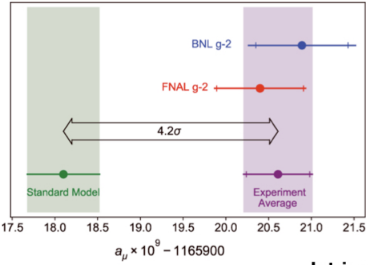
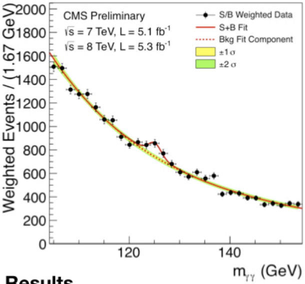
Our primary goal

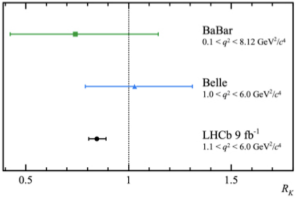
Experimental Measurements



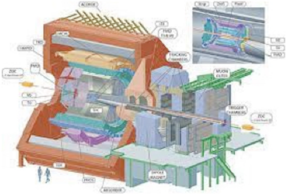
Discovery

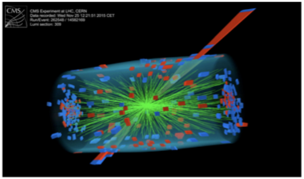


Intriguing Results

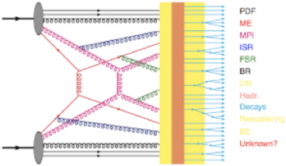


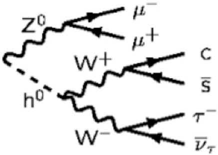
Initial ingredients





Guidance by existing theory





How to proceed ?

Outline

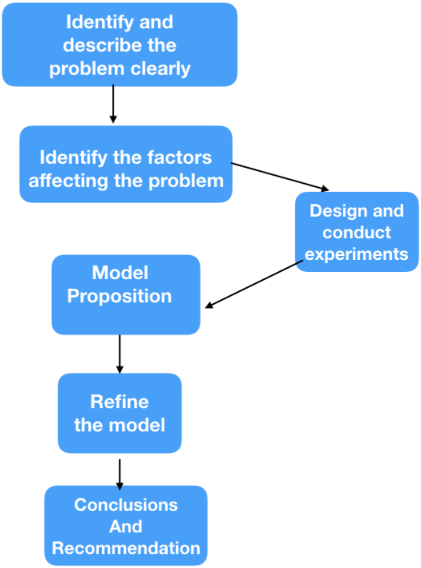


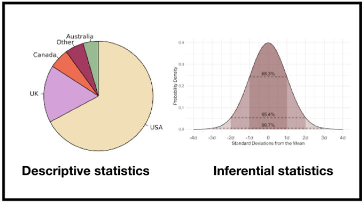
Figure: A schematic of scientific method

Statistics : Definition

The art of learning from data. It is concerned with the collection of data, its subsequent description, and its analysis, which often leads to the drawing of conclusions.

Two Types :

1. Descriptive Statistics : The part of statistics, concerned with the description and summarization of data, is called descriptive statistics.
2. Inferential Statistics : The part of statistics, concerned with the drawing of conclusions, is called inferential statistics.



Population and Sample

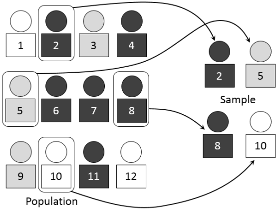


Figure: A schematic of population and sample

Population is the entire set of possible cases for which a study is conducted.
Sample refers to a subgroup of population from which we try to learn about the population.
A measure concerning a population is called **parameter** while that of a sample is called a **statistic**.

Sample Spaces and Events

Random Experiment : An experiment that can result in different outcomes, even though it is repeated in similar manner every time, is called a random experiment.

Sample Space : The set of all possible outcomes of a random experiment is called the sample space of the experiment. The sample space is denoted as S

Example : Consider an experiment in which the thickness of a silicon wafer is measured and a test is carried out to see whether it conforms to the required specifications or not. The outcome can be yes or no. If two wafers are selected, the sample space can be represented by

$S = (yy, yn, ny, nn)$

Roll a fair dice. The sampl e space consists of

$S = (1, 2, 3, 4, 5, 6)$

Event : An event is a subset of the sample space of a random experiment .

Why probability ?

Probability is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur. The likelihood of an outcome is quantified by assigning a number from 0 to 1 to the outcome (or a percentage from 0 to 100%). The higher numbers indicate that the outcome is more likely than lower numbers.
Suppose we perform a random experiment, the sample space consists of all possible outcomes of the random experiment. An event is a subset of the sample space.
How do we assign probability to the occurrence of an event ?

- Subjective Approach
- Frequency Approach
- Bayesian Approach

Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties: If S is the sample space and E is any event in a random experiment,

- P(S) = 1
- $0 \leq P(E) \leq 1$

- If two events E_1 and E_2, which have no outcomes in common , $P(E_1 U E_2) = P(E_1) + P(E_2)$

Conditional Probability

The conditional probability of an event F given an event E has already occurred, denoted by $P(F|E)$, is given by
$P(F|E) = P(E \cap F)/P(E)$, for $P(E) > 0$
The validity of the definition can be easily shown for a special case in which all outcomes of a random experiment are equally likely.
If there are n outcomes,
$P(E) = \frac{\text{number of outcomes in E}}{n}$
$P(E \cap F) = \frac{\text{number of outcomes in E} \cap \text{F}}{n}$
Therefore, $\frac{P(E \cap F)}{P(E)} = \frac{\text{number of outcomes in E} \cap \text{F}}{\text{number of outcomes in E}}$
Hence, $P(F|E)$ can be interpreted as the relative frequency of event F among the trials that produce an outcome in event E.

Multiplication Rule

The general expression for the probability of the intersection of two events is called the multiplication rule for probabilities.
$P(E \cap F) = P(F|E)P(E) = P(E|F)P(F)$
Total Probability Rule For any two events E and F, one can express E as $E = (E \cap F) \cup (E \cap F^{'})$
The value to be in event E, it must be either in both E and F or be in E but not in F. As EF and $EF^{'}$ are mutually exclusive sets, one can write that $P(E) = P(E \cap F) + P(E \cap F^{'})$
$= P(E|F)P(F) + P(E|F^{'})P(F^{'})$
The above expression can be generalized for k mutually exclusive and exhaustive sets. If F_1, F_2, .., F_k are k mutually exclusive and exhaustive sets, then
$P(E) = P(E \cap F_1) + P(E \cap F_2) + ... + P(E \cap F_k)$
$= P(E|F_1)P(F_1) + P(E|F_2)P(F_2). + ... + P(E|F_k)P(F_k)$

Baye's Theorem

According to the definition of conditional probability,
$P(E \cap F) = P(F|E)P(E) = P(E|F)P(F)$
We can write
$P(E|F) = \frac{P(F|E)P(E)}{P(F)}$, for $P(F) > 0$
If $E_1, E_2, .., E_k$ are k mutually exclusive and exhaustive sets and F is any event,
$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F|E_1)P(E_1)+P(F|E_2)P(E_2).+....+P(F|E_k)P(E_k)}$, for $P(F) > 0$
Example: Consider a beam which has 90% pions and 10% kaons. The kaons have 95% probability of giving no Cherenkov signal while pions have 5% probability of giving none. What is the probability that a particle that gave no signal is a kaon?

Bayesian Philosophy

$P(Theory|Data) = \frac{P(Data|Theory)P(Theory)}{P(Data)}$
Posterior $\sim Likelihood \times Prior$

Random Variable : Definition

A **Random Variable** is a variable that associates a number with the outcome of a random experiment.
A Random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.
They are generally denoted by upper case alphabets **X or Y** to distinguish from algebraic variables.
After an experiment is conducted, the measured value of the random variable is denoted by a lower case letter.
The **range** of a random variable X, shown by Range(X) or R_X, is the set of possible values of X.

Type of Random Variables :

Discrete Random Variables

Random variables whose set of possible values can be written either as a finite sequence $x_1, ..., x_n$ or as an infinite sequence $x_1, ...$ are said to be discrete.

A discrete random variable is a random variable with a finite (or countably infinite) range.

For instance, a random variable whose set of possible values is the set of non-negative integers is a discrete random variable.

Example: outcome of coin toss or random dice experiments, number of persons affected with covid-19 in a year etc.

Continuous Random Variables

Random variables that take on a continuum of possible values are known as continuous random variables.

A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

We can think of lifetime of a laboratory instrument, when the lifetime is assumed to take on any value in some interval (a, b).

Probability Distributions

The probability distribution of a random variable X is a description of the probabilities associated with the possible values of X.

Probability Mass Function (PMF) For a discrete random variable **X** with possible values $x_1, x_2, ..x_n$, a **probability mass function** is a function such that

- $f(x_i) \geq 0$
- $\sum_{i=0}^{\infty} f(x_i) = 1$
- $f(x_i) = P(X = x_i)$

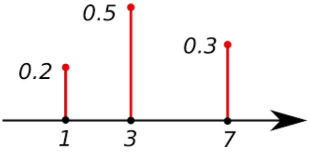


Figure: PMF of a discrete random variable

Cumulative Distribution Function (CDF) of Discrete random variable

The cumulative distribution function (or simply the distribution function) of a discrete random variable X, denoted as F(x), is the probability that the random variable X takes on a value that is less than or equal to x

$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$

For a discrete random variable X, F(x) satisfies the following properties.

- $F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$
- $0 \leq F(x) \leq 1$
- If x $\leq$ y, then F(x) $\leq$ F(y)

Probability Density Function (PDF)

For a continuous random variable X, a probability density function is a function such that

- $f(x) \geq 0$
- $\int\limits_{-\infty}^{\infty} f(x)dx = 1$
- $P(a \leq X \leq b) = \int\limits_{a}^{b} f(x)dx =$ area under $f(x)$ from a and b

  It also follows

  $P(X = a) = \int\limits_{a}^{a} f(x)dx = 0$

  The probability that a continuous random variable will assume any particular value is **zero**.

  If X is a continuous random variable, for any x_1 and x_2,

  $P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2)$

The cumulative distribution function (cdf) of a continuous random variable X is

$$F(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx \text{ , for } -\infty < a < \infty$$

Differentiating both the sides:

$$\frac{d}{da}F(a) = f(a)$$

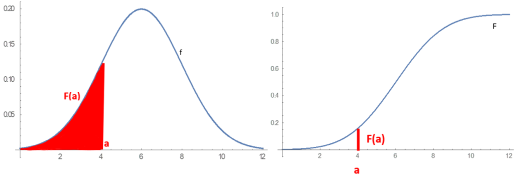


Figure: (Left panel) : PDF (Right panel) : CDF of a continuous random variable

$$P\left\{a - \epsilon/2 \leq X \leq a + \epsilon/2\right\} = \int_{a-\epsilon/2}^{a+\epsilon/2} f(x)dx \sim \epsilon f(a)$$ when ϵ is small. In other words, the probability that X will be contained in an interval of length ϵ around the point a is approximately $\epsilon f(a)$.

Expectation Value

If X is a discrete random variable taking on the possible values $x_1, x_2, ..$, the mean or expected value of the discrete random variable X, denoted as μ or E[X], is

$$\mu = E[X] = \sum_i x_i f(x_i)$$

Thus, the expected value of X is a weighted average of the possible values that X can take on, each value being weighted by the probability associated with X.

If X is a continuous random variable with the probability density function $f(x)$, the mean or expected value of X, denoted as E[X], is given by

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

Variance

Variance quantifies the variation, or spread, of the values associated with the random variable X. It measures the dispersion, or variability in the distribution.

If X is a discrete random variable with mean μ, then the variance of X , denoted by Var(X) or σ^2, is defined by

$Var(X) = \sigma^2 = E[(X - \mu)^2]$

$= \sum_x (x - \mu)^2 f(x) = \sum_x x^2 f(x) - \mu^2$

If X is a continuous random variable with probability density function f(x), V(X) or σ^2 is defined as

$$Var(X) = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

The standard deviation, σ is

$\sigma = \sqrt{\sigma^2}$

It can be shown

$Var(X) = E[X^2] - \mu^2$

If a and b are constants, $Var(aX + b) = a^2 Var(X)$

$Var(b) = 0$

Covariance

The covariance between the random variables X and Y, denoted as $cov(X,Y)$ or σ_{XY}, is

$\sigma_{XY} = Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$=$ E[XY] - $\mu_x E[Y] - \mu_y E[X] + \mu_x \mu_y = E[XY] - E[X]E[Y]$

$\sigma_{XY} = \sigma_{YX}$

$\sigma_{XX} = Var(X)$

Properties :

- Cov(X,X)=Var(X)
- Cov(X,Y) = Cov(Y,X)
- Cov(aX,Y) = aCov(X,Y)
- Cov(X+c,Y) = Cov(X,Y)
- Cov(X+Y, Z) = Cov(X,Z) + Cov(Y,Z)
  Cov(X+Y, Z)= E[(X+Y)Z] - E(X+Y)E[Z]
  =E[XZ + YZ] - (E[X]+E[Y])E[Z]
  =E[XZ] - E[X]E[Z] + E[YZ] - E[Y]E[Z]
  = Cov(X,Z) + Cov(Y,Z)

We can show that

$$Cov\Big(\sum_{i=1}^{n} X_i, Y\Big) = \sum_{i=1}^{n} Cov(X_i, Y)$$

$$Cov\Big(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$$

Variance of sum of random variables

$$Var\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} Cov(X_i, X_j)$$

if X and Y are independent, then Cov(X,Y)=0

As they are independent, Cov(X.Y) = E[XY] - E[X]E[Y]

=E[X]E[Y] - E[X]E[Y] = 0

Cov(X,Y) = 0

Therefore, for independent variables $X_1 ... X_n$, $Var\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} Var(X_i)$

Correlation Coefficient

The correlation coefficient, ρ_{XY} of two random variables X and Y is obtained by dividing the covariance by the product of standard deviations of X and Y .

$\rho_{XY} = Corr(X, Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

Properties of the correlation coefficient:

- $-1 \leq \rho_{X,Y} \leq 1$
- If $\rho_{X,Y} = 1$, then $Y = aX + b$, where $a > 0$
- If $\rho_{X,Y} = -1$, then $Y = aX + b$, where $a < 0$
- $\rho(aX + b, cY + d) = \rho(X, Y)$

If $\rho(X, Y) = 0$, we say that X and Y are uncorrelated.
If $\rho((X, Y) > 0$, we say that X and Y are positively correlated.
If $\rho(X, Y) < 0$, we say that X and Y are negatively correlated.

Moment Generating function

Moments: Let X be any random variable. The moments are the expected values of various powers of X.
E[X] = first moment
E[X^2] = second moment
E[X^k] = k^{th} moment
The moment generating function (MGF), $M_X(t)$ of a random variable X is defined for all the values of t (provided the expectation exists for some t in a neighbourhood of zero) by

$$\mathrm{M}_X(t) = E[e^{tX}] = \begin{cases} \sum\limits_x e^{tx} f(x) \text{ , if X is a discrete rv} \\ \int\limits_{-\infty}^{+\infty} e^{tx} f(x) dx \text{ , if X is continuous rv} \end{cases}$$

Finding Moments

One can write

$e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} = \sum_{k=0}^{\infty} \frac{X^k t^k}{k!}$

$M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} \frac{E[X^k]t^k}{k!}$

$E[X^k] = \frac{d^k}{dt^k} M_X(t)|_{t=0}$

Properties

(1)If X is a random variable and a is a constant, then

(i) $M_{X+a}(t) = e^{at} M_X(t)$

(ii)$M_{aX}(t) = M_X(at)$

(2) The moment generating function of the sum of independent random variables is just the product of the individual moment generating functions.

$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$

(3) The moment generating function uniquely determines the distribution.

Markov's Inequality

If X is a random variable that takes only non-negative values, then for any value $a > 0$

$P[X \geq a] \leq \frac{E[X]}{a}$

Let us show when X is a continuous random variable with pdf, f(x)

$$E[X] = \int_0^\infty xf(x)dx$$

$$= \int_0^a xf(x)dx + \int_a^\infty xf(x)dx$$

$$\geq \int_a^\infty xf(x)dx$$

$$\geq \int_a^\infty af(x)dx$$

$$= \text{a} \int_a^\infty f(x)dx = aP[X \geq a]$$

Chebyshev's Inequality

If X is a random variable with mean μ and variance σ^2, then for any value $k > 0$

$P[|X - \mu| \geq k] \leq \frac{\sigma^2}{k^2}$

Proof:

As $(X - \mu)^2$ is a non-negative random number, once can apply Markov's inequality

$P[(X - \mu)^2 \geq k^2] \leq \frac{E[(X-\mu)^2]}{k^2}$

$P[|X - \mu| \geq k] \leq \frac{E[(X-\mu)^2]}{k^2} = \frac{\sigma^2}{k^2}$

One can also replace k by $k\sigma$ to get : $P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$

The Law of Large Numbers

Let $X_1, X_2, ...X_n$, be a sequence of independent and identically distributed random variables, each having mean $E[X_i] = \mu$ and variance, σ^2. Then, for any $\epsilon > 0$,
$P[|\frac{X_1+...+X_n}{n} - \mu| > \epsilon] \to 0$ as n $\to \infty$

Sequential Deduction : One can show,
$E[\frac{X_1+...X_n}{n}] = \mu$
$Var(\frac{X_1+...X_n}{n}) = \frac{\sigma^2}{n}$
From Chebyshev's inequality , one can write
$P[|\frac{X_1+...+X_n}{n} - \mu| > \epsilon] \leq \frac{\sigma^2}{n\epsilon^2}$
which tends to zero as n $\to \infty$

Bernoulli Random Variable : Definition

A Bernoulli trial (named after the 17th century Swiss mathematician Jacob Bernoulli), is an experiment that can result in two outcomes, which can be denoted as a success and as a failure. If we let $X = 1$ when the outcome is a success and $X = 0$ when it is a failure, then the probability mass function of X is given by

$P(X = 0) = 1 - p$

$P(X = 1) = p$

where p, $0 < p < 1$, is the probability that the trial is a success.

A random variable X is said to be a Bernoulli random variable if its probability mass function is given by above equation for some $p \in (0, 1)$.

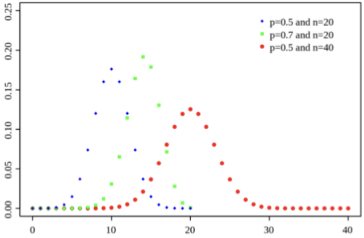
The expectation of a Bernoulli random variable is the probability that the random variable equals 1.

$E[X] = 1P(X = 1) + 0P(X = 0) = p$

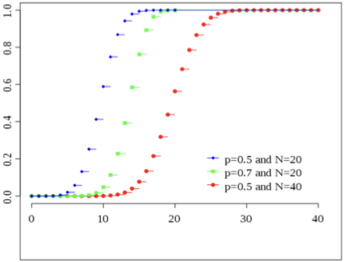
The probability of a success is denoted p, and the probability of failure is therefore $1 - p$.

Binomial Random Variable : Definition

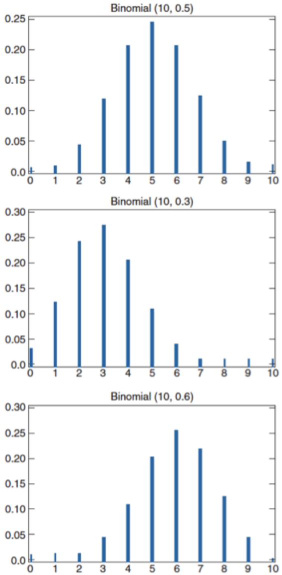
A random experiment consists of n Bernoulli trials such that
(1) The trials are independent
(2) Each trial results in only two possible outcomes, labeled as success and failure
(3) The probability of a success in each trial, denoted as p, remains constant
The random variable X that equals the number of trials that result in a success has a binomial random variable with parameters $0 < p < 1$ and $n = 1, 2, \ldots.$ The probability mass function of X is
$f(x) = \binom{n}{x} \cdot p^x (1-p)^{n-x}$, x = 0,1,2, ...n



pmf of Binomial Distribution



cdf of Binomial Distribution



Properties of Binomial Random Variable

(1) $f(x) > 0$

(2) $\sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = [p + (1-p)]^n = 1$

If X is a binomial random variable with parameters p and n,

$E(X) = np$

We know that X is the sum of n identical Bernoulli random variables, each with expected value p.

$X = X_1 + \cdots + X_n$

From the linearity of the expected values

$\mathrm{E}[\ X\ ] = \mathrm{E}\ [\mathrm{X}_1 + X_2 \cdots + X_n]$

$= \mathrm{E}\ [\mathrm{X}_1] + \cdots + E[X_n]$

$= \mathrm{p} + \cdots + p = np$

Similarly we can show that

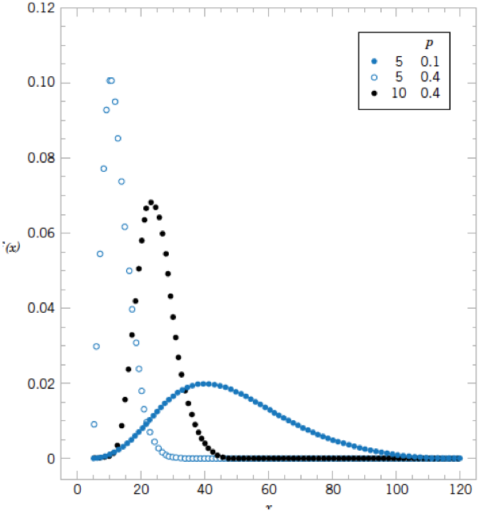
$\mathrm{Var(X)} = \mathrm{np(1\text{-}p)}$

(the variance of a sum of independent random variables is the sum of the variances.)

Negative Binomial Random Variable : Definition

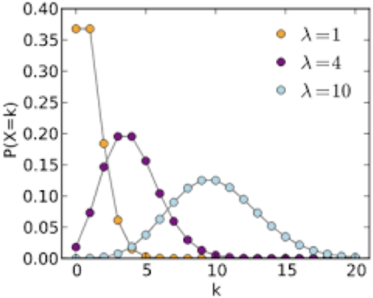
In a series of Bernoulli trials (independent trials with constant probability p of a success), let the random variable X denote the number of trials until r successes occur. Then X is a negative binomial random variable with parameters p and $r = 1, 2, 3, \cdots$ and
$f(x) = \binom{x-1}{r-1} \cdot p^r (1-p)^{x-r}$, $x = r, r+1, r+2, ...$
If X is a negative binomial random variable with parameters p and r,
$E(X) = r/p$
$Var(X) = (r(1-p))/p^2$

pmf of Negative Binomial Random Variable

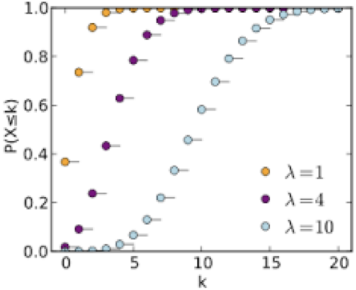


Poisson Distribution

Given an interval of real numbers, assume counts occur at random throughout the interval. If the interval can be partitioned into subintervals of small enough length such that
(1) the probability of more than one count in a subinterval is zero,
(2) the probability of one count in a subinterval is the same for all subintervals and proportional to the length of the subinterval, and
(3) the count in each subinterval is independent of other subintervals, the random experiment is called a Poisson process.
The random variable X that equals the number of counts in the interval is a Poisson random variable with parameter $\lambda > 0$, and the probability mass function of X is
$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, ..$



pmf of Poisson Random Variable



cdf of Poisson Random Variable

Figure: A schematic of pdf and cdf of Poisson distribution

A continuous random variable X has a uniform distribution if the probability density function is
$f(x) = \frac{1}{(b-a)}$, for $a \leq x \leq b$

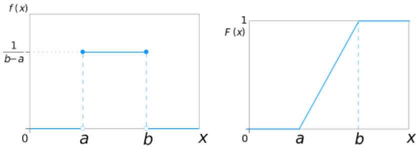


Figure: A schematic of pdf and cdf of uniform continuous distribution

The CDF of a uniform random variable is

$$F(x) = \begin{cases} 0, x < a \\ \frac{(x-a)}{(b-a)}, \text{a} \leq x \leq b \\ 1, x > b \end{cases}$$

Continued :

The moment generating function of a continuous uniform random variable defined over the support. $a < x < b$ is:

$M_X(t) = \frac{e^{bt}-e^{at}}{t(b-a)},\ t > 0$

The mean of the continuous random variable. is

$$E[X] = \int_a^b \frac{x}{(b-a)}dx = \frac{0.5x^2}{(b-a)}\Big|_a^b. = \frac{(a+b)}{2}$$

The variance is

$$Var[X] = \int_a^b \frac{(x-(\frac{a+b}{2}))^2}{(b-a)}dx$$

$$= \frac{(x-(\frac{a+b}{2}))^3}{3(b-a)}\Big|_a^b$$

$$= \frac{(b-a)^2}{12}$$

Example:

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0, 20 mA], and that the probability density function of X is uniform. Estimate the probability of measuring the current between 5 and 10 mA. Calculate the mean and standard deviation.

Exponential Random Variable :

A continuous random variable X is said to have an exponential distribution with parameter $\lambda > 0$, if its PDF is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, x > 0 \\ 0, otherwise \end{cases}$$

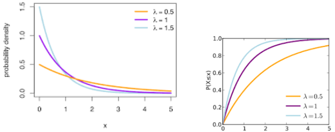


Figure: A schematic of pdf and cdf of exponential continuous distribution

The CDF of the exponential variable is given by

$F(x) = P(X \leq x)$

$= \int_0^x \lambda e^{-\lambda t} dt$

$= 1 - e^{-\lambda x}, \; x \geq 0$

Normal Random Variable

A random variable X is normal random variable if the pdf is given by
$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$
with parameters μ ($-\infty < \mu < \infty$) and $\sigma > 0$.

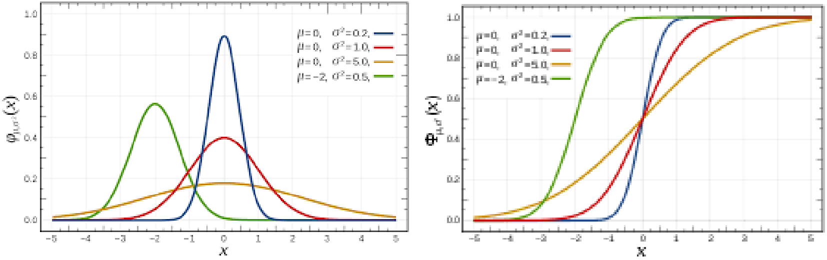


Figure: A schematic of pdf and cdf of normal continuous distribution

A simple way to see De Moivre's approximation

Lets us consider a binomial distribution for large n

$$P(X = k) = \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \quad (1)$$

We can use Sterlings approximation for the factorials which is generally good for large n
$n! \approx \sqrt{2\pi n} n^n e^{-n}$
Substituting for factorials in Equation (1)

$$P(X = k) \approx \frac{\sqrt{2\pi n} n^n e^{-n}}{\sqrt{2\pi(n-k)}(n-k)^{(n-k)} e^{-(n-k)} \sqrt{2\pi k} k^k e^{-k}} p^k (1-p)^{n-k} \quad (2)$$

$$\approx \sqrt{\frac{n}{2\pi k(n-k)}} \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k} \quad (3)$$

Substitute k = x + np and n - k = nq - x in Equation (3)
$P(X = np + x) \approx \frac{1}{\sqrt{2\pi npq}} (\frac{np}{np+x})^{np+x} (\frac{nq}{nq-x})^{nq-x}$

Continued :

$$P(X = np + x) \approx \frac{1}{\sqrt{2\pi npq}}\left(\frac{np}{np+x}\right)^{np+x}\left(\frac{nq}{nq-x}\right)^{nq-x} \quad (4)$$

$$\approx C\left(\frac{np+x}{np}\right)^{-np-x}\left(\frac{nq-x}{nq}\right)^{-nq+x} \quad (5)$$

$$\approx C\left(1+\frac{x}{np}\right)^{-np-x}\left(1-\frac{x}{nq}\right)^{-nq+x} \quad (6)$$

We can take logarithm of both the sides

$$lnP(X = np + x) \approx lnC - (np + x)ln\left(1 + \frac{x}{np}\right) - (nq - x)ln\left(1 - \frac{x}{nq}\right) \quad (7)$$

We can use

$ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots (-1)^n \frac{1}{n}x^n$

$ln(1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots - \frac{1}{n}x^n$

Continued :

Keeping only the first two terms of the expansion in Equation (7)

$$\approx lnC - (np+x)(\frac{x}{np} - \frac{x^2}{2(np)^2}) - (nq-x)(-\frac{x}{nq} - \frac{x^2}{2(nq)^2}) \quad (8)$$

$$\approx lnC - x - \frac{x^2}{np} + \frac{x^2}{2np} + \frac{x^3}{2(np)^3} + x - \frac{x^2}{nq} + \frac{x^2}{2nq} - \frac{x^3}{2(nq)^3} \quad (9)$$

$$\approx lnC - \frac{x^2}{2npq} \quad (10)$$

$$P(X=k) \approx \frac{1}{\sqrt{2\pi npq}} exp(-\frac{1}{2}\frac{(k-np)^2}{npq}) \quad (11)$$

One can notice, for a binomial distribution, $\mu = np$ and $\sigma^2 = npq$
$P(X=k) \approx \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{1}{2}\frac{(k-\mu)^2}{\sigma^2})$

Properties of a normal random variable

- $f(x) > 0$ for all x
- $\int\limits_{-\infty}^{\infty} f(x) = 1$

$\int\limits_{-\infty}^{\infty} f(x) = \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$

Substituting $\frac{x-\mu}{\sigma} = z$, $dx = \sigma$ dz

I $= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$

Squaring both the sides

$\mathrm{I}^2 = [\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx][\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy]$

$= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dxdy$

continued

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dxdy$$

Let us do a change of variable in polar coordinates ; $x = rcos\theta$, $y = rsin\theta$

$$\mathrm{I}^2 = \frac{1}{2\pi} \int_{0}^{2\pi} (\int_{0}^{\infty} e^{-\frac{r^2}{2}} rdrd\theta)$$

let $u = r^2/2$, $du = rdr$

$$\mathrm{I}^2 = \frac{1}{2\pi} \int_{0}^{2\pi} (\int_{0}^{\infty} e^{-u} du) d\theta$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} -(0-1) d\theta = 1$$

$\mathrm{I} = \pm 1$

But $f(x) > 0$, $\mathrm{I} = +1$.

continued

All normal random variables are symmetric about μ.

$f(\mu + x) = f(\mu - x)$

$f(\mu + x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x+\mu-\mu)^2}{2\sigma^2}}$

$= \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}}$

$f(\mu - x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(\mu-x-\mu)^2}{2\sigma^2}}$

$= \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}}$

The point of inflection of a normal distribution is $\mu \pm \sigma$

Moment Generating Function of a Normal Random Variable

$M_X(t) = E[e^{tX}]$

$= \frac{1}{\sqrt{2\pi}\sigma} \int\limits_{-\infty}^{\infty} e^{tx} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$

Substituting $\frac{x-\mu}{\sigma} = z$, $dx = \sigma$ dz

$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int\limits_{-\infty}^{\infty} e^{t\sigma z} e^{\frac{-z^2}{2}} dz$

$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{[-(\frac{z^2-2t\sigma z}{2})]} dz$

$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{[-\frac{(z-t\sigma)^2}{2} + \frac{t^2\sigma^2}{2}]} dz$

$= exp(\mu t + \frac{t^2\sigma^2}{2}) \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{\frac{-(z-t\sigma)^2}{2}} dz$

$= exp(\mu t + \frac{t^2\sigma^2}{2})$

Mean and Variance

$$M_X(t) = exp(\mu t + \frac{t^2\sigma^2}{2}) \tag{12}$$

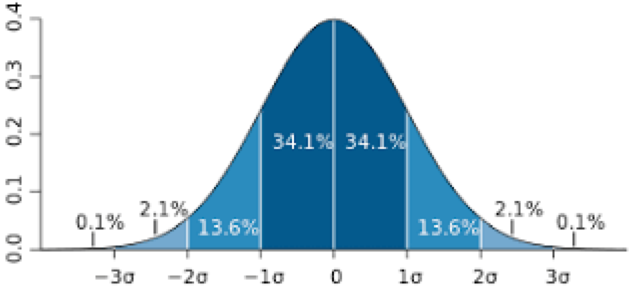
Differentiating w.r.t t

$$M_X^{'}(t) = (\mu + t\sigma^2)exp(\mu t + \frac{t^2\sigma^2}{2}) \tag{13}$$

$$M_X^{''}(t) = \sigma^2 exp(\mu t + \frac{t^2\sigma^2}{2}) + (\mu + t\sigma^2)^2 exp(\mu t + \frac{t^2\sigma^2}{2}) \tag{14}$$

$E[X] = M_X^{'}(0) = \mu$
$E[X^2] = M_X^{''}(0) = \sigma^2 + \mu^2$
$Var[X] = E[X^2] - (E[X])^2 = \sigma^2$

For a normal distribution one can show that
approximately 68% of the data fall within one standard deviation of the mean
approximately 95% of the data fall within two standard deviations of the mean
approximately 99.7% of the data fall within three standard deviations of the mean



Some more properties:

If X is normal with mean μ and variance, σ^2, then $Y = \alpha X + \beta$ is normal with mean $\alpha\mu + \beta$ and variance $\alpha^2\sigma^2$.

$$E[e^{t(\alpha X+\beta)}] = e^{t\beta}E[e^{\alpha t X}] \tag{15}$$

$$= e^{t\beta}exp[\mu\alpha t + \frac{\sigma^2\alpha^2 t^2}{2}] \tag{16}$$

$$= exp[(\beta + \mu\alpha)t + \frac{\sigma^2\alpha^2 t^2}{2}] \tag{17}$$

The last expression is the moment generating function of a normal random variable with mean $\alpha\mu + \beta$ and variance $\alpha^2\sigma^2$.

The sum of independent normal random variables is also a normal random variable with mean $\mu = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$

Let X_i , i = 1, $\cdots$, n, are independent normal random variables with mean, μ_i and variance, σ_i^2.

The moment generating function of $\sum_{i=1}^{n} X_i$ is

$$E[e^{(t\sum_{i=1}^{n} X_i)}] = E[e^{tX_1} \cdots e^{X_n}] \tag{18}$$

$$= \prod_{i=1}^{n} E[e^{tX_i}] = \prod_{i=1}^{n} e^{\mu_i t + (\sigma_i^2 t^2)/2} = e^{\mu t + (\sigma^2 t^2)/2} \tag{19}$$

where $\mu = \sum_{i}^{n} \mu_i$ and $\sigma^2 = \sum_{i}^{n} \sigma_i^2$

Standard Normal Random Variable

A normal random variable with $\mu = 0$ and $\sigma^2 = 1$ is called a **standard normal random variable** and is denoted by $Z \sim N(0, 1)$

$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$, $-\infty < z < \infty$

The cumulative distribution function of a standard normal random variable is denoted as $\Phi(x)$ is

$$\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{\frac{-t^2}{2}} dt$$

Properties

- $\lim\limits_{x \to \infty} \Phi(x) = 1$
- $\lim\limits_{x \to -\infty} \Phi(x) = 0$
- $\Phi(0) = 1/2$
- $\Phi(-x) = 1 - \Phi(x)$

We can obtain any normal random variable by shifting and scaling a standard normal random variable.

$X = \sigma Z + \mu$, $\sigma > 0$
$E[X] = \sigma E[Z] + \mu = \mu$
$Var[X] = \sigma^2 Var(Z) = \sigma^2$

.

If Z is a standard normal random variable and $X = \sigma Z + \mu$, then X is a normal random variable with mean μ and variance $\sigma^2 i.e$,X $\sim N(\mu, \sigma^2)$
If $X \sim N(\mu, \sigma^2)$, the random variable defined by $Z = \frac{(X-\mu)}{\sigma}$ is a standard normal random variable, i.e., $Z \sim N(0, 1)$.

$$F_X(x) = P(X \le x) = P(\sigma Z + \mu \le x) \tag{20}$$

$$= P(Z \le \frac{(x-\mu)}{\sigma}) \tag{21}$$

$$= \Phi(\frac{x-\mu}{\sigma}) \tag{22}$$

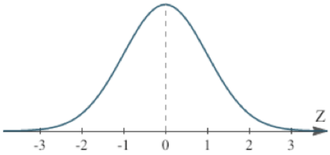
For any $a < b$

$$P(a < X < b) = P(\frac{(a-\mu)}{\sigma} < \frac{(X-\mu)}{\sigma} < \frac{(b-\mu)}{\sigma}) \tag{23}$$

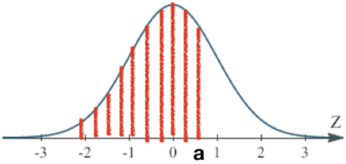
$$= P(\frac{(a-\mu)}{\sigma} < Z < \frac{(b-\mu)}{\sigma}) \tag{24}$$

$$= P(Z < \frac{(b-\mu)}{\sigma}) - P(Z < \frac{(a-\mu)}{\sigma}) \tag{25}$$

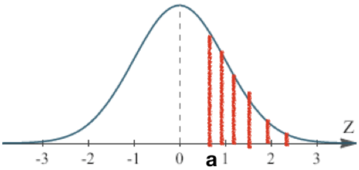
$$P(a < X < b) = \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma}) \tag{26}$$



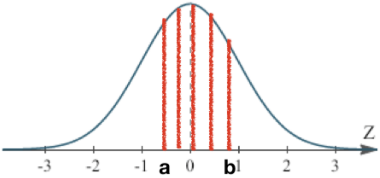
$$P(Z = z) = \phi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2} z^2}$$



$$P(Z \leq a) = \phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{\frac{-1}{2} t^2} dt$$



$$P(Z \geq a) = 1 - \phi(a) = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{\frac{-1}{2} t^2} dt$$



$$P(a \leq z \leq b) = \phi(b) - \phi(a) = \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{\frac{-1}{2} t^2} dt$$

Example :

If $X \sim N(-5, 4)$

Find $P(X < 0)$

Find $P(-7 < X < -3)$

Find $P(X > -3|X > -5)$

Solution

X is a random variable with mean -5 and $\sigma = 2$

(1) $P(X < 0) = F(0) = \Phi(\frac{0-(-5)}{2}) = \Phi(2.5)$

(2) $P(-7 < X < -3) = \Phi(\frac{-3-(-5)}{2}) - \Phi(\frac{-7-(-5)}{2})$

$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1$

(3) $P(X > -3|X > -5) = \frac{P(X>-3,X>-5)}{P(X>-5)}$

$= \frac{P(X>-3)}{P(X>-5)}$

$= \frac{1-\Phi(1)}{1-\Phi(0)}$

Example :

Let the background noise follows a normal distribution with a mean of 0 volt and standard deviation of 0.4 volt in the detection of a digital signal. The system assumes that a digital 1 has been transmitted when the voltage exceeds 0.8. What is the probability of detecting a digital 1 when none was sent?

Solution:

Let X denote the random variable which denote the voltage of noise.

$P(X > 0.8) = P(\frac{X-0}{0.4} > 0.8/0.4) = P(Z > 2) = 1 - \Phi(2) = 0.022$

The Chi-Square Distribution:

If Z_1, Z_2, ... , Z_n are independent standard normal random variables, the random variable Y defined as

$Y = Z_1^2 + Z_2^2 + \cdots + Z_n^2$

is said to have a chi-square (χ_n^2) distribution with n degrees of freedom shown by $Y \sim \chi_n^2$

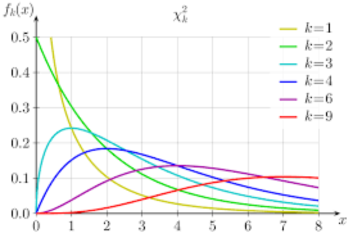
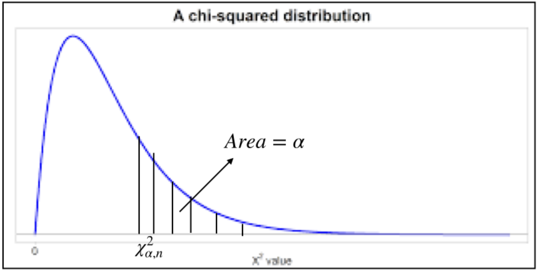


Figure: Schematic of χ_n^2 distribution for different degrees of freedom.

Properties:

If Y_1 and Y_2 are independent chi-square random variables with n_1 and n_2 degrees of freedom, respectively, then $Y_1 + Y_2$ is a chi-square random variable with $n_1 + n_2$ degrees of freedom.

If X is a chi-square random variable with n degrees of freedom, then for any $\alpha \in (0,1)$ the quantity $\chi^2_{\alpha,n}$ is defined to be such that
$P(X \geq \chi^2_{\alpha,n}) = \alpha$



The Moment Generating Function:

The moment generating function of a chi-square random variable with n degrees of freedom is

$M_X(t) = E[e^{tX}] = E[e^{tZ^2}]$ where Z $\sim$ N(0,1)

$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{tx^2} e^{\frac{-x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{\frac{-x^2(1-2t)}{2}} dx$

if $\overline{\sigma}^2 = (1-2t)^{-1}$

$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{\frac{-x^2}{2\overline{\sigma}^2}} dx = (1-2t)^{-1/2} \frac{1}{\sqrt{2\pi\overline{\sigma}^2}} \int\limits_{-\infty}^{\infty} e^{\frac{-x^2}{2\overline{\sigma}^2}} dx$

$= (1-2t)^{-1/2}$

When X $= \sum\limits_{i=1}^{n} Z_i^2$

E[e^{tX}] $= (1-2t)^{-n/2}$

E[X] = n

Var[X] = 2n

Central moments

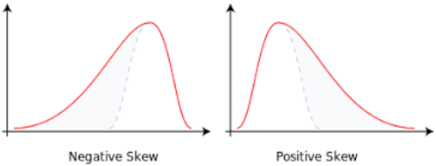
Central moments are moments about the mean, μ
μ_n = E[(X - μ)^n]
Raw moments
$\mu_n^{'}$ = E[(X)^n]
First central moment $= \mu_1 = 0$
Second central moment $= \mu_2 = \sigma^2 = \mu_2^{'} - \mu^2$
Third central moment $= \mu_3 = \mu_3^{'} - 3\mu_1^{'}\mu_2^{'} + 2\mu^3$
Fourth central moment $= \mu_4 = \mu_4^{'} - 4\mu\mu_3^{'} + 6\mu^2\mu_2^{'} - 3\mu^4$
The standardized moment of degree n is

$$\text{standardized moment} = \frac{\mu_n}{\sigma^n} \tag{27}$$

where σ is the standard deviation.

Skewness

The term skewness is a measure of lack of symmetry or departure from symmetry of the probability distribution function. If the distribution is not symmetrical (or is asymmetrical) about the mean, it is called a skewed distribution.
If the bulk of the data is at the left and the right tail is longer, the distribution is skewed right or positively skewed.
If the bulk of the data is towards the right and the left tail is longer, we say that the distribution is skewed left or negatively skewed.



Measure of Skewness

The skewness is quantified as the third standardized moment.

$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$

Pearson's coefficient of skewness

$\gamma_1 = \sqrt{\beta_1}$

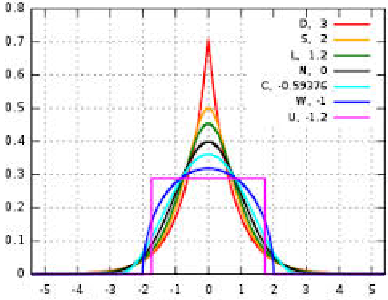
Estimate the skewness of normal and exponential distribution.

Kurtosis

Kurtosis refers to the height and sharpness of the peak relative to a normal distribution. Sometimes, it is also interpreted as the tailedness of the pdf. Higher values indicate a higher, sharper peak; lower values indicate a lower, less distinct peak.
The reference standard is a normal distribution, which has a kurtosis of 3. Therefore, excess kurtosis is defined as kurtosis - 3.
A normal distribution has kurtosis exactly 3 (excess kurtosis $=0$).
Any distribution with kurtosis $= 3$ (excess kurtosis $= 0$) is called mesokurtic.
A distribution with $kurtosis < 3$ (excess kurtosis < 0) is called platykurtic.
A distribution with $kurtosis > 3$ (excess kurtosis > 0) is called leptokurtic.

Measure of Kurtosis

$\beta_2 = \frac{\mu_4}{\mu_2^2}$
Excess Kurtosis
$\gamma_2 = \beta_2 - 3$



Estimate the kurtosis of a normal distribution

Summarizing Data Sets

Sample Mean :
The sample mean characterizes the central tendency in the data by the arithmetic average.
Let us consider a sample of size n with numerical values, $x_1, x_2, x_3 ... x_n$.
The arithmetic average of the values is called sample mean.

$$\mathbf{\bar{x}} = \frac{\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{x_i}}{\mathbf{n}} \tag{28}$$

Example
Lets consider the eight observations , 13.8, 14.9, 13.4, 15.3, 13.6, 13.5, 12.6, and 13.1 which constitutes a sample . Calculate the sample mean.

If a data set is presented in a frequency table where the k distinct values, $v_1, ..., v_k$, having corresponding frequencies $f_1, ..., f_k$ such that $n = \sum_{i=1}^{k} f_i$, the sample mean of these n data values is

$$\mathbf{\bar{x}} = \frac{\sum\limits_{\mathbf{i=1}}^{\mathbf{k}} \mathbf{v_i f_i}}{\mathbf{n}} \tag{29}$$

Therefore, the sample mean is a weighted average of the distinct values, where the weight given to the value v_i is equal to the proportion of the n data values that are equal to v_i , i = 1, . . . , k.

Sample median

Sample median is the middle value when the data set is arranged in increasing order.
Determination of median :
Order the values of a data set of size n from smallest to largest.
If n is odd, the sample median is the value in position $(n + 1)/2$
if n is even, it is the average of the values in positions $n/2$ and $n/2 + 1$.

Sample mode

The value that occurs with the greatest frequency.

Sample Variance

This statistic describes the variability or scatter in the data.
The sample variance , s^2 (of a data set $x_1, x_2, x_3 ... x_n$) is given by
,

$$\mathbf{s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{(n-1)}} \tag{30}$$

The sample standard deviation is given by

$$\mathbf{s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{(n-1)}}} \tag{31}$$

Show that
$\sum_{i=1}^{n}(x_i - \overline{x})^2 = (\sum_{i=1}^{n} x_i^2) - n\overline{x}^2$ Note: While calculating variance for population, one divides by N, size of population .

Sample Correlation Coefficient

If s_x and s_y denote, respectively, the sample standard deviations of the x values and the y values in a paired data set, the sample correlation coefficient, r , of the data pairs (x_i, y_i) , i = 1, . . . , n is defined by

$$\mathbf{r} = \frac{\sum_{\mathbf{i=1}}^{\mathbf{n}} (\mathbf{x_i} - \overline{\mathbf{x}})(\mathbf{y_i} - \overline{\mathbf{y}})}{(\mathbf{n-1})\mathbf{s_x s_y}} \tag{32}$$

When $r > 0$, the sample data pairs are positively correlated, and when $r < 0$, they are negatively correlated.

Properties of r

1. $-1 \leq r \leq 1$
2. if $y_i = a + bx_i$, for i = 1,2,.. n then $r = 1$ (for $b > 0$) and $r = -1$ (for $b < 0$) where a and b are constants.
3. If r is the sample correlation coefficient for the data pairs (x_i, y_i) , i= 1,..., n then it is also the sample correlation coefficient for the data pairs $a + bx_i, c + dy_i$, i=1,...,n provided that b and d are both positive or both negative constants.

Sample Correlation Coefficient