Nambu-Goldstone theorem and integrability

Ivan Arraut

OUHK

Based on:

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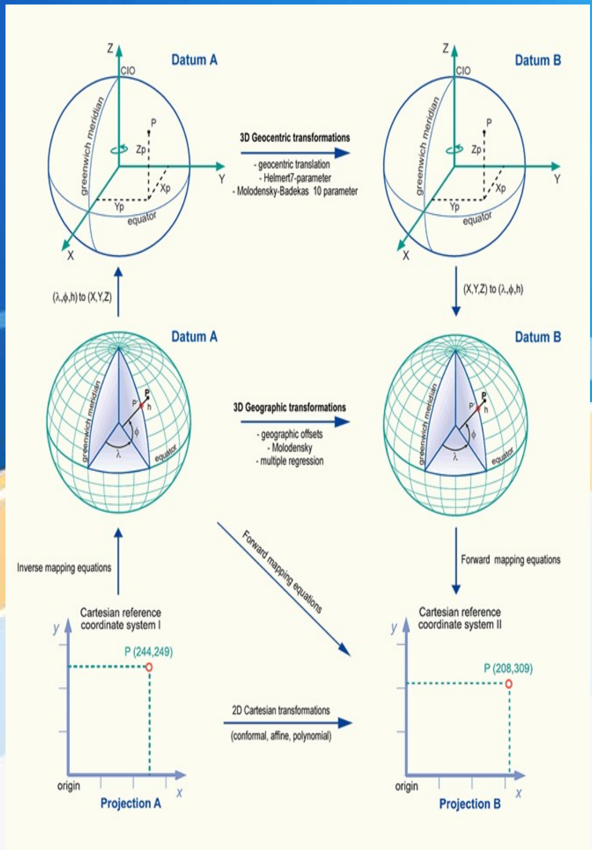
CONTENT

- 1). Internal and external symmetries.
- 2). Spontaneous symmetry breaking.
- 3). Nambu-Goldstone theorem: Classical and Quantum version.
- 4). Discrepancies between the relativistic and non-relativistic situations: Type I and type II NGB.
- 5). Generic character of the NGB Type I and type II: Connection with the Quantum Yang Baxter Equations.

Motivation

- 1) Understand in deeper details the Nambu-Goldstone theorem
  2) Solve the apparent failure of the Nambu-Goldstone theorem
  3) Find an alternative tool for analyzing more complex problems where spontaneous symmetry breaking is involved

INTERNAL AND EXTERNAL SYMMETRIES

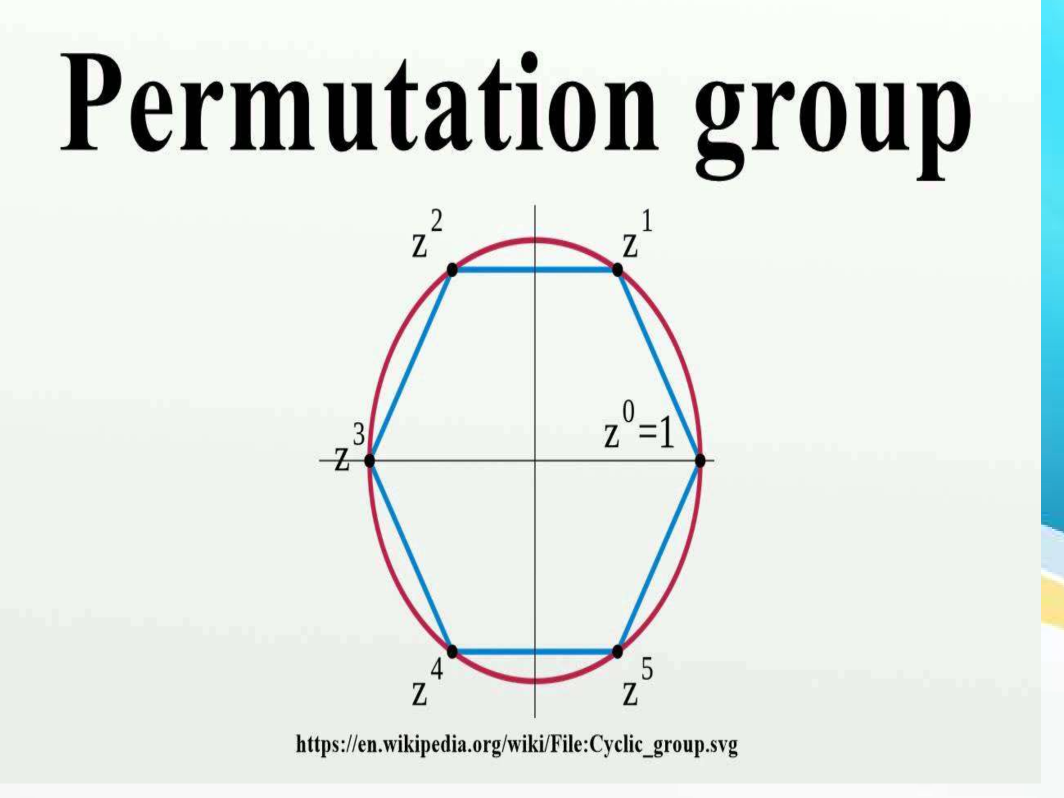


$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R$$

Coordinate transformations.

Example of external symmetries.

INTERNAL AND EXTERNAL SYMMETRIES



SPONTANEOUS SYMMETRY BREAKING

$$\pounds = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}[(\phi^i)^2]^2,$$

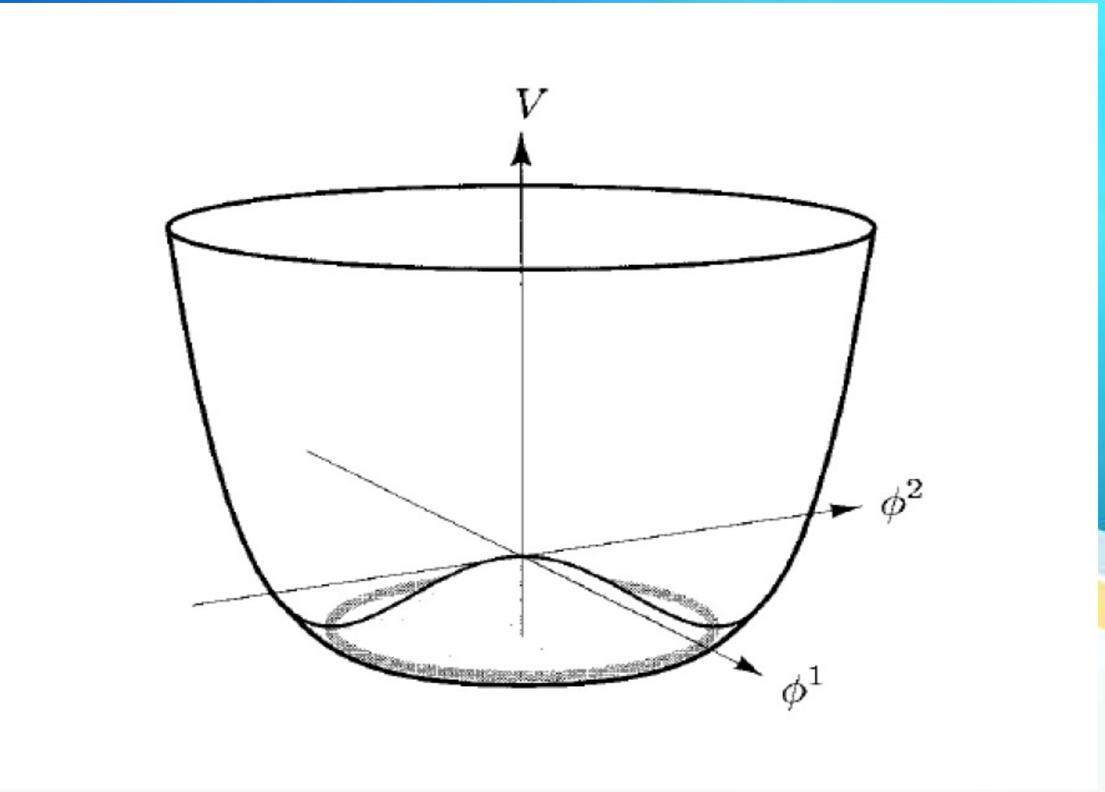
$$\phi^i \to R^{ij}\phi^j.$$

Representation of $O(N)$ group

$$V(\phi^i) = -\frac{1}{2}\mu^2(\phi^i)^2 + \frac{\lambda}{4}[(\phi^i)^2]^2.$$

The Lagrangian is invariant under the transformation.

SPONTANEOUS SYMMETRY BREAKING



SPONTANEOUS SYMMETRY BREAKING

$$(\phi_0^i)^2 = \frac{\mu^2}{\lambda}.$$

Vacuum condition

$$\phi_0^i = (0, 0, ..., 0, v), \; v = \mu/\sqrt{\lambda}.$$

Arbitrary selection of the direction

$$\phi(x) = (\pi^k(x), v + \sigma(x)),$$

Vacuum redefinition

$$k = 1, ..., N-1.$$

$$\pounds = \frac{1}{2}(\partial_\mu \pi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu(\pi^k)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi^k)^2\sigma^2 - \frac{\lambda}{4}[(\pi^k)^2]^2.$$

Nambu-Goldstone Theorem: Classical approach

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)^a(\phi - \phi_0)^b \left(\frac{\partial^2}{\partial\phi^a\partial\phi^b}V\right)_{\phi_0} + ...,$$

$$\partial V/\partial\phi^a = 0 \qquad \left(\frac{\partial^2}{\partial\phi^a\partial\phi^b}V\right)_{\phi_0} = m_{ab}^2 \geq 0.$$

Nambu et al. 1961

Vacuum condition plus mass matrix

$$\pounds = (kinetic\ \ terms) - V(\phi),$$

$$G: \quad \phi_0^{a'} = U(g)\phi_0^a \neq \phi_0^a,$$

$$H: \quad \phi_0^{a'} = U(h)\phi_0^a = \phi_0^a,$$

Vacuum is not invariant under the action of the whole group.

Nambu-Goldstone Theorem: Quantum version

$$U(g) = e^{T^a \alpha} \approx \hat{I} + \alpha T^a \to U(h)\phi_0^a = \phi_0^a + \alpha T^a(\phi_0),$$

$$T^a(\phi_0)T^b(\phi_0)\frac{\partial^2}{\partial\phi^a\partial\phi^b}V(\phi) = T^a(\phi_0)T^b(\phi_0)m_{ab}^2 = 0.$$

We can define a conserved current as

$$j_\mu^a(x) = \frac{\partial \pounds}{\partial(\partial^\mu\phi)}\frac{\delta\phi(x)}{\delta\alpha^a},$$

$$Q^a(x) = \int d^3x j_0^a(x).$$

Nambu-Goldstone Theorem: Quantum version

$$[Q^a, Q^b] = C^{abc} Q^c,$$ Lie algebra structure

$$U = e^{iQ^a \alpha^a}.$$ $$Q^a|0>= 0.$$ Standard definition of vacuum

$$U \neq e^{iQ^a \alpha^a}, \quad Q^a|0>\neq 0.$$ Degenerate vacuum condition

$$[Q^a, \phi'(x)] \backsim \phi(x).$$ Order parameter

$$< 0|Q^a \phi'(x) - \phi'(x) Q^a|0 >\neq 0.$$ Nambu-Goldstone field

Nambu-Goldstone Theorem: Quantum version

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2).$$

Convenient separation

$$< 0_{SV}|\phi_1|0_{SV} >= \pm\mu/\sqrt{\lambda}.$$

Example of order parameter for the case of $U(1)$ symmetry.

$$\sum_n \int d^3y \, [< 0|j_0^a(y)|n >< n|\phi'(x)|0 > - < 0|\phi'(x)|n >< n|j_0^a(y)|0 >]_{x^0=y^0} \neq 0,$$

$$(2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n) \left[< 0|j_0^a(0)|n >< n|\phi'(x)|0 > e^{iM_n y_0} - < 0|\phi'(x)|n >< n|j_0^a(0)|0 > e^{-iM_n y^0}\right]_{x^0=y^0},$$

Spacetime translational invariance

The Nambu-Goldstone theorem suggests that the number of Nambu-Goldstone bosons is equal to the number of broken symmetry generators; besides this, the Nambu-Goldstones, being massless, have a linear dispersion relation $E_n \backsim \mathbf{p}_n$

Systems where this does not apply

- 1) Heisenberg Ferromagnet
  2) Any system where the Lorentz invariance is violated explicitly (Many examples in condensed matter Physics)

For these systems two things occur, they are:
1) Two broken symmetries are related to one Nambu-Goldstone boson
2) The dispersion relation of the mode is quadratic and not linear as it is supposed to be

DISCREPANCIES BETWEEN TYPE I AND II NGB

$$n_{\mathrm{BS}} - n_{\mathrm{NG}} = \frac{1}{2}\,\mathrm{rank}\,\rho,$$

- 1). Dispersion relation.
- 2). Counting rule. **Watanabe and Murayama**.

$$< 0_{SV}|\phi_{a,b}(x)|0_{SV} > \neq 0.$$

$$\phi_{a,b}(\vec{x}) = \phi_{a,b}^{m,l}\epsilon_m \otimes \epsilon_l$$

Generic form of order parameter

$$[Q_{m,k}(y), \phi_{a,b}(x)] \backsim \phi'_{a,b}(x).$$

$$< 0_{SV}|[Q_{m,k}(y), \phi_{a,b}(x)]|0_{SV} > \neq 0.$$

$$Q_{m,l} = Q_{m,l}^{p,k}\epsilon_p \otimes \epsilon_k.$$

QUANTUM YANG BAXTER EQUATIONS

$$< 0_{SV} | \, [\phi_{b,a}(x), [Q_{p,k}(y), Q_{l,m}(z)]] \, | 0_{SV} > \neq 0.$$

$$Q_{k,p}(y) = e^{-ipy} Q_{k,p}(0) e^{ipy}$$

$$\begin{aligned}
&\sum_{n,n'} < 0_{SV} | \phi_b(x) | n >< n | \, Q_p(0) | n' >< n' | Q_l(0) | 0_{SV} > e^{-i(p_n - p_{n'})y} e^{-i\tilde{p}_{n'} z} \\
&\quad - < 0_{SV} | \phi_b(x) | n' >< n' | \, Q_l(0) | n >< n | Q_p(0) | 0_{SV} > e^{-i(\tilde{p}_{n'} - \tilde{p}_n) z} e^{-ip_n y} \\
&\quad - < 0_{SV} | \, Q_p(0) | n >< n | Q_l(0) | n' >< n' | \phi_b(x) | 0_{SV} > e^{-i(\tilde{p}_n - \tilde{p}_{n'}) z} e^{ip_n y} \\
&+ < 0_{SV} | Q_l(0) | n' >< n' | \, Q_p(0) | n >< n | \phi_b(x) | 0_{SV} > e^{-i(p_{n'} - p_n) y} e^{i\tilde{p}_{n'} z} \neq 0.
\end{aligned}$$

QUANTUM YANG BAXTER EQUATIONS

$Q_{p,k} = Q_{p,m} = Q_{m,p} = Q_{k,p} = Q_p$; $Q_{l,m} = Q_{l,a} = Q_{a,l} = Q_{m,l} = Q_l$, and $\phi_{b,a} = \phi_{b,k} =$

$\phi_{k,b} = \phi_{a,b} = \phi_b$

$$R^{0,n'}_{m,l} =< 0_{DV}|Q_{m,l}(y)|n' >, \quad R^{n,k}_{p,n'} =< n'|Q_{k,p}(z)|n >,$$
$$R^{a,b}_{n,0} =< n|\phi_{a,b}(x)|0_{DV} >, \quad R^{n,0}_{p,m} =< n|Q_{p,m}(z)|0_{DV} >,$$
$$R^{a,n'}_{n,l} =< n'|Q_{l,a}(y)|n >, \quad R^{b,k}_{0,n'} =< 0_{DV}|\phi_{b,k}(x)|n' > .$$

$R_{(1,2)}R_{(1,3)}R_{(2,3)} = R_{(2,3)}R_{(1,3)}R_{(1,2)}$, $R : M \otimes V = R^{a,b}_{c,d} m_a \otimes m_b$. $R^{s_2,s_3}_{j,k} R^{s_1,c}_{i,s_3} R^{a,b}_{s_1,s_2} = R^{r_1,r_2}_{i,j} R^{a,r_3}_{r_1,k} R^{b,c}_{r_2,r_3}$.

$$R^{0,n'}_{m,l} R^{n,k}_{p,n'} R^{a,b}_{n,0} = R^{n,0}_{p,m} R^{a,n'}_{n,l} R^{b,k}_{0,n'},$$

QUANTUM YANG BAXTER EQUATIONS

$$\sum_{0,n,n'} < 0_{DV}|Q_{m,l}(0)|n' >< n'|Q_{k,p}(0)|n >< n|\phi_{a,b}(x)|0_{DV} >= \sum_{0,n,n'} < 0_{DV}|\phi_{b,k}(x)|n' > \times$$

$$< n'|Q_{l,a}(0)|n >< n|Q_{p,m}(0)|0_{DV} > ($$

In simplified notation

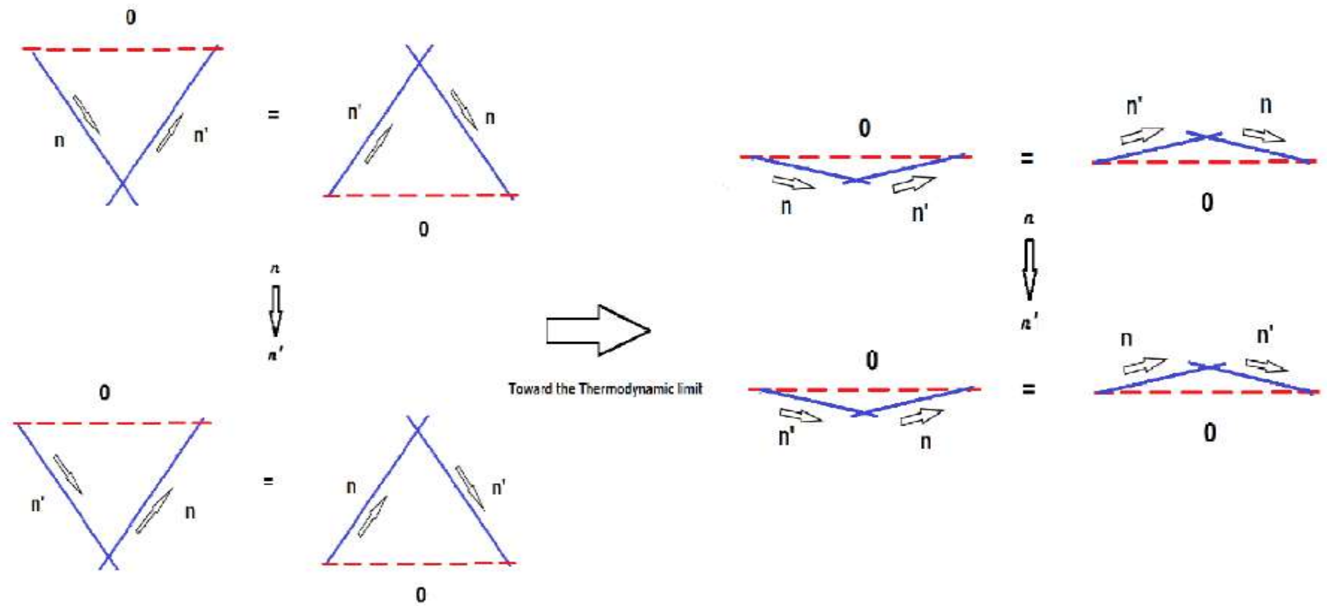
$$\sum_{0} < 0_{DV}|Q_l(0)Q_p(0)\phi_b(x)|0_{DV} >= \sum_{0} < 0_{DV}|\phi_b(x)Q_l(0)Q_p(0)|0_{DV} > .$$

QUANTUM YANG BAXTER EQUATIONS

$n \rightarrow n'$ Exchange of particles

$$R^{0,n}_{m,p} R^{n',a}_{l,n} R^{k,b}_{n',0} = R^{n',0}_{l,m} R^{k,n}_{n',p} R^{b,a}_{0,n},$$

$$R_{(3,2)} R_{(3,1)} R_{(2,1)} = R_{(2,1)} R_{(3,1)} R_{(3,2)}.$$



Witten 2016,

Kostello 2013

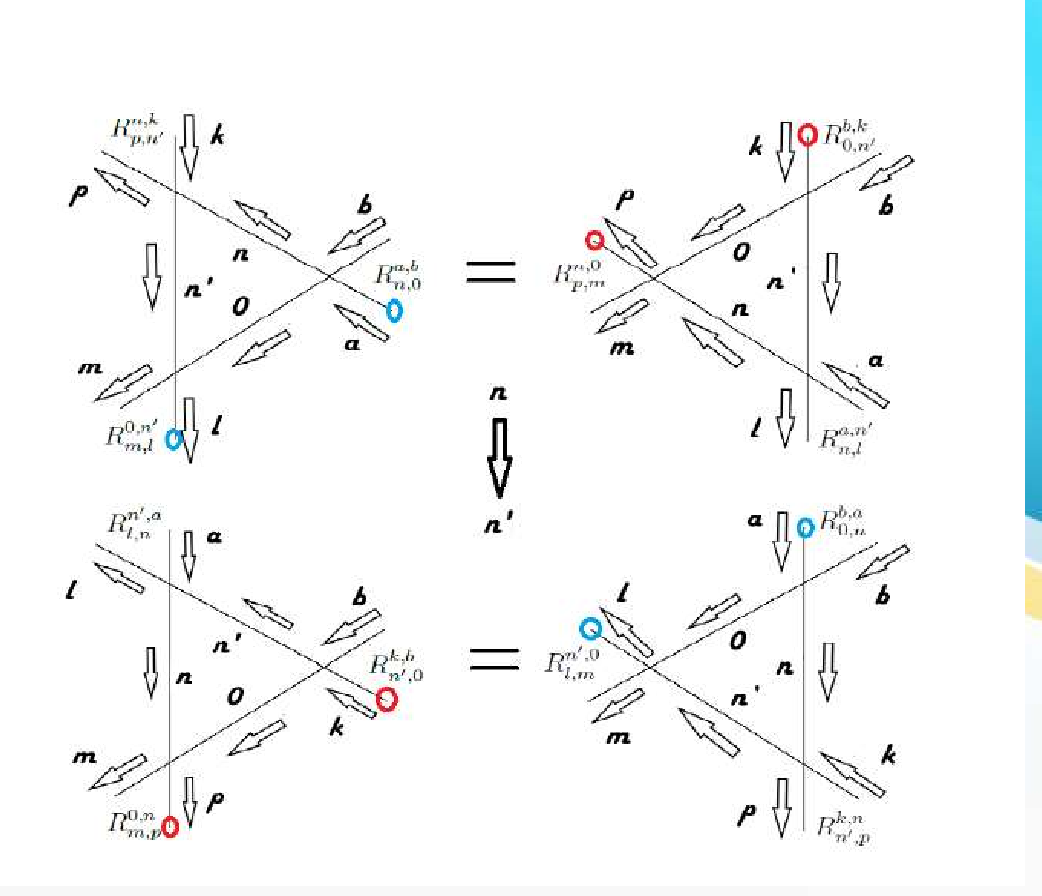
QUANTUM YANG BAXTER EQUATIONS

$$\sum_{0,n,n'} < 0_{DV}|Q_{m,p}(0)|n >< n|Q_{a,l}(0)|n' >< n'|\phi_{k,b}(x)|0_{DV} >= \sum_{0,n,n'} < 0_{DV}|\phi_{b,a}(x)|n > \times$$
$$< n|Q_{p,k}(0)|n' >< n'|Q_{l,m}(0)|0_{DV} >$$

In simplified notation we get

$$\sum_{0} < 0_{DV}|Q_p(0)Q_l(0)\phi_b(x)|0_{DV} >= \sum_{0} < 0_{DV}|\phi_b(x)Q_p(0)Q_l(0)|0_{DV} > .$$

QUANTUM YANG BAXTER EQUATIONS



M. Jimbo 1989,

QUANTUM YANG BAXTER EQUATIONS: TYPE B NGB

$$\sum_{0,n,n'} < 0_{DV}|\phi_b(x)|n >< n|\ Q_p(0)|n' >< n'|Q_l(0)|0_{DV} > 2e^{-i\tilde{E}_{n'}z_0} cos\,(\tilde{\mathbf{p}}_n \cdot \mathbf{z})$$
$$- < 0_{DV}|\ Q_p(0)|n >< n|Q_l(0)|n' >< n'|\phi_b(x)|0_{DV} > 2e^{iE_n y_0} cos\,(\mathbf{p}_n \cdot \mathbf{y}) = 0.$$

$p_n = p_{n'}$ and $\tilde{p}_n = \tilde{p}_{n'}$ $\quad n = n'$ $\quad p_n y = E_n y_0 - \vec{p_n} \cdot \vec{y},$

$$2\sum_{0,n,n'} < 0_{DV}|\phi_b(x)|n >< n|\ Q_p(0)|n' >< n'|Q_l(0)|0_{DV} > (e^{-i\tilde{E}_{n'}z_0} cos\,(\tilde{\mathbf{p}}_n \cdot \mathbf{z}) \times$$
$$-e^{iE_n z_0} cos\,(\mathbf{p}_n \cdot \mathbf{y})) = 0,$$

Spatial integration

QUANTUM YANG BAXTER EQUATIONS: TYPE B NGB

Under the conditions $z_0 = y_0$

If $\mathbf{y} \to \mathbf{z}$

$$4i \sum_{0,n,n'} < 0_{DV}|\phi_b(x)|n >< n|\ Q_p(0)|n' >< n'|Q_l(0)|0_{DV} > sin(E_n z_0) cos\left(\mathbf{p}_n \cdot \mathbf{z}\right)_{\mathbf{y}\to\mathbf{z}} = 0$$

$E_n \to 0$ $\qquad$ $p_n \to 0$

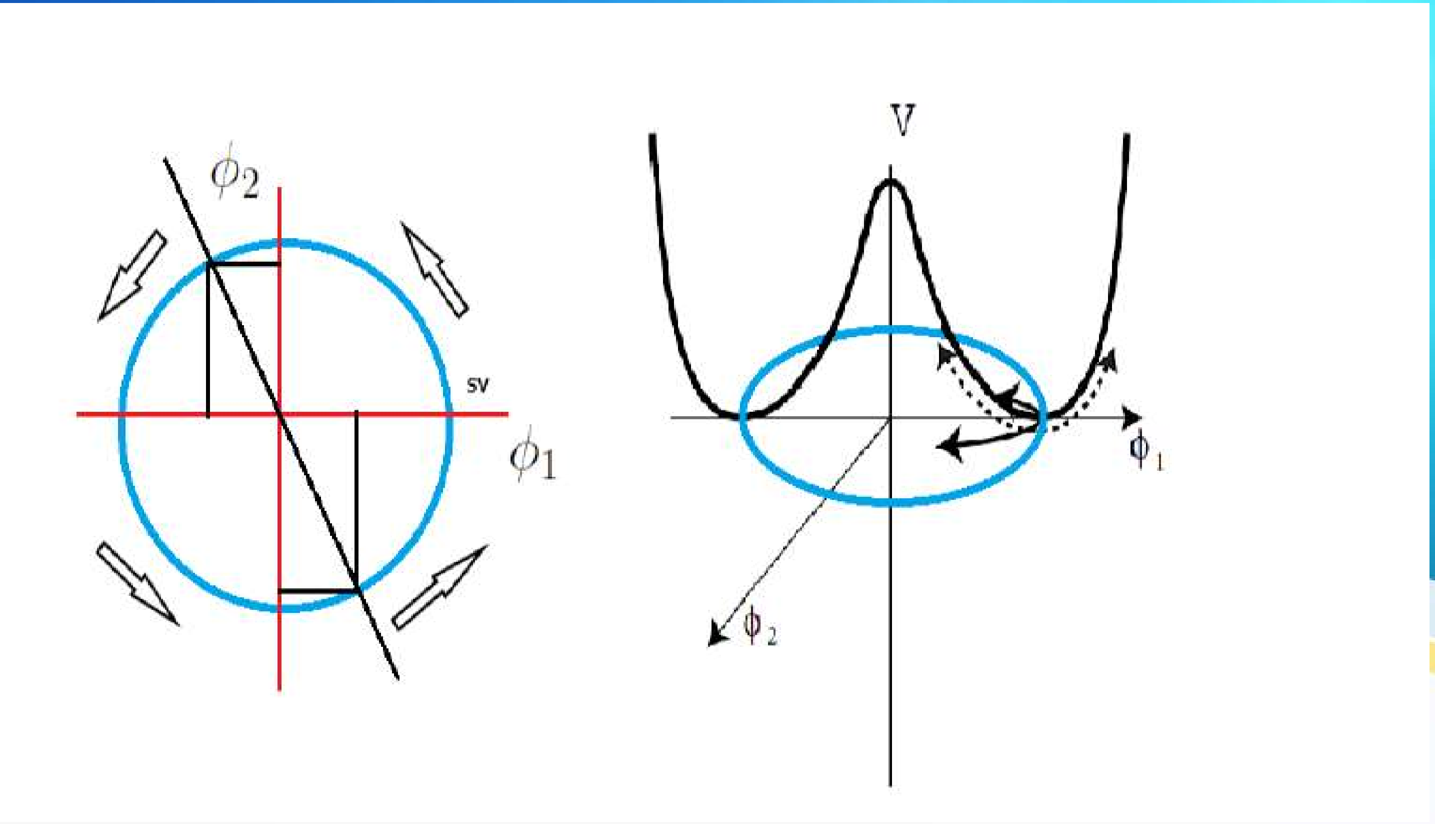
QUANTUM YANG BAXTER EQUATIONS: TYPE A NGB

$$\sum_{0,n,n'} < 0_{DV}|\phi_b(x)|n >< n|\ Q_p(0)|n' >< n'|Q_l(0)|0_{DV} > (e^{-i(p_n - p_{n'})y} e^{-i\tilde{p}_{n'}z}$$
$$-e^{-i(\tilde{p}_n - \tilde{p}_{n'})z} e^{ip_n y}) - < 0_{DV}|\phi_b(x)|n' >< n'|\ Q_l(0)|n > \times$$
$$< n|Q_p(0)|0_{DV} > (e^{-i(\tilde{p}_{n'} - \tilde{p}_n)z} e^{-ip_n y} - e^{-i(p_{n'} - p_n)y} e^{i\tilde{p}_{n'}z}) = 0.$$

$$\textbf{typeA} : n_A = Rank(R^{i,j}_{k,l})_{n \neq n'} \text{ and } \textbf{TypeB} : n_B = \tfrac{1}{2} Rank(R^{i,j}_{k,l})_{n=n'}$$

$$N_{NG} = n_A + n_B \text{ and } N_{BG} = Rank(R^{i,j}_{k,l})_{n=n'} + Rank(R^{i,j}_{k,l})_{n \neq n'} = 2n_B + n_A$$

QUANTUM YANG BAXTER EQUATIONS: JUSTIFICATION OF THE SUM OF VACUUMS



QUANTUM YANG BAXTER EQUATIONS: JUSTIFICATION OF THE SUM OF VACUUMS

$$\sum_{0} < 0_{DV}|\phi_1|0_{DV} >= \bar{\phi}_1 - \bar{\phi}_1 + \bar{\phi}_2 - \bar{\phi}_2 + ... + \bar{\phi}_n - \bar{\phi}_n = \sum_{n=1}^{N} \bar{\phi}_i = 0.$$

$$< 0_{SV}|[Q, \phi_2]|0_{SV} >= i < 0_{SV}|\phi_1|0_{SV} > .$$

$$\sum_{0} < 0_{DV}|[Q, \phi_2]|0_{DV} >= 0.$$

Why can we sum over different vacuums which repreesnt in principle different Hilbert spaces?

CONCLUSIONS

- 1). The Nambu-Goldstone theorem is a natural consequence of the constraints coming from the Quantum Yang Baxter Equations. The same constraints illustrate the generic character of the Type I and Type II NG.
- 2). The QYBE suggest that the interaction of N Nambu-Goldstone bosons can be decomposed in the interaction of pairs.

  3). The interaction of a pair of Nambu-Goldstone bosons is represented by 4 different histories of interaction. The QYBE reduce the total independent histories to 2.

Conclusions

- 4). In some systems, the interaction of a pair of NGB is reduced to only 1 history. In such a case, the two NGB become a single degree of freedom with quadratic dispeersion relation.
  5). For the systems where the histories can be only reduced to 2 forr the interaction of 2 NGB, the number of NGB is equal to the number of broken symmetries and the NGB have a linear dispersion relation.