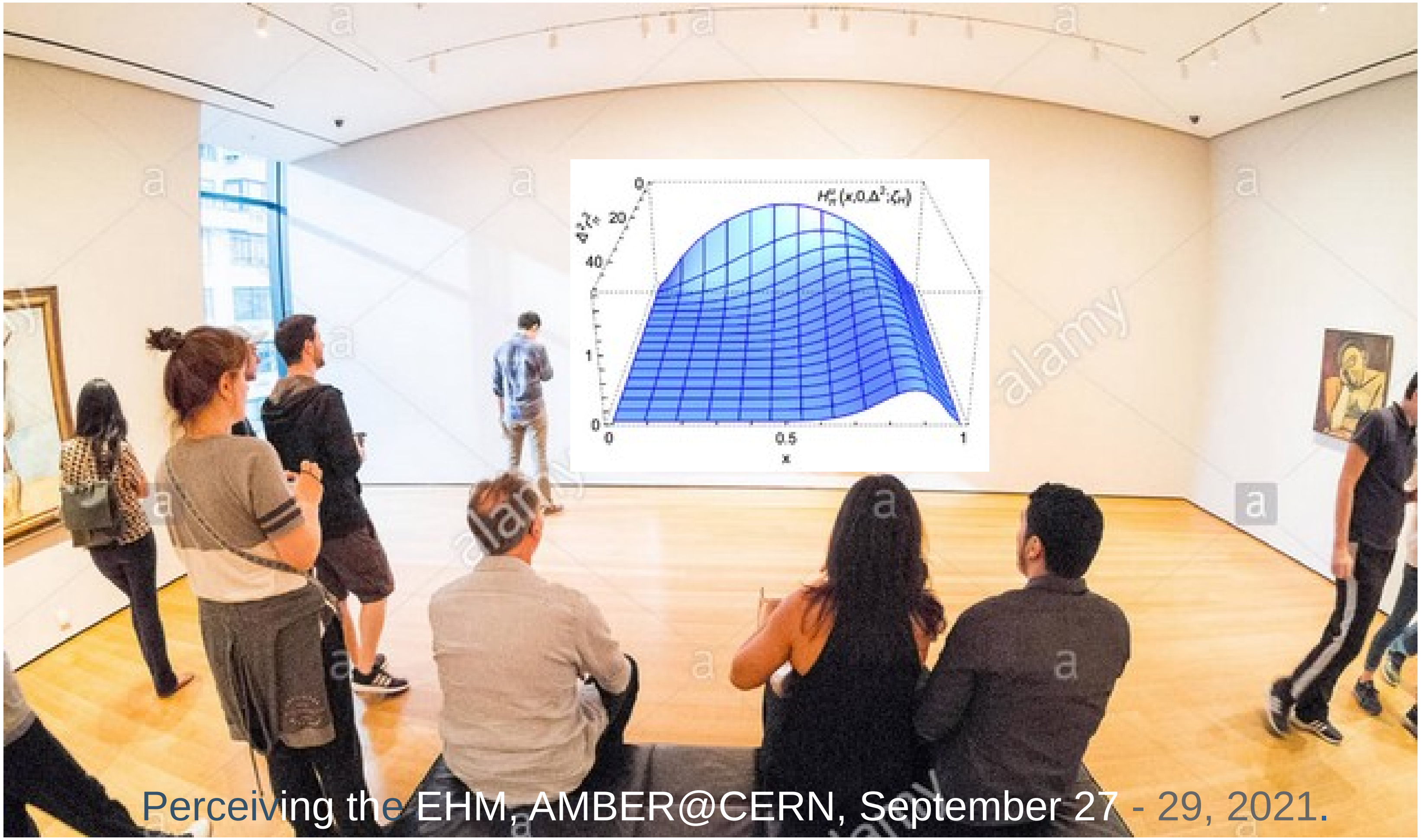
Pion and kaon LFWFs and GPDs

J. Rodríguez-Quintero



Perceiving the EHM, AMBER@CERN, September 27 - 29, 2021.

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Based on:

Jin-Li Zhang, Khépani Raya, Lei Chang, Zhu-Fang Cui, José Manuel Morgado, Craig D. Roberts, José Rodríguez-Quintero; Physics Letters B815 (2021) 136158; [arXiv:2101.12286]

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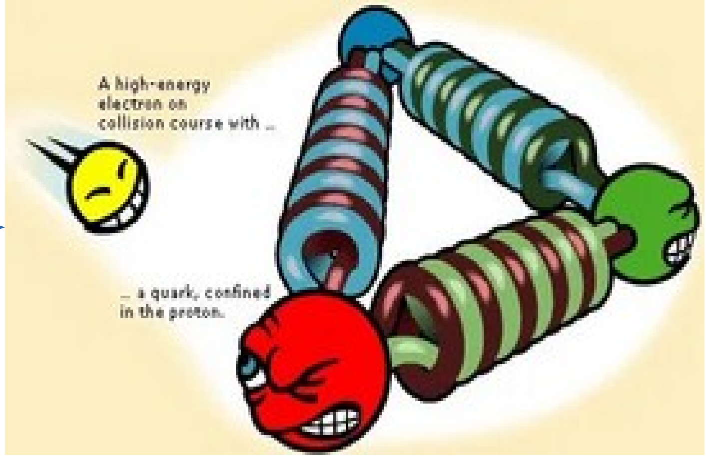
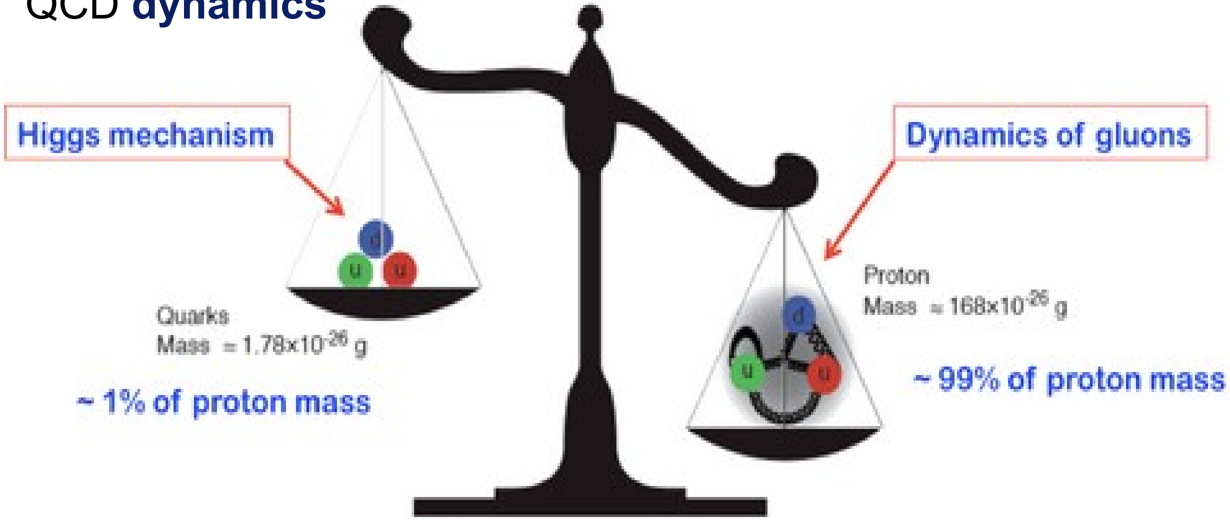
QCD and hadron physics

- **QCD** is characterized by two **emergent** phenomena: confinement and dynamical generation of mass (DGM).

Gluons and quarks have never been seen isolated in nature; only colorless bound states (hadrons) have.

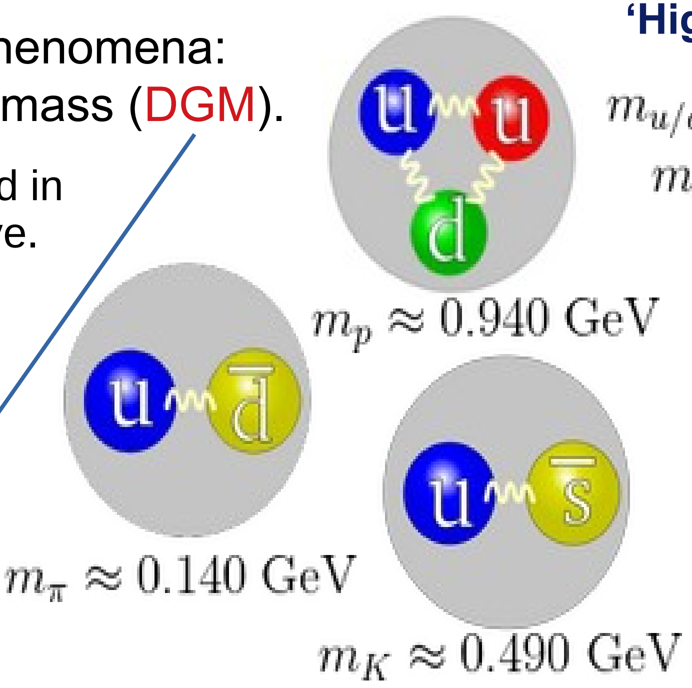


Emergence of hadron masses (EHM) from QCD **dynamics**



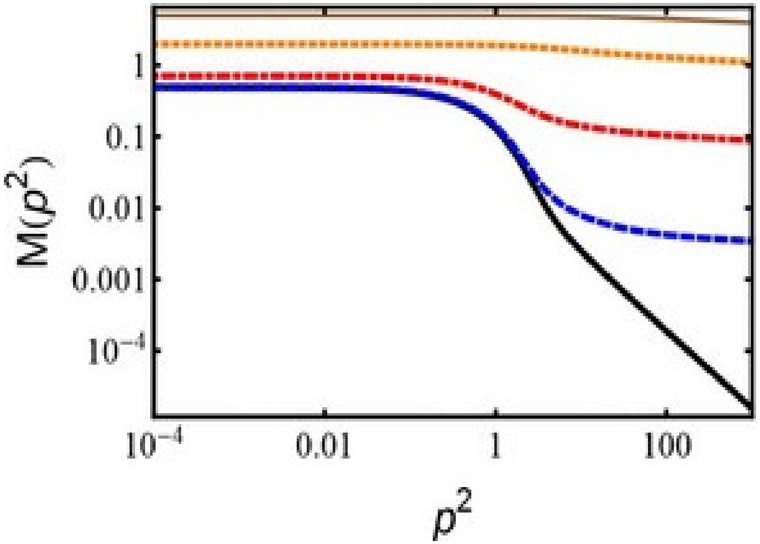
'Higgs' masses

$$m_{u/d} \approx 0.004 \text{ GeV}$$
$$m_s \approx 0.095 \text{ GeV}$$



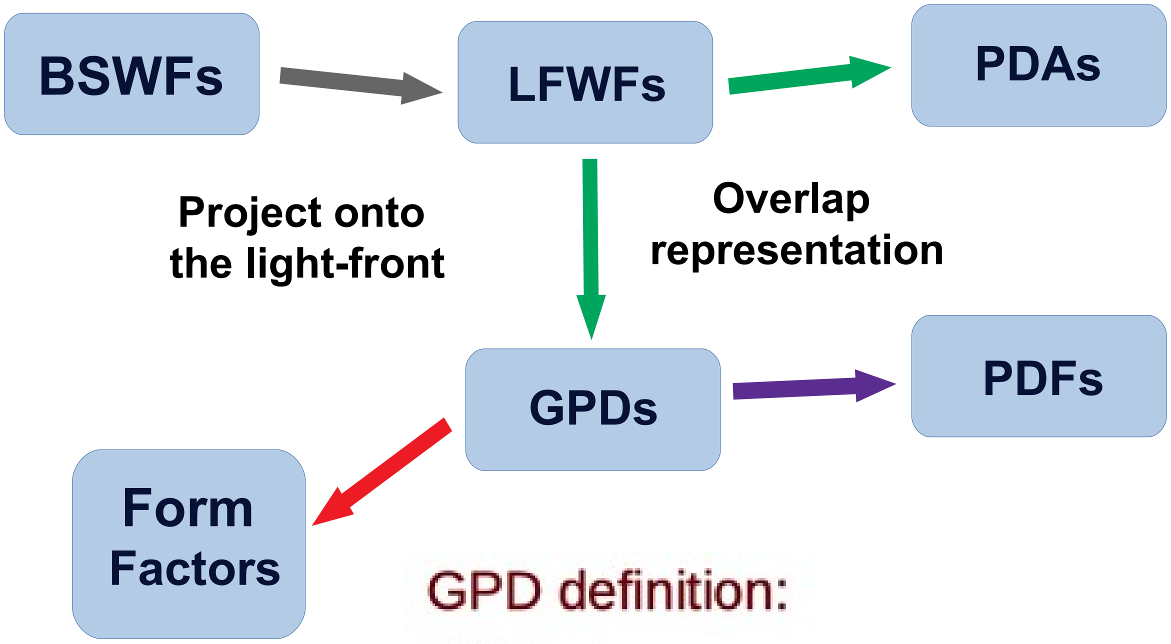
Pions and **Kaons** emerge as QCD's (pseudo)-**Goldstone** bosons.

Dynamical Chiral Symmetry Breaking (**DCSB**)



LFWFs, PDFs and PDAs from DSEs

- **Goal:** get a **broad picture** of the pion/Kaon structure.



The approach:
Compute ***everything*** from the **LFWF,** obtained from ***solutions*** of quark **DSE** and meson **BSE**.

- ✓ Already on the market:
  PDAs, PDFs, Form factors...
  K. Raya et al., arXiv: 1911.12941 [nucl-th]
  Z-F Cui et al., arXiv: 2006.14075 [hep-ph]

GPD definition:

$$H_\pi^q(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2}\right| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \left|\pi, P-\frac{\Delta}{2}\right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

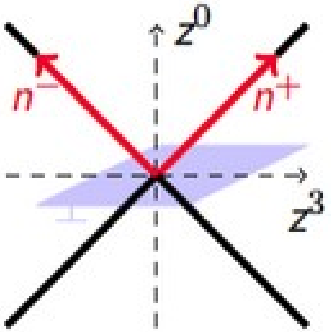
with $t=\Delta^2$ and $\xi = -\Delta^+/(2P^+)$.

🟩 $\int dk_\perp$

🟥 $\int dx$

🟪 $t=0, \xi=0$

Muller et al., Fortchr. Phys. 42 (1994) 101
Radyushkin, Phys. Lett. B380 (1996) 417
Ji, Phys. Rev. Lett. 78 (1997) 610



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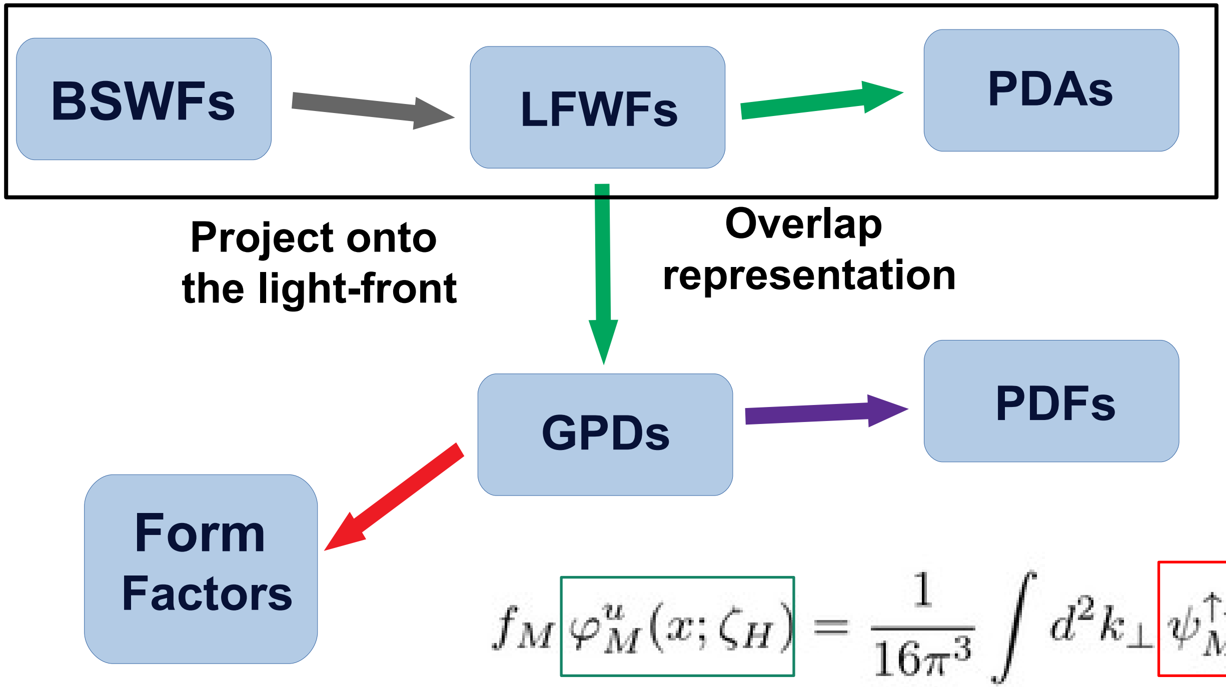

BSWFs

LFWFs

PDAs

Project onto the light-front

Overlap representation

GPDs

PDFs

Form Factors

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ζ_H : hadron scale

$$f_M \boxed{\varphi_M^u(x;\zeta_H)} = \frac{1}{16\pi^3}\int d^2k_\perp \boxed{\psi_{M_u}^{\uparrow\downarrow}(x,k_\perp^2;\zeta_H)}$$

$$= N_c \mathrm{tr} Z_2(\zeta_H,\Lambda)\int_{dk}^{\Lambda}\delta_n^x(k_\eta)\gamma_5\gamma\cdot n\boxed{\chi_M(k_{\eta\bar{\eta}};P;\zeta_H)}$$

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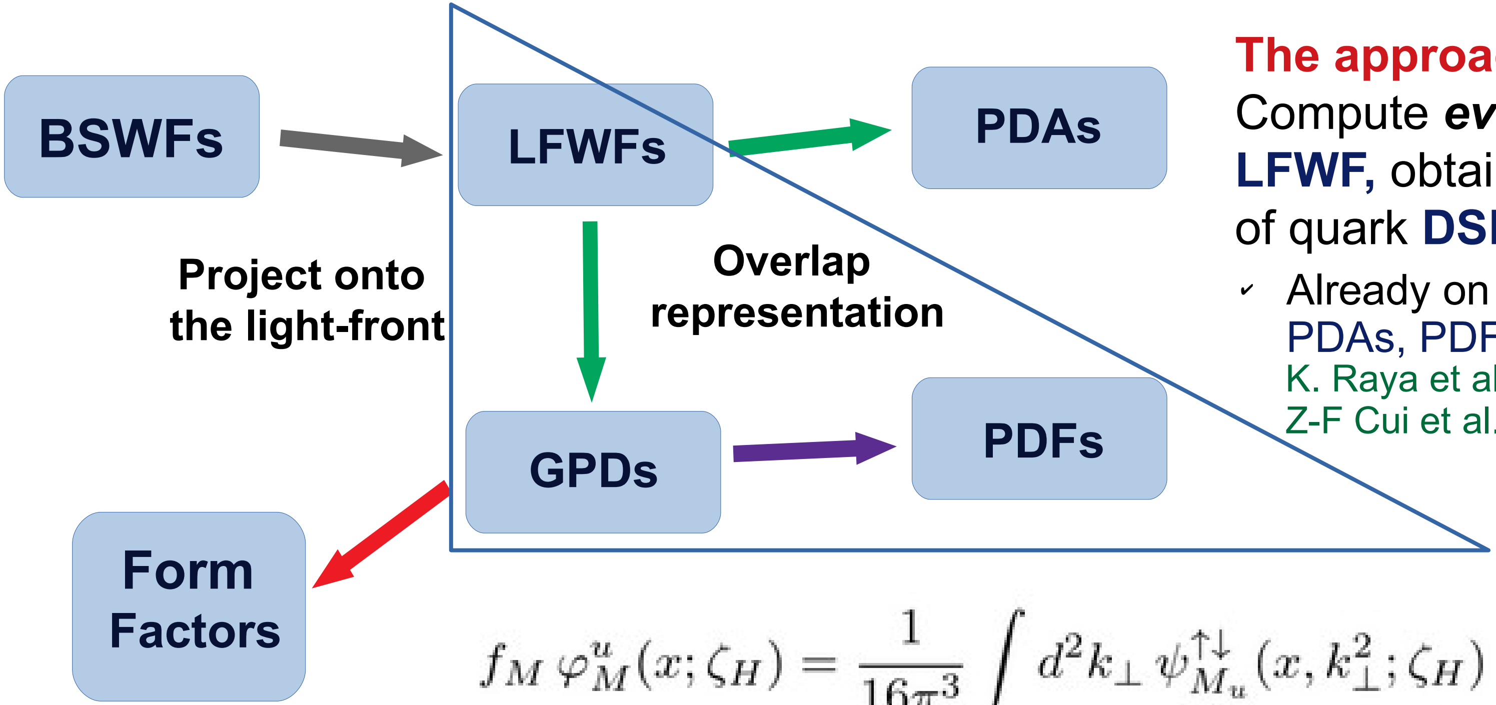

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$$\boxed{u^M(x;\zeta_H)} = \boxed{H_M^u(x,0,0;\zeta_H)} = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3}\boxed{\left|\psi_{Mu}^{\uparrow\downarrow}(x,\mathbf{k}_\perp^2;\zeta_\mathbf{H})\right|^2}$$

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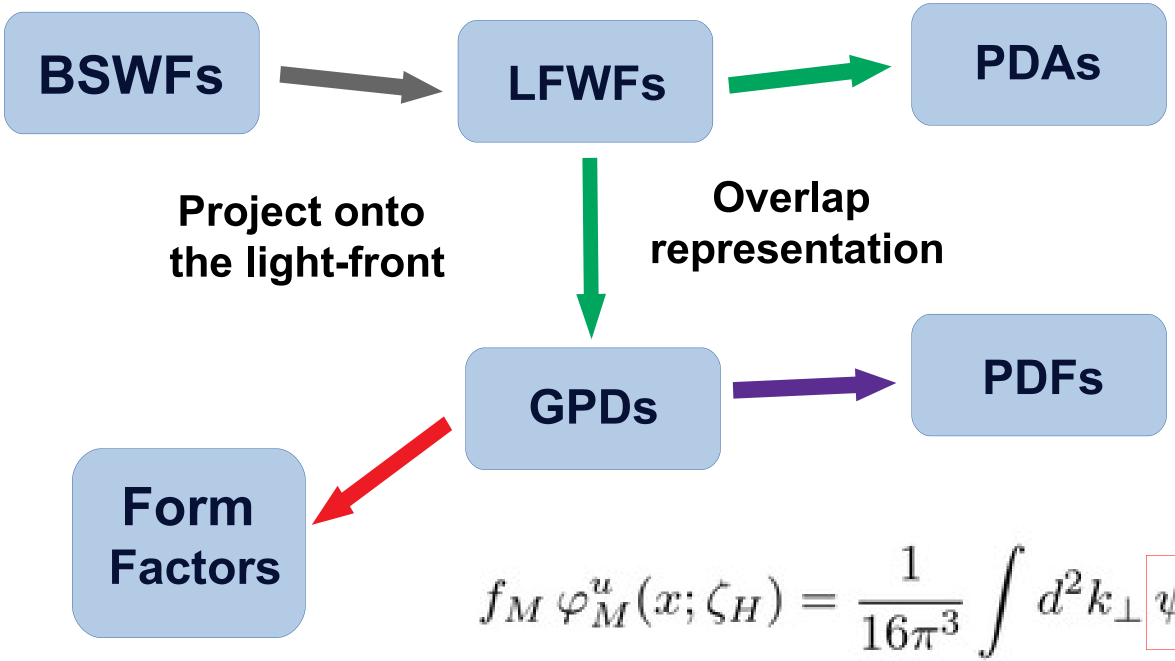

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S.-S. Xu et al,. Phys.Rev.D97094014(2018)

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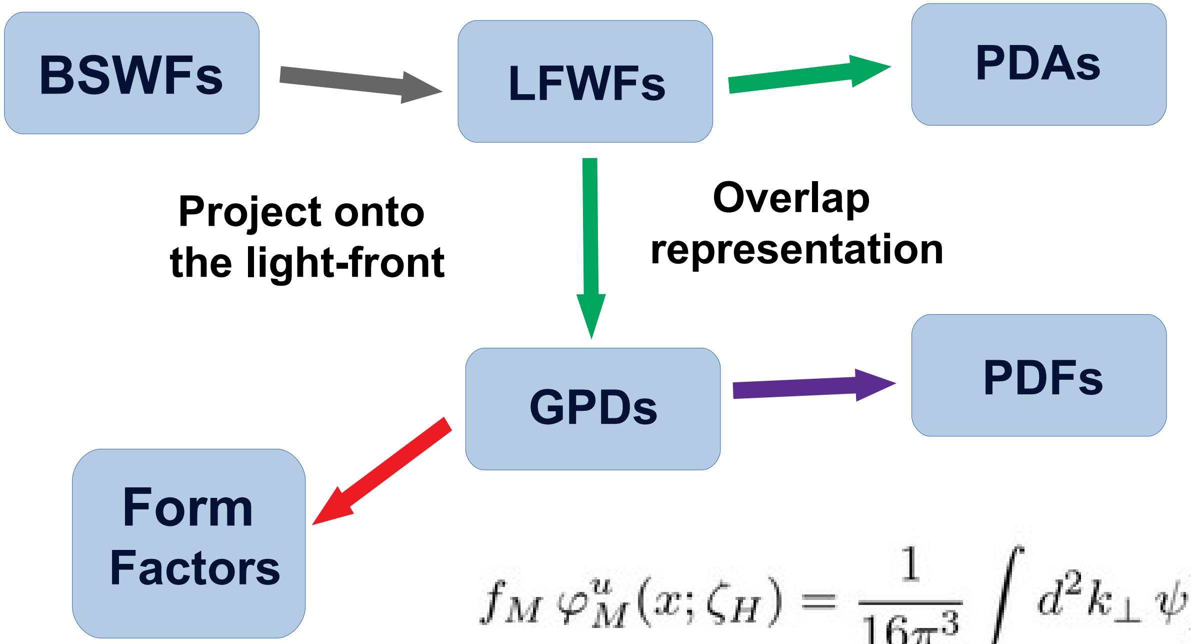

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$$u^M(x;\zeta_H) = H_M^u(x,0,0;\zeta_H) = |\varphi_M^u(x;\zeta_H)|^2\int\frac{d^2\mathbf{k}_\perp}{16\pi^3}\left|\psi_{M_u}^{\uparrow\downarrow}(\mathbf{k}_\perp^2;\zeta_\mathbf{H})\right|$$

$$\bar{h}^M(x;\zeta_H) = u^M(1-x;\zeta_H) \qquad (h=d,s)$$

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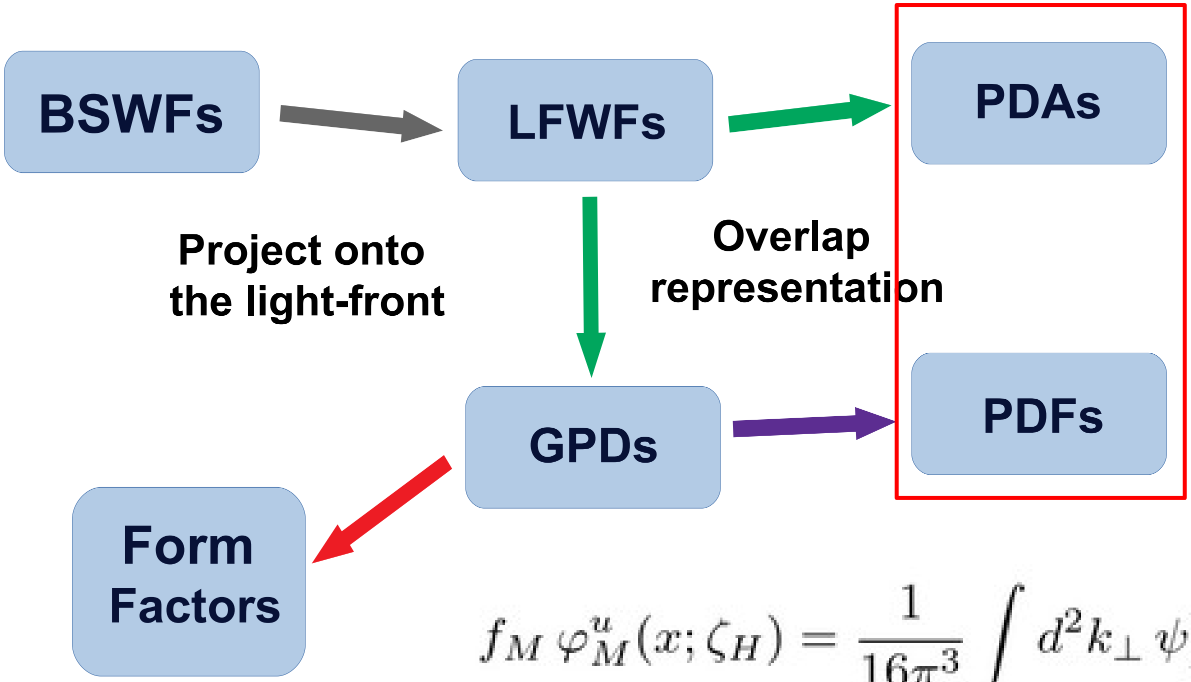

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$$\boxed{u^M(x;\zeta_H) = H_M^u(x,0,0;\zeta_H) \propto |\varphi_M^u(x;\zeta_H)|^2}$$

Direct connection between meson PDAs and PDFs at the hadronic scale, ζ_H, grounded on the *factorization approximation*, only valid for integrated quantities and not beyond ζ_H due to parton splitting effects.

Beyond factorization: PTIR

Modeling the LFWF:

S-S Xu et al., PRD 97 (2018) no.9, 094014.

- Considering the Kaon as a example, we employ a Nakanishi-like representation:

$$n_K \chi_K^{(2)}(k_-^K; P_K) = \underbrace{\mathcal{M}(k;P_K)}_{1} \int_{-1}^{1} d\omega \; \underbrace{\rho_K(\omega)}_{2}\underbrace{\mathcal{D}(k;P_K)}_{3} \, ,$$

1: Matrix structure (leading BSA):

$$\mathcal{M}(k;P_K) = -\gamma_5[\gamma \cdot P_K M_u + \gamma \cdot k (M_u - M_s) + \sigma_{\mu\nu} k_\mu P_{K\nu}] \, ,$$

2: Sprectral weight: To be described later.

3: Denominators: $\mathcal{D}(k;P_K) = \Delta(k^2, M_u^2)\Delta((k-P_K)^2, M_s^2)\hat{\Delta}(k_{\omega-1}^2, \Lambda_K^2) \, ,$

$$\text{where: } \Delta(s,t) = [s+t]^{-1}, \; \hat{\Delta}(s,t) = t\Delta(s,t) \, .$$

- **Algebraic** manipulation yields:

$$\chi_K^{(2)}(k_-^K; P_K) = \mathcal{M}(k;P_K) \int_0^1 d\alpha \; 2\boxed{\chi_K(\alpha; \sigma^3(\alpha))} \, , \; \sigma = (k - \alpha P_K)^2 + \Omega_K^2 \, ,$$

- **$\rho_K(\omega)$** will play a **crucial role** in determining the meson's observables.
- Realistic **DSE predictions** will help us to shape it.

Scalar function:

$$\chi_K(\alpha;\sigma^3) = \left[\int_{-1}^{1-2\alpha} d\omega \int_{1+\frac{2\alpha}{\omega-1}}^{1} dv + \int_{1-2\alpha}^{1} d\omega \int_{\frac{\omega-1+2\alpha}{\omega+1}}^{1} dv\right] \frac{\rho_K(\omega)}{n_K} \frac{\Lambda_K^2}{\sigma^3} \, .$$

Beyond factorization: PTIR

Modeling the LFWF:

S-S Xu et al., PRD 97 (2018) no.9, 094014.

- The **pseudoscalar LFWF** can be written:

$$f_K \psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \mathrm{tr}_{CD}\int_{dk_\parallel} \delta(n\cdot k - x\, n\cdot P_K)\gamma_5\gamma\cdot n\,\chi_K^{(2)}(k_-^K;P_K)\,.$$

- The **moments** of the distribution:

$$< x^m >_{\psi_K^{\uparrow\downarrow}} = \int_0^1 dx\, x^m \psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \frac{1}{f_K n\cdot P}\int_{dk_\parallel}\left[\frac{n\cdot k}{n\cdot P}\right]^m \gamma_5\gamma\cdot n\,\chi_K^{(2)}(k_-^K;P_K)$$

$$\int_0^1 d\alpha\,\alpha^m\left[\frac{12}{f_K}\mathcal{Y}_K(\alpha;\sigma^2)\right]\,,\qquad \mathcal{Y}_K(\alpha;\sigma^2) = [M_u(1-\alpha)+M_s\alpha]\mathcal{X}(\alpha;\sigma_\perp^2)\,.$$

Uniqueness of Mellin moments ➡

$$\boxed{\psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \frac{12}{f_K}\mathcal{Y}_K(x;\sigma_\perp^2)}$$

- ✓ Compactness of this result is a merit of the algebraic model.

- The explicit form of **$\rho_K(\omega)$ controls** the shape of **PDAs, PDFs**, **GPDs,** etc.

$$\boxed{\psi_K^{\uparrow\downarrow}(x,k_\perp^2)\sim\int d\omega\,\cdots\rho_K(\omega)\cdots}$$

Beyond factorization: PTIR

Modeling the LFWF:

- *Asymptotic* model:

$$\rho_\pi(\omega) \sim (1-\omega^2) \longrightarrow \begin{cases} \phi(x) \sim x(1-x) & \textit{Asymptotic} \text{ PDA} \\ q(x) \sim [x(1-x)]^2 & \textit{Free-scale} \text{ PDF} \end{cases}$$

C. Mezrag et al., PLB 741 (2015) 190-196.
C. Mezrag et al., FBS 57 (2016) no.9, 729-772

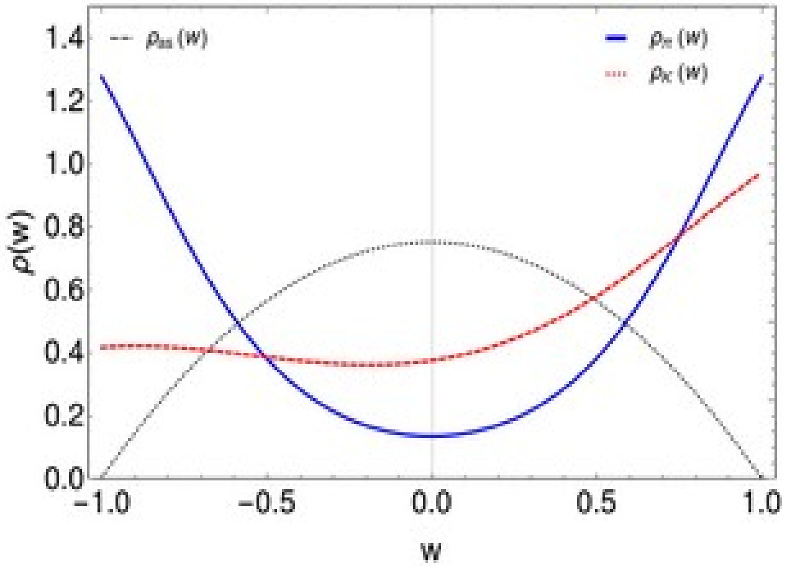
- Experience and careful analysis lead us to the following **flexible** parametrization intended to a realistic description of meson DFs:

$$\rho_{\mathsf{P}}(\omega) = \frac{1+\omega\, v_{\mathsf{P}}}{2a_{\mathsf{P}} b_0^{\mathsf{P}}} \left[\operatorname{sech}^2\left(\frac{\omega-\omega_0^{\mathsf{P}}}{2b_0^{\mathsf{P}}}\right) + \operatorname{sech}^2\left(\frac{\omega+\omega_0^{\mathsf{P}}}{2b_0^{\mathsf{P}}}\right)\right]$$

- Employing PDFs and PDAs as ***benchmarks:***

P	m_{P}	M_u	M_h	Λ_{P}	b_0^{P}	ω_0^{P}	v_{P}
π	0.14	0.31	M_u	M_u	0.316	1.23	0
K	0.49	0.31	$1.2M_u$	$3M_s$	0.1	0.625	0.41

Typical values of **constituent** quark masses, from **realistic** DSEs **solutions**.



Off-forward extension of PDFs

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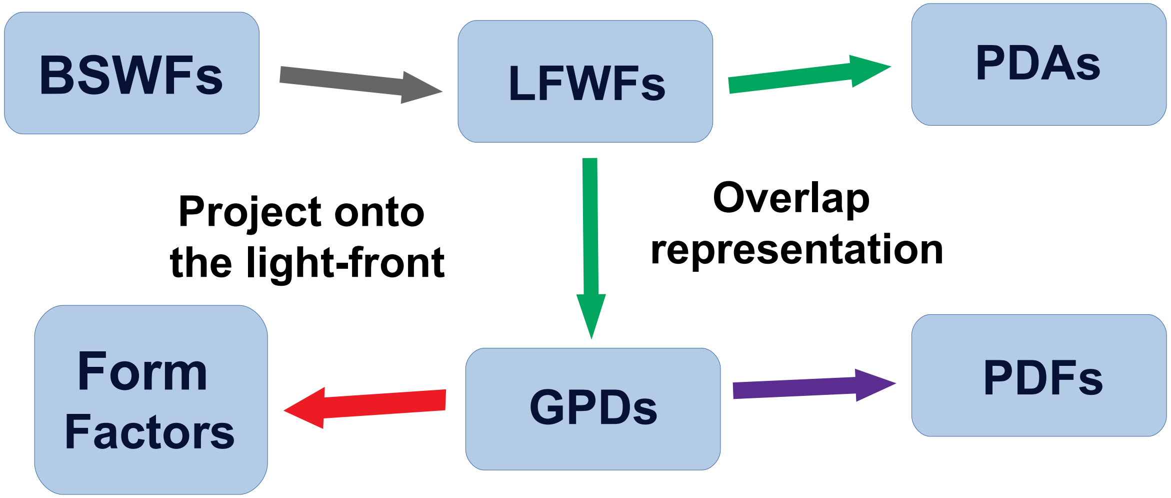

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Let us first apply the factorization approximation:

$x-\xi \geq 0 \, ; \, \xi \geq 0$

$$H_M^u(x,\xi,t;\zeta_H) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3} \psi_{Mu}^{\uparrow\downarrow *}\left(\frac{x-\xi}{1-\xi}, \left(\mathbf{k}_\perp + \frac{\mathbf{1-x}}{\mathbf{1}-\xi}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2 ; \zeta_H\right) \psi_{Mu}^{\uparrow\downarrow}\left(\frac{x+\xi}{1+\xi}, \left(\mathbf{k}_\perp - \frac{\mathbf{1-x}}{\mathbf{1}+\xi}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2 ; \zeta_H\right)$$

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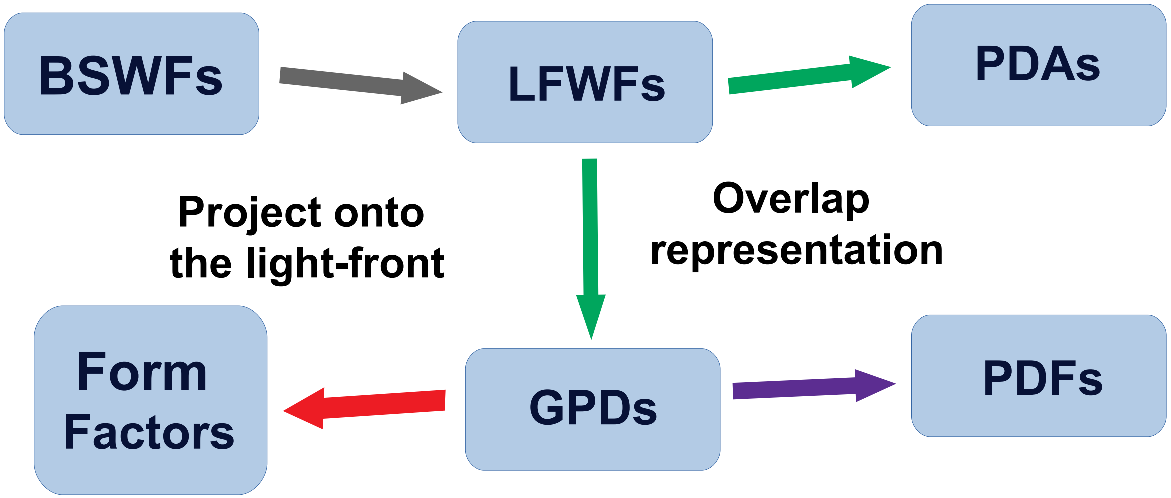

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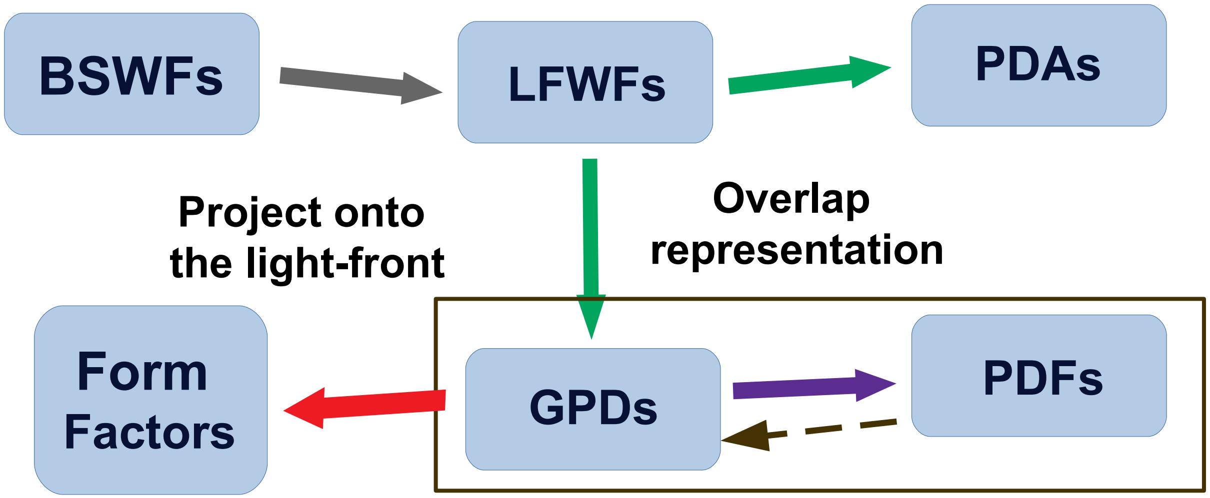

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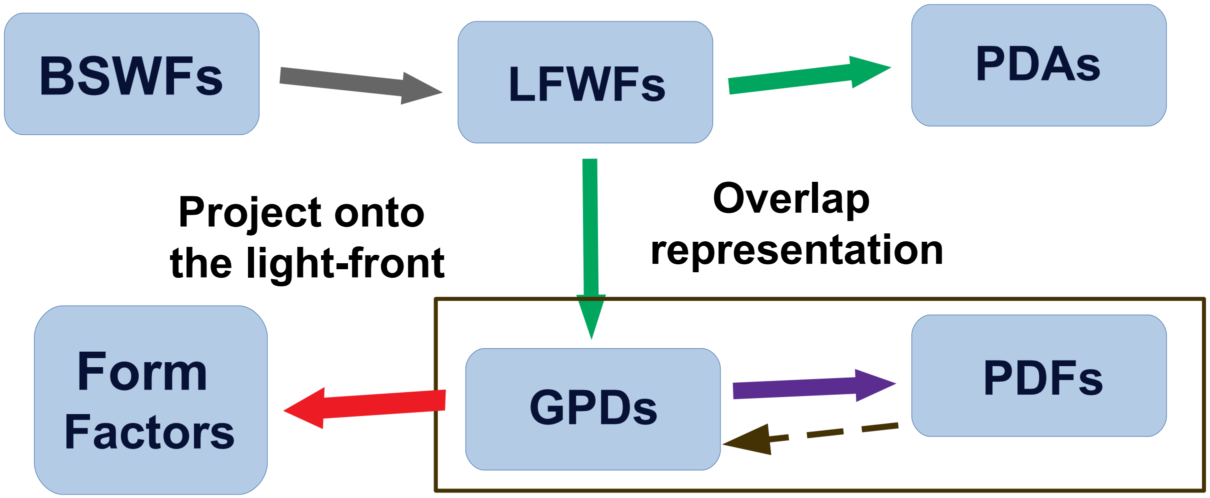

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M being a **Goldstone** boson in the **chiral limit.**

$$\psi^{\uparrow\downarrow}_{M_u}(x,k_\perp^2;\zeta_H)=\varphi^u_M(x;\zeta_H)\psi^{\uparrow\downarrow}_{M_u}(k_\perp^2;\zeta_H)$$

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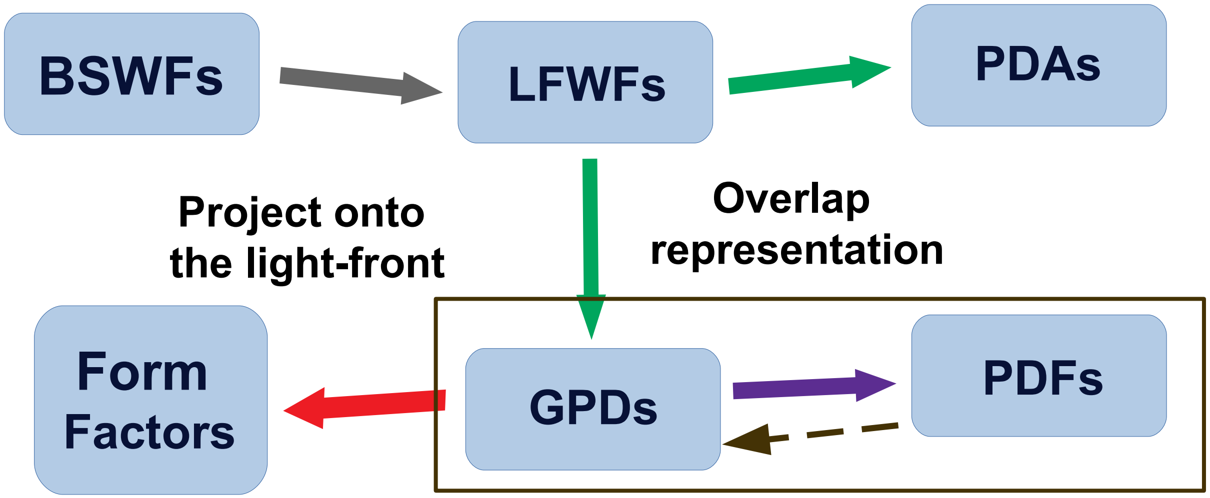

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- Already on the market:
  PDAs, PDFs, Form factors...
  K. Raya et al., arXiv: 1911.12941 [nucl-th]
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Let us first apply the factorization approximation:

$x-\xi \geq 0\,;\,\xi \geq 0$

$$\boxed{H^u_M(x,\xi,t;\zeta_H) = \sqrt{u^M\left(\frac{x-\xi}{1-\xi};\zeta_H\right) u^M\left(\frac{x+\xi}{1+\xi};\zeta_H\right)}\, \Phi^u_M\left(\frac{-t(1-x)^2}{(1-\xi^2)};\zeta_H\right)}$$

M being a **Goldstone** boson in the **chiral limit.**

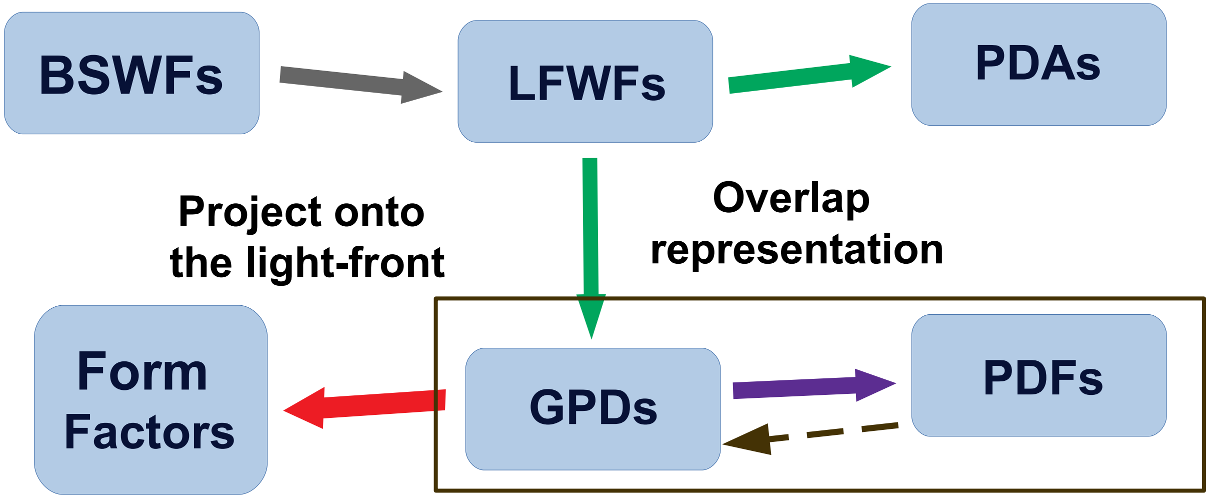
- The **positivity condition** is made manifest $\Phi^u_M(z) \leq 1$

$$\boxed{\psi^{\uparrow\downarrow}_{M_u}(x,k_\perp^2;\zeta_H) = \varphi^u_M(x;\zeta_H)\psi^{\uparrow\downarrow}_{M_u}(k_\perp^2;\zeta_H)}$$

$$\boxed{u^M(x;\zeta_H) \propto |\varphi^u_M(x;\zeta_H)|^2}$$

Off-forward extension of PDFs

- **Goal:** get a **broad picture** of the pion/Kaon structure.



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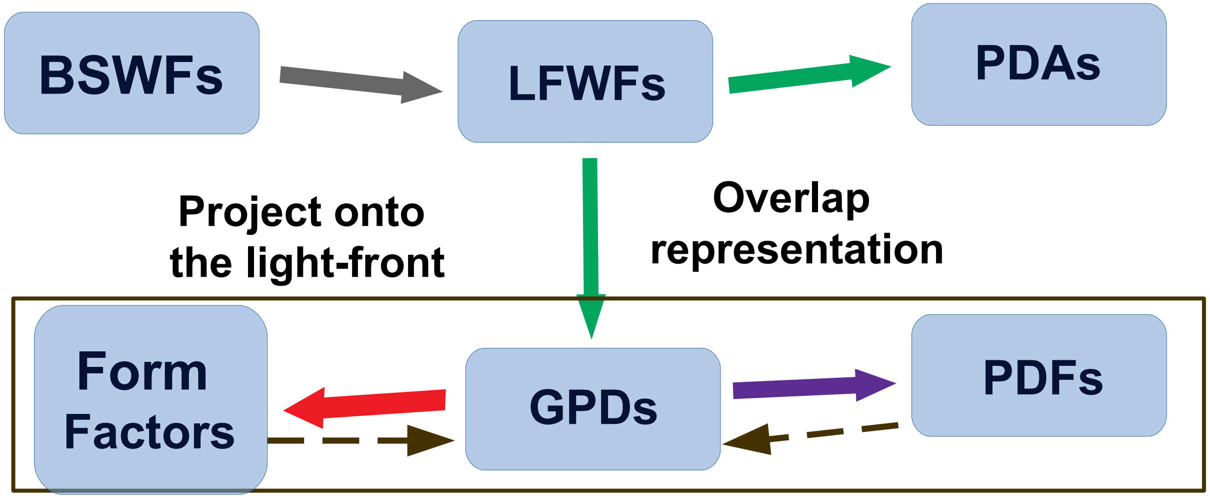

BSWFs

LFWFs

PDAs

Project onto the light-front

Overlap representation

Form Factors

GPDs

PDFs

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$$\int_{-1}^{1} dx H_M^u(x,\xi=0,t) = \boxed{\int_0^1 dx u^M(x;\zeta_H)\Phi_M^u\left(-t(1-x)^2;\zeta_H\right) = F_M^u(-t)}$$

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Off-forward extension of PDFs

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$$\big\downarrow$$

$$\langle x^{2n}\rangle_{\bar{h}}^{\zeta_H} = \int_0^1 dx x^{2n}\bar{h}^M(x;\zeta_H) = \int_0^1 dx(1-x)^{2n}u^M(x;\zeta_H)$$

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Realistic (CSF) PDF
\+
Realistic (CSF) FF

$\longrightarrow$ Realistic GPD prediction

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- Only the first derivative implies: [combining quark and antiquark GPDs]

$$\frac{\partial}{\partial z}\Phi_M^u(z;\zeta_H)\Big|_{z=0} = -\frac{r_M^2}{4\langle x^2\rangle_{\bar{s}}^{\zeta_H} + 2(1+\delta)\langle x^2\rangle_u^{\zeta_H}}$$

in terms of the meson’s EM charge radius

$$\propto M_u - M_h$$

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- The impact-parameter GPD

$$u^M(x,b_\perp^2;\zeta_H)=\int_0^\infty\frac{d\Delta_\perp}{2\pi}\Delta_\perp J_0(b_\perp\Delta_\perp)\;H^u_M(x,\xi,t;\zeta_H)|_{\xi=0}$$

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$$u^M(x,b_\perp^2;\zeta_H)=\frac{\boxed{u^M(x;\zeta_H)}}{(1-x)^2}\int_0^\infty\frac{sds}{2\pi}\Phi_M^u(s^2;\zeta_H)J_0\left(\frac{b_\perp}{1-x}s\right)$$

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Compact expressions in terms of the PDF and Φ_M^u

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$$\boxed{H_M^u(x,\xi,t;\zeta_H) = \sqrt{u^M\left(\frac{x-\xi}{1-\xi};\zeta_H\right) u^M\left(\frac{x+\xi}{1+\xi};\zeta_H\right)}\,\Phi_M^u\left(\frac{-t(1-x)^2}{(1-\xi^2)};\zeta_H\right)}$$

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Pion's case

$$\frac{\partial}{\partial z}\Phi_\pi^u(z;\zeta_H)\Big|_{z=0} = -\frac{r_\pi^2}{6\langle x^2\rangle_u^{\zeta_H}}$$

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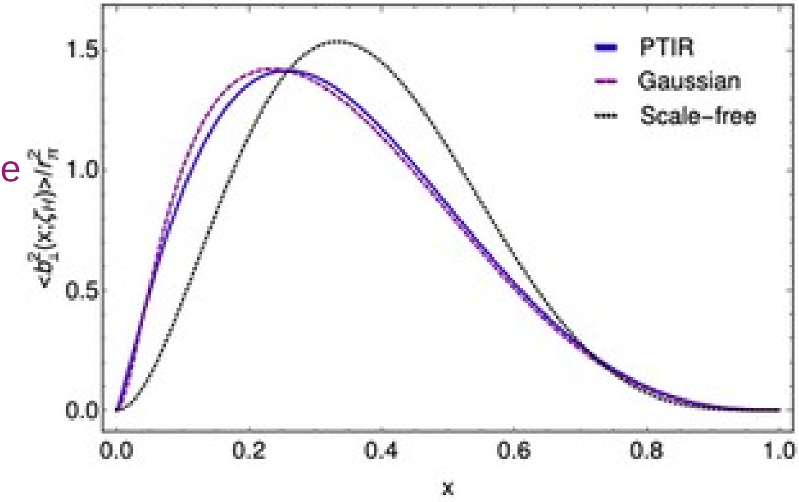


Compact expressions in terms of the PDF and Φ_M^u

Mean-squared transverse extent

Inputs: PDFs and PDAs from CSF

Symmetry-preserving DSE computation of the valence-quark PDF:

[L. Chang et al., Phys.Lett.B737(2014)23]
[M. Ding et al., Phys.Rev.D101(2020)054014

$$q^{\pi}(x;\zeta) = N_c \mathrm{tr} \int_{dk} \delta_n^x(k_\eta) \Gamma_\pi^P(k_{\bar\eta\eta};\zeta)\, S(k_{\bar\eta};\zeta) \times \{ n \cdot \frac{\partial}{\partial k_\eta} [\Gamma_\pi^{-P}(k_{\eta\bar\eta};\zeta) S(k_\eta;\zeta)] \}$$

$$\varphi_\pi^{\surd}(x;\zeta_H) = 15.271\, x(1-x) \times [1 - 2.9342\sqrt{x(1-x)} + 2.2911 x(1-x)]^{1/2}$$

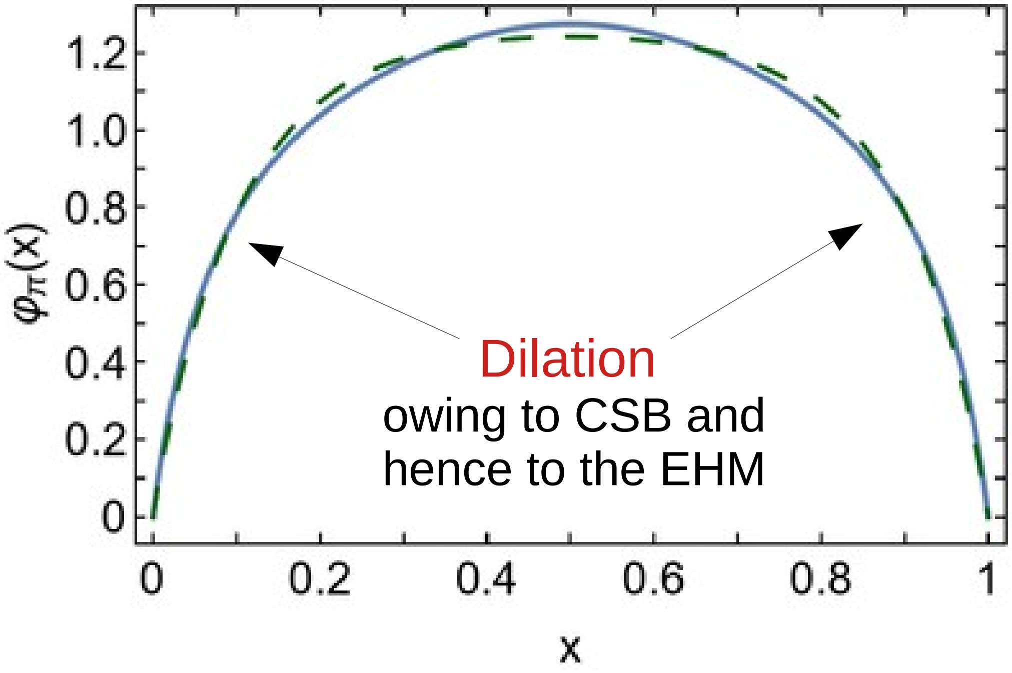
$$\boxed{u^M(x;\zeta_H) = H_M^u(x,0,0;\zeta_H) \propto |\varphi_M^u(x;\zeta_H)|^2}$$

PDA computation using the BSA obtained with the DB kernel:

[L. Chang et al., Phys.Rev.Lett.110(2013)132001]

$$f_M\, \varphi_M^u(x;\zeta_H) = N_c \mathrm{tr} Z_2(\zeta_H,\Lambda) \int_{dk}^{\Lambda} \delta_n^x(k_\eta) \gamma_5 \gamma \cdot n \chi_M(k_{\eta\bar\eta};P;\zeta_H)$$

$$\varphi_\pi^{\mathrm{DB}}(x;\zeta_H) = 20.227\, x(1-x) \times [1 - 2.5088\sqrt{x(1-x)} + 2.0250\, x(1-x)]$$



Dilation
owing to CSB and
hence to the EHM

$$u^{\mathrm{P}}(x;\zeta_{\mathcal{H}}) = n_{\mathrm{P}} x^2 (1-x)^2 \times \left[1 + \rho_{\mathrm{P}} x^{\frac{\alpha_{\mathrm{P}}}{2}} (1-x)^{\frac{\beta_{\mathrm{P}}}{2}} + \gamma_{\mathrm{P}} x^{\alpha_{\mathrm{P}}} (1-x)^{\beta_{\mathrm{P}}}\right]^2$$

Cui et al., arXiv:2006.14075

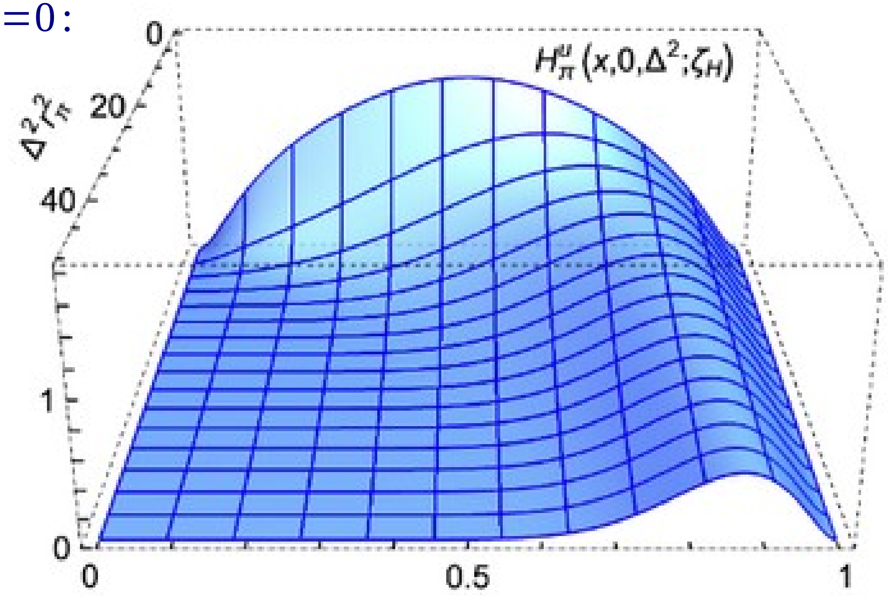
P	n_{P}	ρ_{P}	γ_{P}	α_{P}	β_{P}
π	375.3	−2.51	2.03	1.0	1.0
K	299.2	5.00	−5.97	0.064	0.048

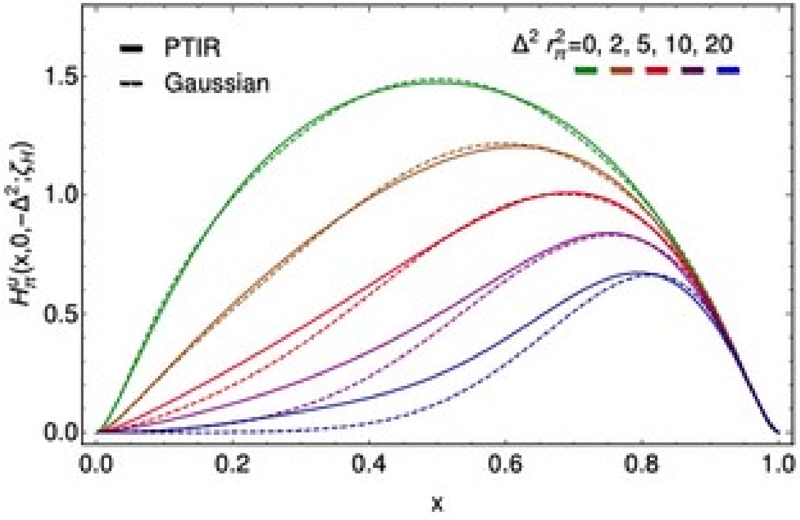
Both computations scale-independent, albeit describing properly the hadronic degrees of freedom only at the hadronic scale, at which they can be successfully comparable with each other!

GPDs from LFWFs

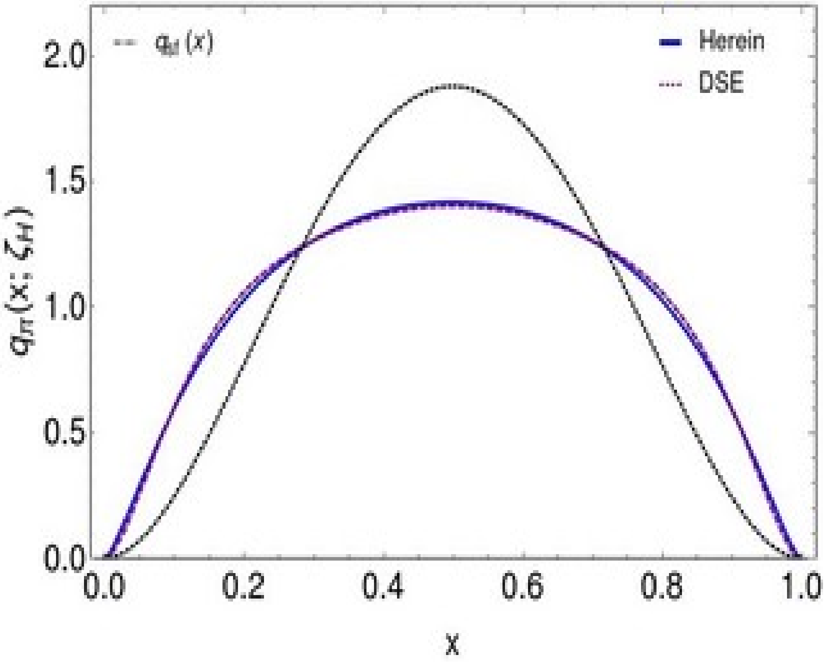
Pion GPD: $$H_\pi^u(x,\xi,t;\zeta_H) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3}\psi^{\uparrow\downarrow *}_{\pi\,u}\left(\frac{x-\xi}{1-\xi},\left(\mathbf{k}_\perp+\frac{\mathbf{1-x}}{\mathbf{1-\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2;\zeta_H\right)\psi^{\uparrow\downarrow}_{\pi\,u}\left(\frac{x+\xi}{1+\xi},\left(\mathbf{k}_\perp-\frac{\mathbf{1-x}}{\mathbf{1+\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2;\zeta_H\right)$$

$\xi=0$:





Valence-quark overlap GPD and forward PDF limit



Factorized gaussian ansatz:

$$H_\pi^u(x,\xi,t;\zeta_H) \;=\; \theta(x-\xi)\sqrt{u^\pi\left(\frac{x-\xi}{1-\xi}\right)u^\pi\left(\frac{x+\xi}{1+\xi}\right)}\;\exp\left(-\frac{-t\,r_\pi^2(1-x)^2}{6\langle x^2\rangle_u^{\zeta_H}(1-\xi^2)}\right)$$

The only (additional) input needed to fix an approximated compact result is the pion charge radius

$PDG:\ r_\pi = 0.659(8)\,fm$ $\qquad DSE:\ r_\pi = 0.69\,fm\,[PTIR]$

GPDs from LFWFs

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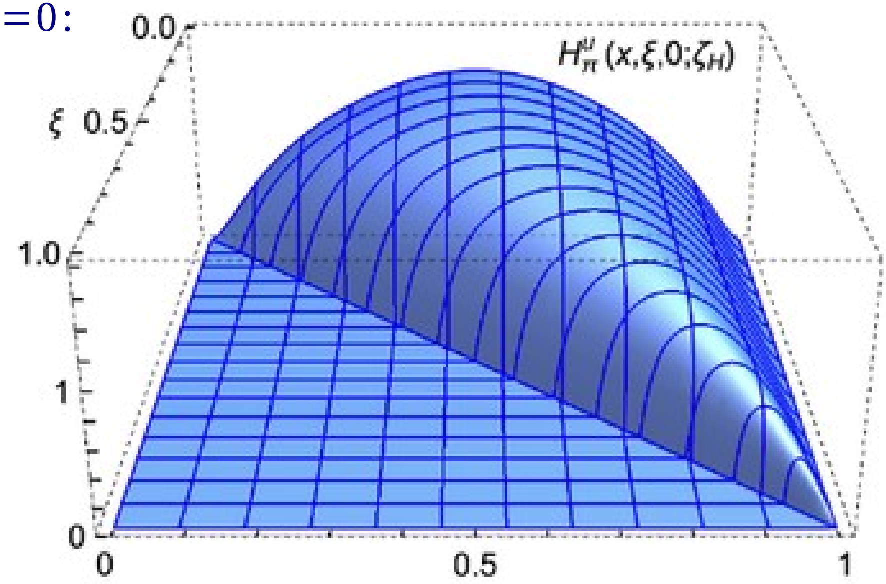
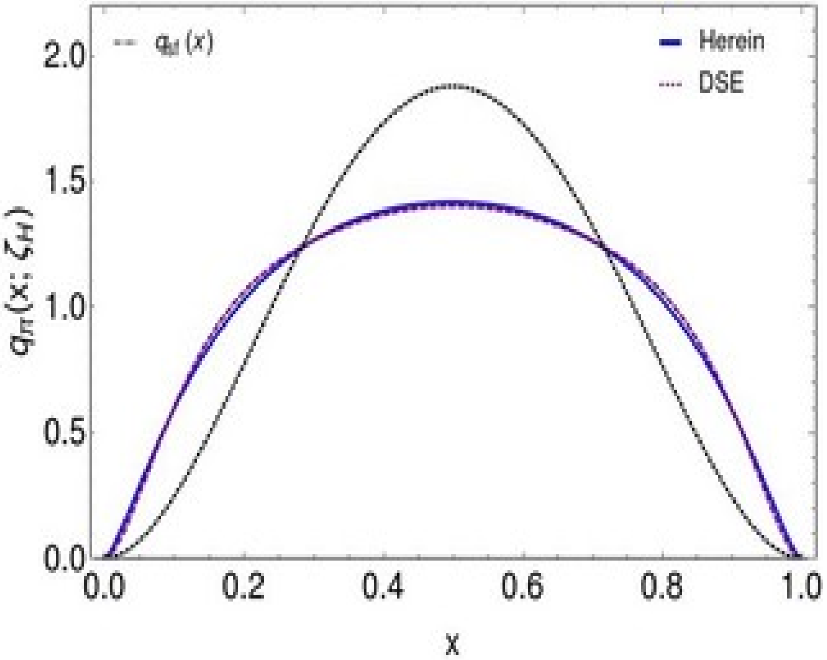
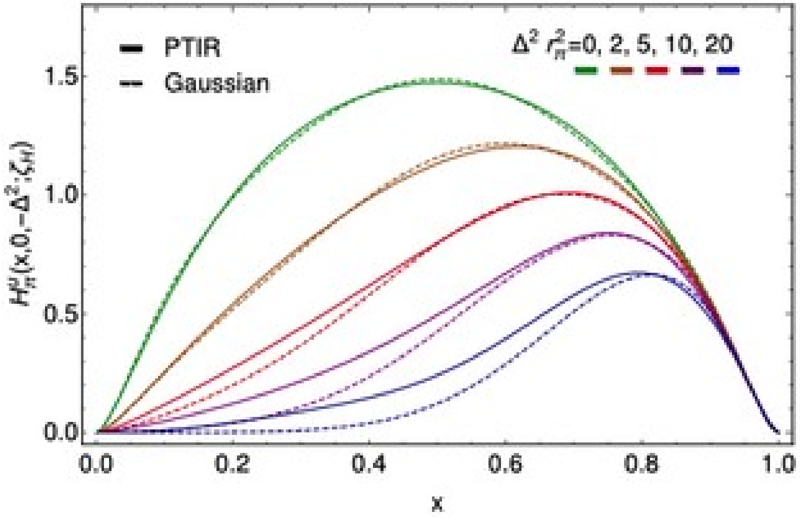
$t=0:$



Valence-quark overlap GPD and forward PDF limit





Factorized gaussian ansatz:

$$H^u_\pi(x,\xi,t;\zeta_H) \;=\; \theta(x-\xi)\sqrt{u^\pi\left(\frac{x-\xi}{1-\xi}\right)u^\pi\left(\frac{x+\xi}{1+\xi}\right)}\;\exp\left(-\frac{-t\,r_\pi^2(1-x)^2}{6\langle x^2\rangle_u^{\zeta_H}(1-\xi^2)}\right)$$

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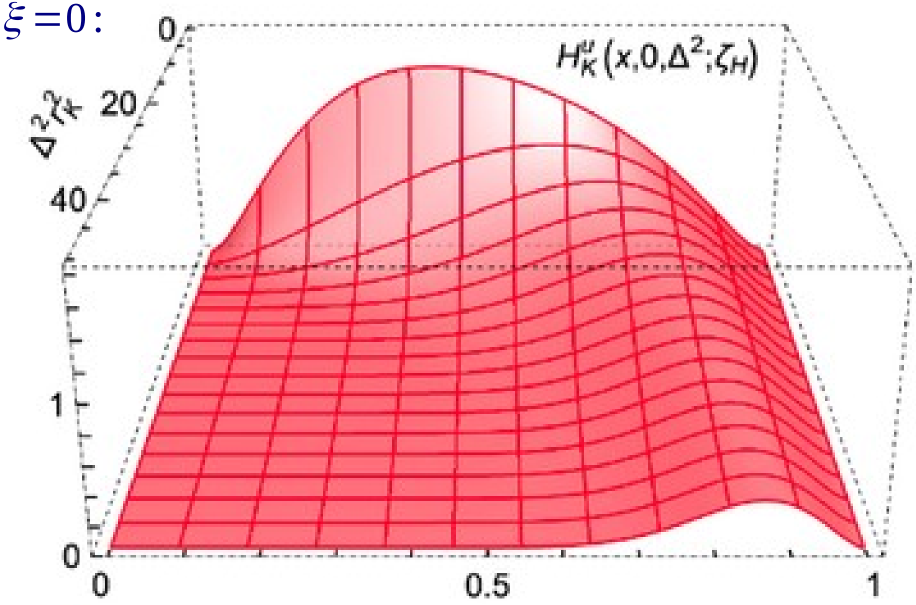
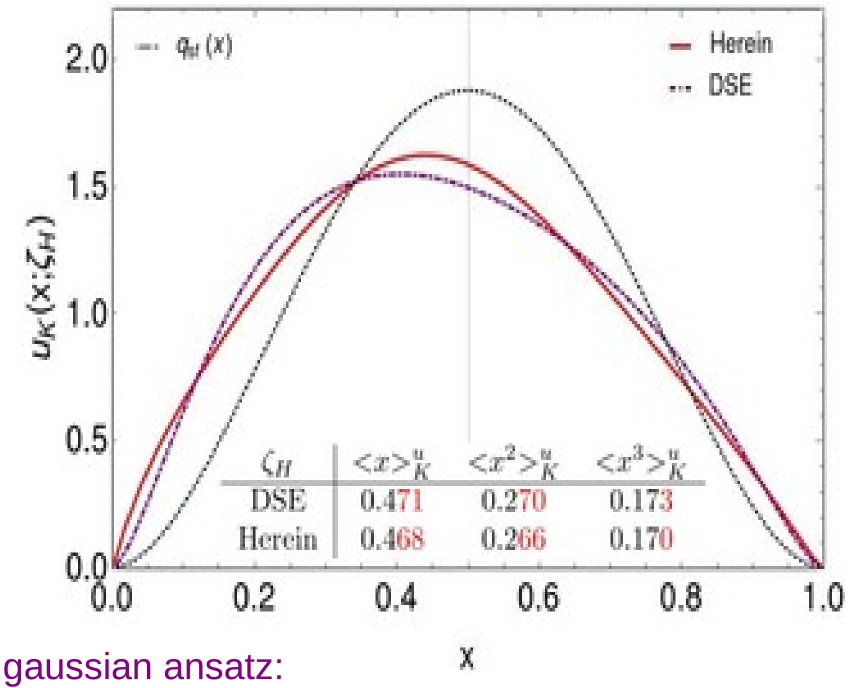
GPDs from LFWFs

Kaon GPD: $H_K^u(x,\xi,t;\zeta_H) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3} \psi_{K^u}^{\uparrow\downarrow *}\left(\frac{x-\xi}{1-\xi}, \left(\mathbf{k}_\perp + \frac{\mathbf{1-x}}{\mathbf{1-\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2; \zeta_H\right) \psi_{K^u}^{\uparrow\downarrow}\left(\frac{x+\xi}{1+\xi}, \left(\mathbf{k}_\perp - \frac{\mathbf{1-x}}{\mathbf{1+\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2; \zeta_H\right)$

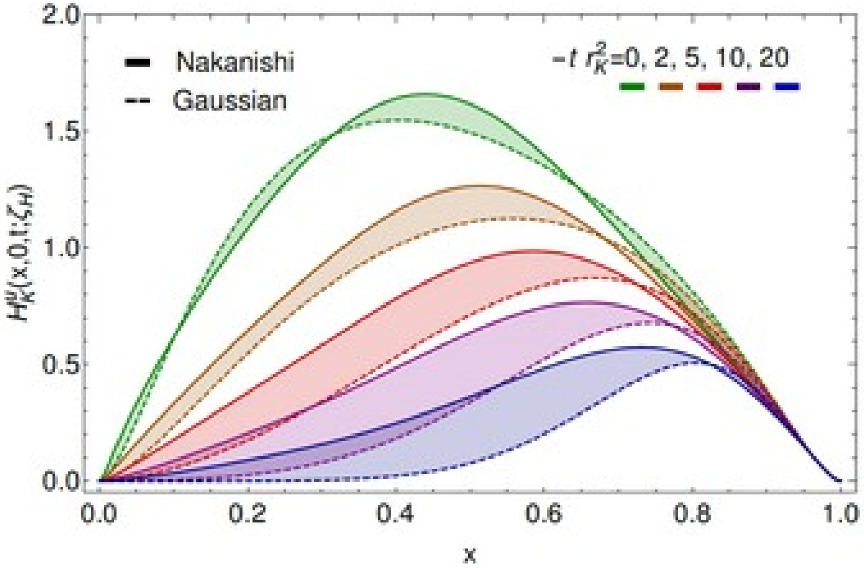
$\xi = 0:$



Valence-quark overlap GPD and forward PDF limit



ζ_H	$<x>_K^u$	$<x^2>_K^u$	$<x^3>_K^u$
DSE	0.471	0.270	0.173
Herein	0.468	0.266	0.170



Factorized gaussian ansatz:

$$H_K^u(x,\xi,t;\zeta_H) = \theta(x-\xi)\sqrt{u^K\left(\frac{x-\xi}{1-\xi}\right)u^K\left(\frac{x+\xi}{1+\xi}\right)} \times \exp\left(-\frac{-t\,r_K^2(1-x)^2}{\left(4\langle x^2\rangle_{\bar{s}}^{\zeta_H} + 2(1+\delta)\langle x^2\rangle_u^{\zeta_H}\right)(1-\xi^2)}\right)$$

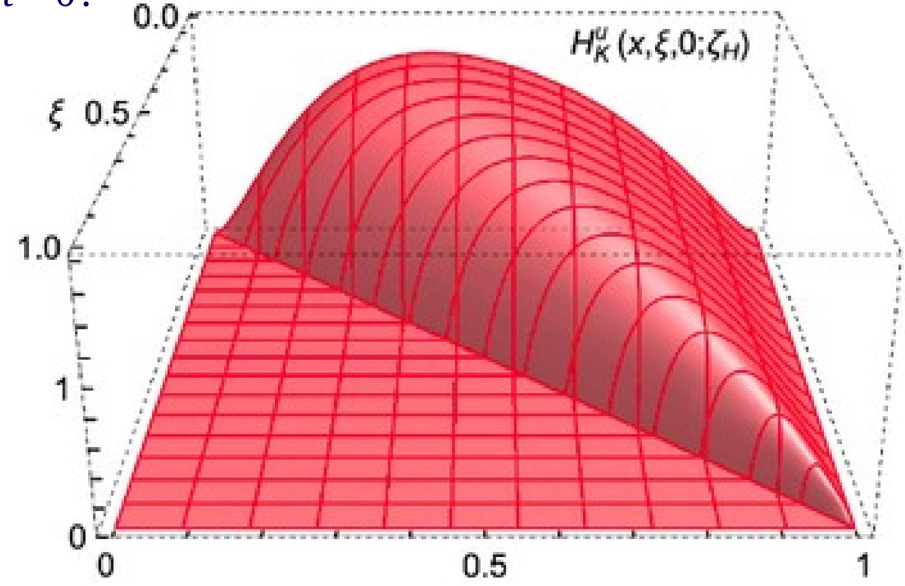
The only (additional) input needed to fix an approximated compact result is the pion charge radius

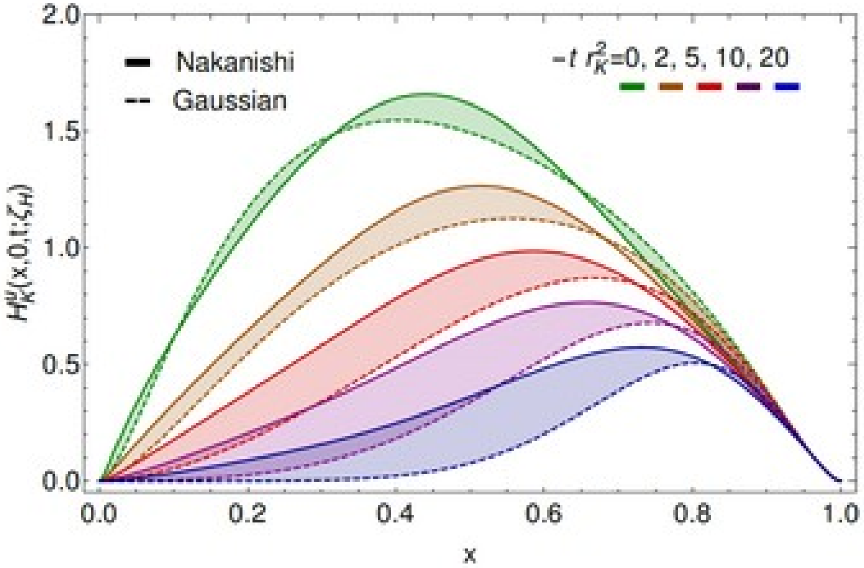
$PDG: \; r_K = 0.560(31)\,fm \qquad DSE: \; r_K = 0.56\,fm\,[PTIR]$

GPDs from LFWFs

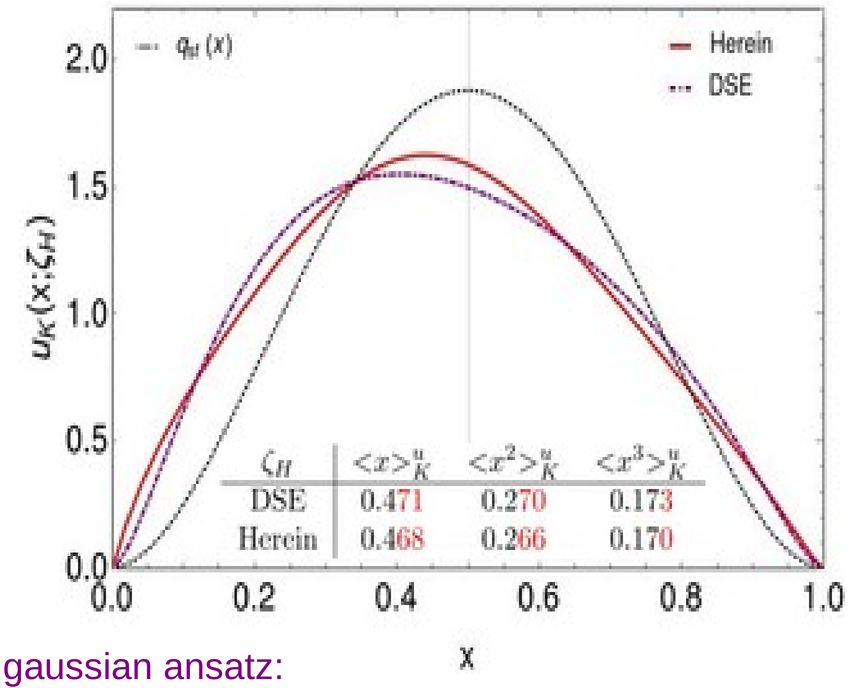
Kaon GPD: $H_K^u(x,\xi,t;\zeta_H) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3}\psi^{\uparrow\downarrow *}_{K^u}\left(\frac{x-\xi}{1-\xi},\left(\mathbf{k}_\perp + \frac{\mathbf{1-x}}{\mathbf{1-\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2;\zeta_H\right)\psi^{\uparrow\downarrow}_{K^u}\left(\frac{x+\xi}{1+\xi},\left(\mathbf{k}_\perp - \frac{\mathbf{1-x}}{\mathbf{1+\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2;\zeta_H\right)$

$t=0$:





Valence-quark overlap GPD and forward PDF limit



ζ_H	$<x>^u_K$	$<x^2>^u_K$	$<x^3>^u_K$
DSE	0.471	0.270	0.173
Herein	0.468	0.266	0.170

Factorized gaussian ansatz:

$$H_K^u(x,\xi,t;\zeta_H) = \theta(x-\xi)\sqrt{u^K\left(\frac{x-\xi}{1-\xi}\right)u^K\left(\frac{x+\xi}{1+\xi}\right)} \\ \times\exp\left(-\frac{-t\,r_K^2(1-x)^2}{\left(4\langle x^2\rangle_{\bar{s}}^{\zeta_H}+2(1+\delta)\langle x^2\rangle_u^{\zeta_H}\right)(1-\xi^2)}\right)$$

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$PDG:\ r_K = 0.560(31)\,fm \qquad DSE:\ r_K = 0.56\,fm\,[PTIR]$

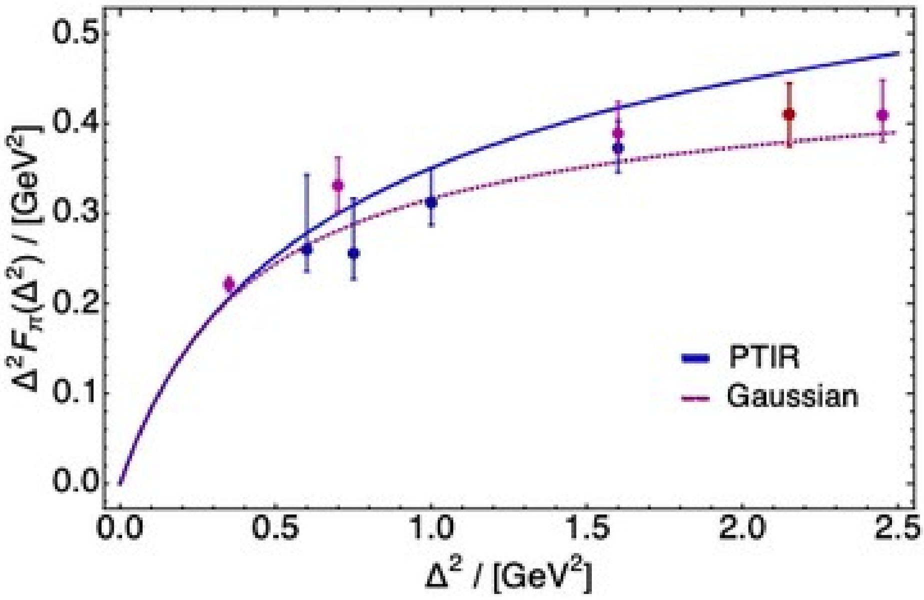
GPDs from LFWFs

Valence-quark overlap GPD and the EM form factors

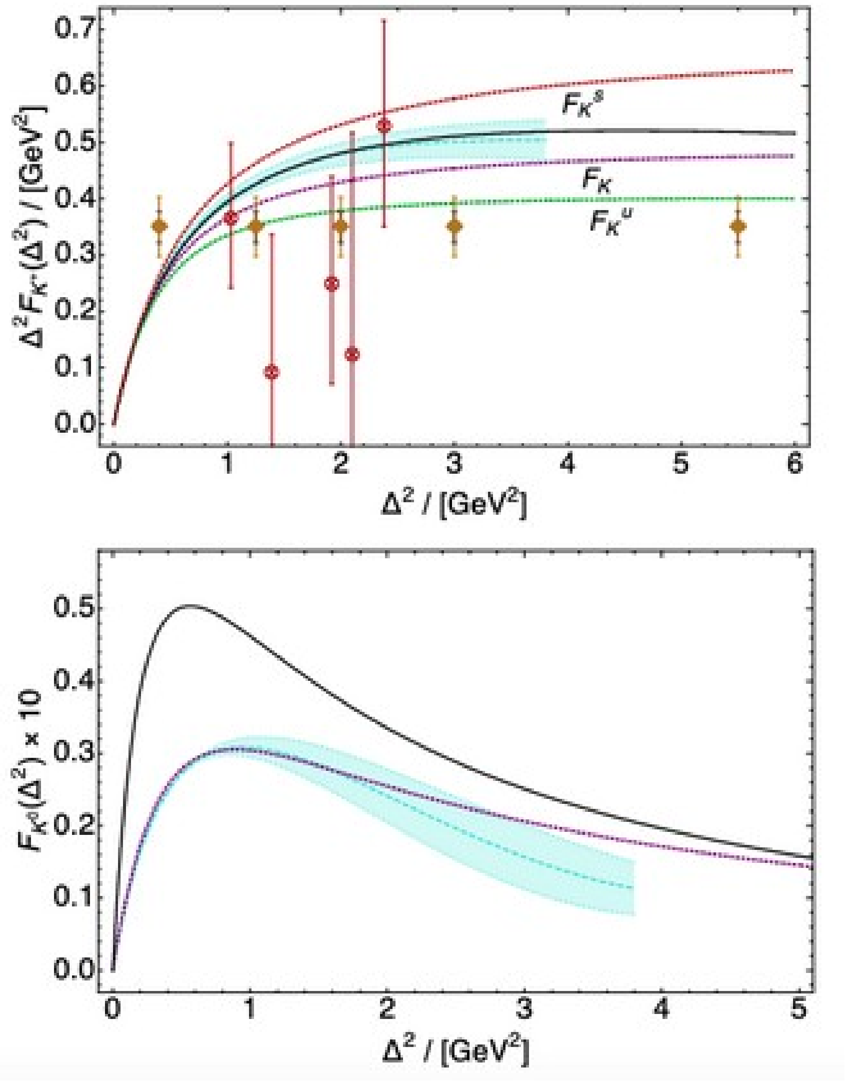
$$F_M(-t) = e_u F_M^u(-t) + e_{\bar{h}} F_M^h(-t)$$

$$F_M^u(-t) = \int_{-1}^{1} dx H_M^u(x, \xi, t; \zeta_H)$$

Pion form factor:



Kaon form factors:



$PDG: \; r_\pi = 0.659(8)\, fm$ $\quad$ $DSE: \; r_\pi = 0.69\, fm\, [PTIR]$ $\quad$ $PDG: \; r_K = 0.560(31)\, fm$ $\quad$ $DSE: \; r_K = 0.56\, fm\, [PTIR]$

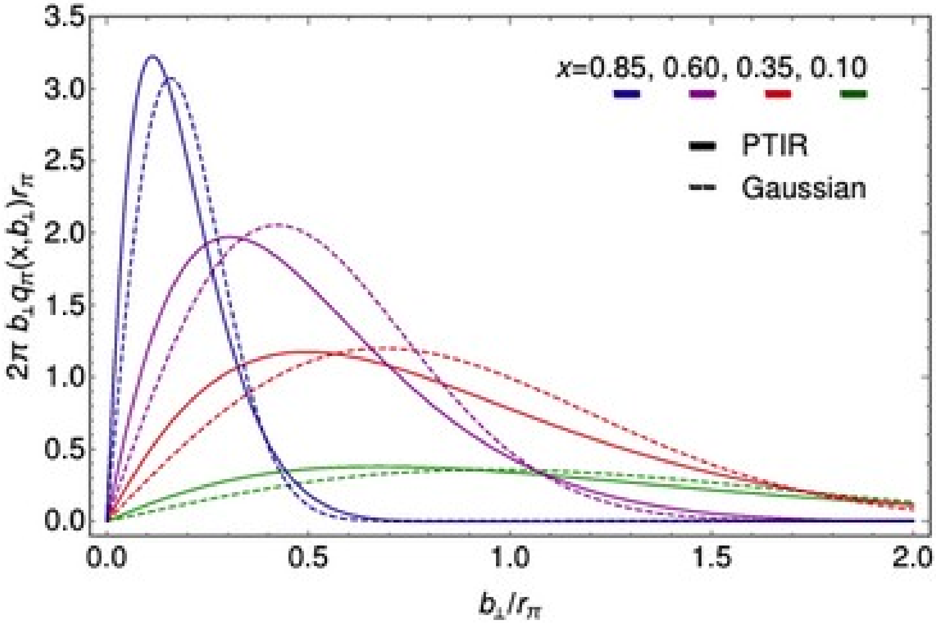
GPDs from LFWFs

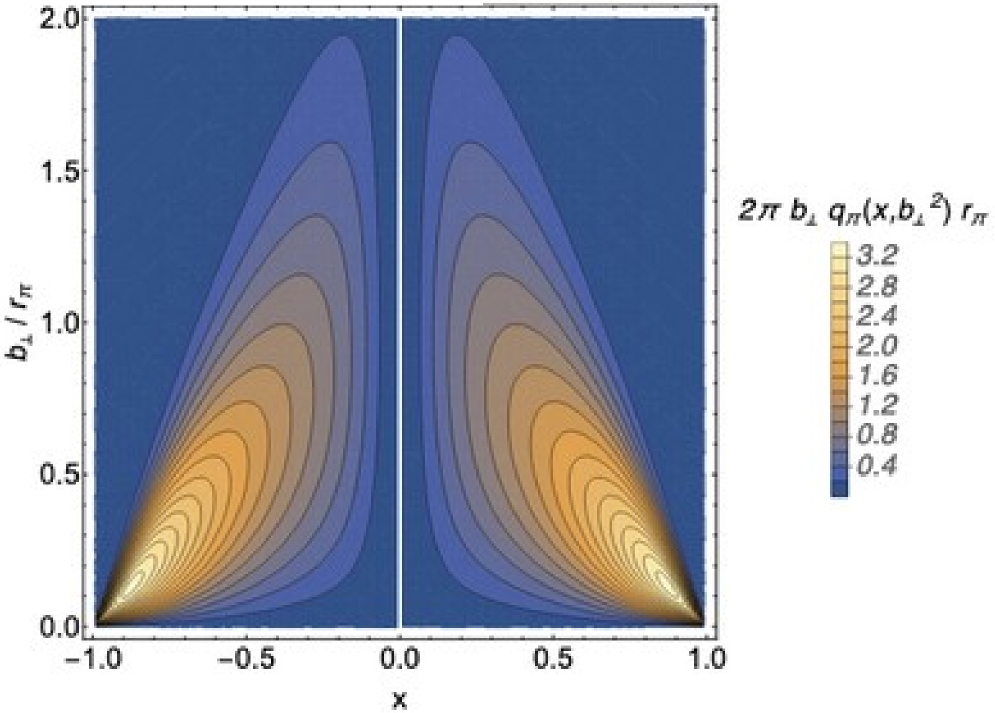
Pion IPS GPD:

$$u^{\pi'}(x, b_\perp^2;\zeta_H) = \int_0^\infty \frac{d\Delta_\perp}{2\pi}\Delta_\perp J_0(b_\perp\Delta_\perp)\ H_{\pi'}^u(x,\xi,t;\zeta_H)|_{\xi=0}$$

The probability of finding the pion’s u-quark (x>0) or d-antiquark (x<0) at a distance $b_\perp$ away from the CoTM peaks up at a small but non-zero value and at |x| near 1.

This probability density at x=cte. peaks around a maximum at non-zero $b_\perp$; the larger is x, the smaller $b_\perp$ and the narrower the distribution. The larger is the momentum fraction carried by the parton, the more it bears on the CoTM definition.





Factorized gaussian ansatz:

$$q^{\pi'}(x, b_\perp^2;\zeta_H) = \frac{\gamma_{\pi'}(\zeta_H)}{\pi r_\pi^2}\frac{q^{\pi'}(|x|;\zeta_H)}{(1-|x|)^2}\exp\left(-\frac{\gamma_\pi(\zeta_H)}{(1-|x|)^2}\frac{b_\perp^2}{r_\pi^2}\right)$$

$$\gamma_{\pi'}(\zeta_H) = \frac{3\langle x^2\rangle_u^{\zeta_H}}{2} \qquad (q=u[x\geq 0], d[x\leq 0])$$

The only (additional) input needed to fix an approximated compact result is the pion charge radius

$PDG:\ r_\pi = 0.659(8)\,fm \qquad DSE:\ r_\pi = 0.69\,fm\,[PTIR]$

GPDs from LFWFs

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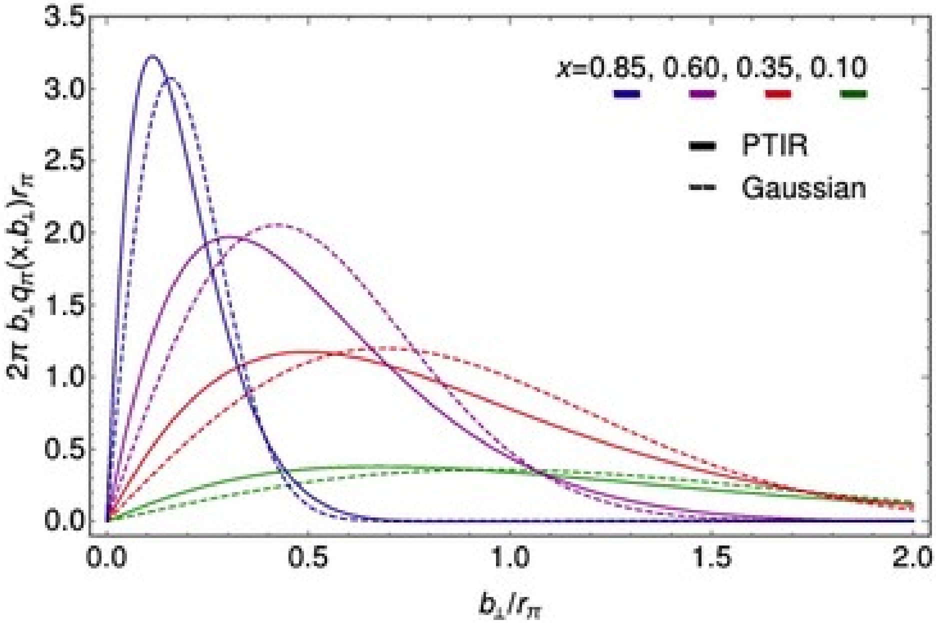
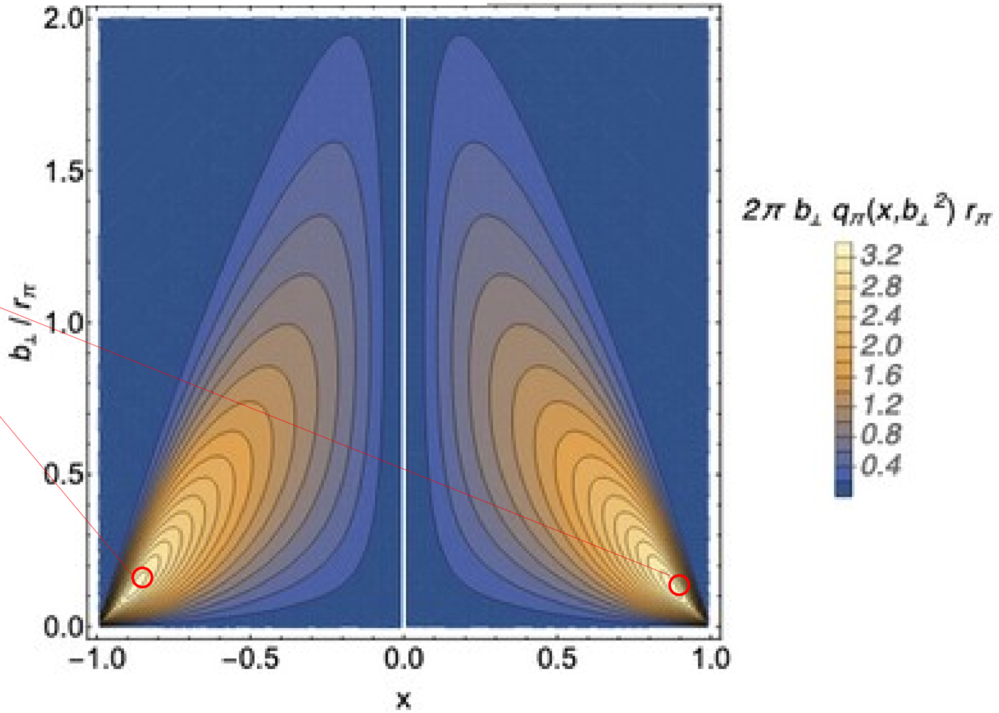
$$u^{\pi'}(x,b_\perp^2;\zeta_H) = \int_0^\infty \frac{d\Delta_\perp}{2\pi}\Delta_\perp J_0(b_\perp\Delta_\perp)\; H^u_{\pi'}(x,\xi,t;\zeta_H)|_{\xi=0}$$

The probability of finding the pion's u-quark (x>0) or d-antiquark (x<0) at a distance $b_\perp$ away from the CoTM peaks up at a small but non-zero value and at |x| near 1.

$$(|x|, b_\perp/r_\pi) = (0.91, 0.065)$$

This probability density at x=cte. peaks around a maximum at non-zero $b_\perp$; the larger is x, the smaller $b_\perp$ and the narrower the distribution.
The larger is the momentum fraction carried by the parton, the more it bears on the CoTM definition.





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GPDs from LFWFs

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$$u^{K}(x,b_\perp^2;\zeta_H) = \int_0^\infty \frac{d\Delta_\perp}{2\pi}\Delta_\perp J_0(b_\perp\Delta_\perp)\; H^u_K(x,\xi,t;\zeta_H)|_{\xi=0}$$

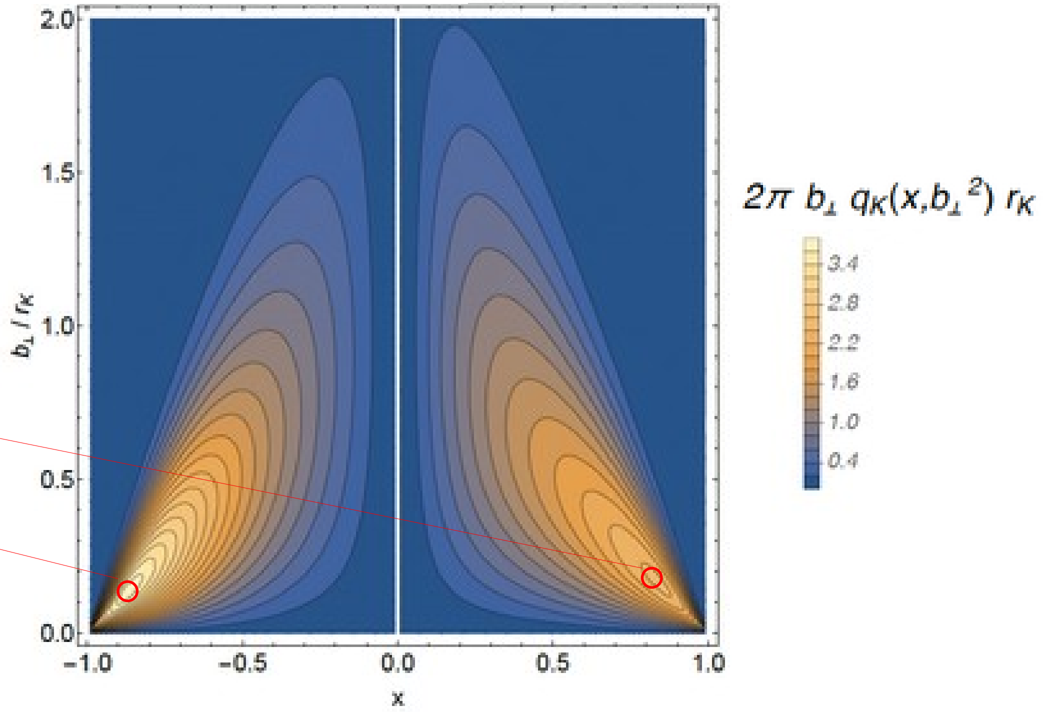
The flavor asymmetry is made manifest by the comparison of u-quark (x>0) and s-antiquark (x<0) probability densities: the heavier parton, carrying a larger momentum fraction, is more probably found close to the CoTM, to the definition of which it contributes more than the lighter.

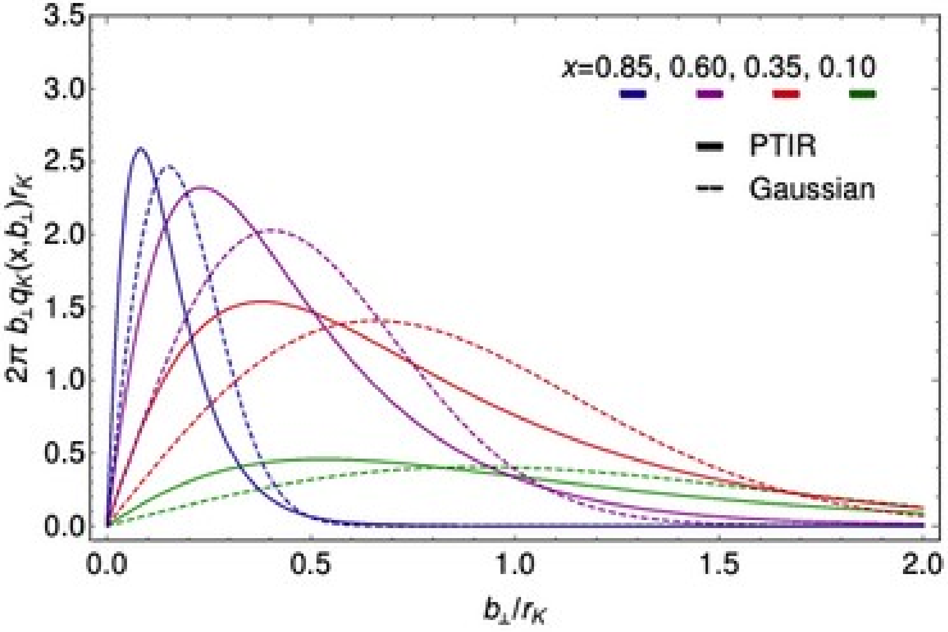
$$(|x|,b_\perp/r_K)=(0.84,0.17)$$

$$(|x|,b_\perp/r_K)=(0.87,0.13)$$

Same picture: qualitative and semi-quantitative agreement!

Gaussian LFWF





Factorized gaussian ansatz:

$$q^K(x,b_\perp^2;\zeta_H) = \frac{\gamma_K(\zeta_H)}{\pi r_K^2}\,\frac{q^K(|x|;\zeta_H)}{(1-|x|)^2}\exp\left(-\frac{\gamma_K(\zeta_H)}{(1-|x|)^2}\frac{b_\perp^2}{r_K^2}\right)$$

$$\gamma_K(\zeta_H) = \langle x^2\rangle_{\bar{s}}^{\zeta_H} + \frac{1+\delta}{2}\langle x^2\rangle_u^{\zeta_H} \qquad (q=u[x\geq 0],\, s[x\leq 0])$$

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Meson gravitational Form Factors

- **Gravitational** form factors connect with Energy-momentum tensor and are obtained from the **t-dependence** of the **GPD’s 1-st Mellin moment**:

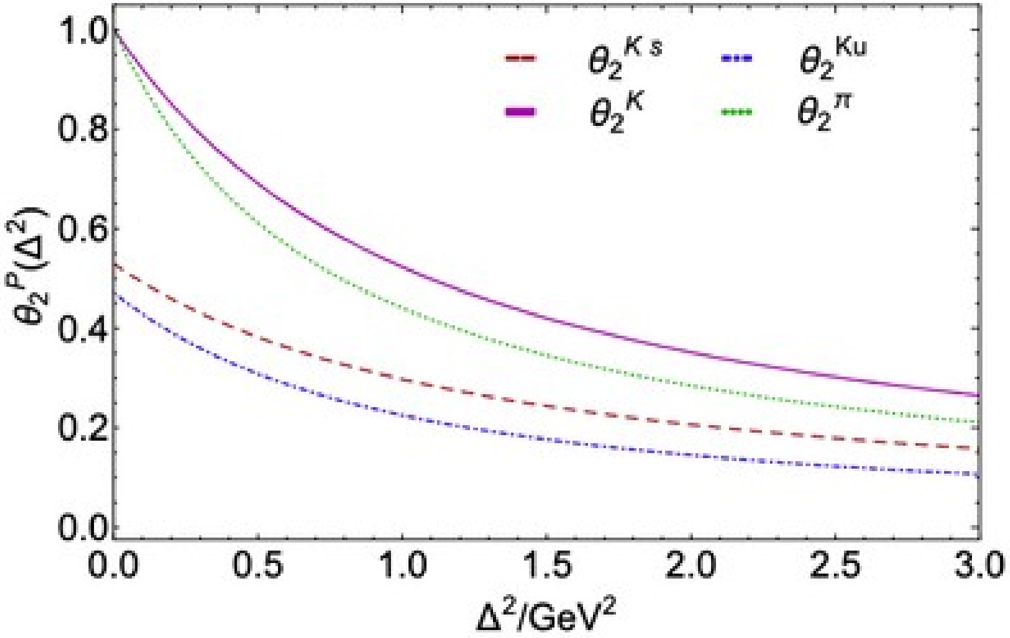
$$\theta^{\mathsf{M}}_{1,2}(-t) = \theta^{\mathsf{M}_u}_{1,2}(-t) + \theta^{\mathsf{M}_{\bar{h}}}_{1,2}(-t)$$

$$\int_{-1}^{1} dx x H^q_{\mathsf{M}}(x,\xi,t;\zeta_H) = \boxed{\theta_2^{\mathsf{M}_q}(-t)} - \xi^2\theta_1^{\mathsf{M}_q}(-t)$$

Owing to GPD’s polynomiality:

mass distribution

$$\int_{-1}^{1} dx x H^q_{\mathsf{M}}(x,0,t;\zeta_H) = \theta_2^{\mathsf{M}_q}(-t)$$



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Owing to GPD’s polynomiality:

mass distribution

$$\int_{-1}^{1} dx\, x H^q_{\mathrm{M}}(x,0,t;\zeta_H) = \theta_2^{\mathrm{M}_q}(-t)$$

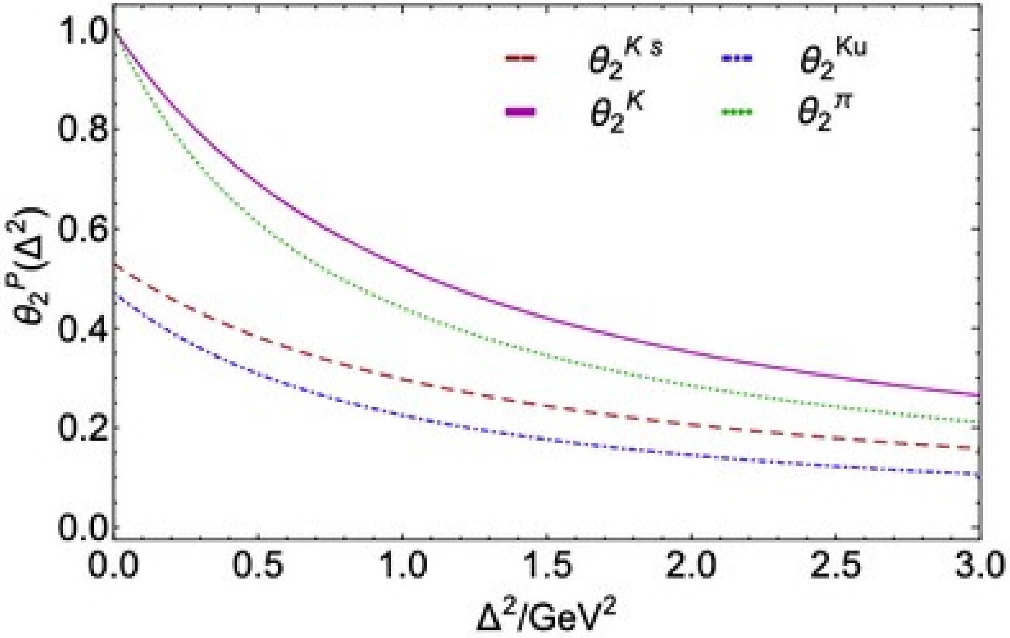
Define the mass-squared radius:

$$\left[r^{\theta_2}_{\mathrm{M}}\right]^2 = -\frac{6}{\theta_2^{\mathrm{M}}(0)}\frac{d}{d(-t)}\theta_2^{\mathrm{M}}(-t)$$

Pion case

$$= r_\pi^2\,\frac{2\langle x^2(1-x)\rangle^{\zeta_{\mathcal{H}}}_{\pi_u}}{\langle x^2\rangle^{\zeta_{\mathcal{H}}}_{\pi_u}} < r_\pi^2$$

$$\frac{2\langle x^2(1-x)\rangle^{\zeta_{\mathcal{H}}}_{\pi_u}}{\langle x^2\rangle^{\zeta_{\mathcal{H}}}_{\pi_u}} = \frac{1/2-\langle x^2\rangle^{\zeta_{\mathcal{H}}}_{\pi_u}}{\langle x^2\rangle^{\zeta_{\mathcal{H}}}_{\pi_u}} = \frac{\langle x(1-x)\rangle^{\zeta_{\mathcal{H}}}_{\pi_u}}{\langle x^2\rangle^{\zeta_{\mathcal{H}}}_{\pi_u}} < 1$$



Also true for the kaon:

$r^{\theta_2}_\pi/r_\pi$	$r^{\theta_2}_K/r_K$
0.81	0.78

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Owing to GPD’s polynomiality:

mass distribution

pressure distribution

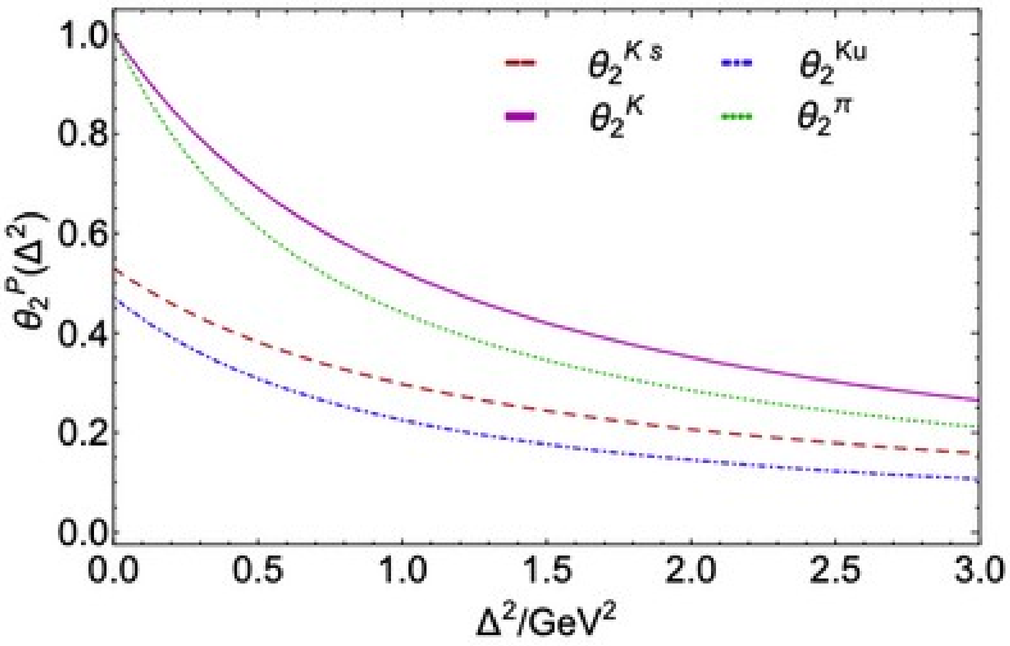
$$\int_{-1}^{1} dx x H^{q}_{\mathsf{M}}(x,0,t;\zeta_H) = \theta_2^{\mathsf{M}_q}(-t)$$

One needs both DGLAP ($|x| \geq \xi$) and ERBL ($|x| \leq \xi$) GPD to derive the pressure distribution.

ERBL completion →

Radon transform inversion [Cédric’s talk]

J-L. Zhang et al., arXiv:2101.12286



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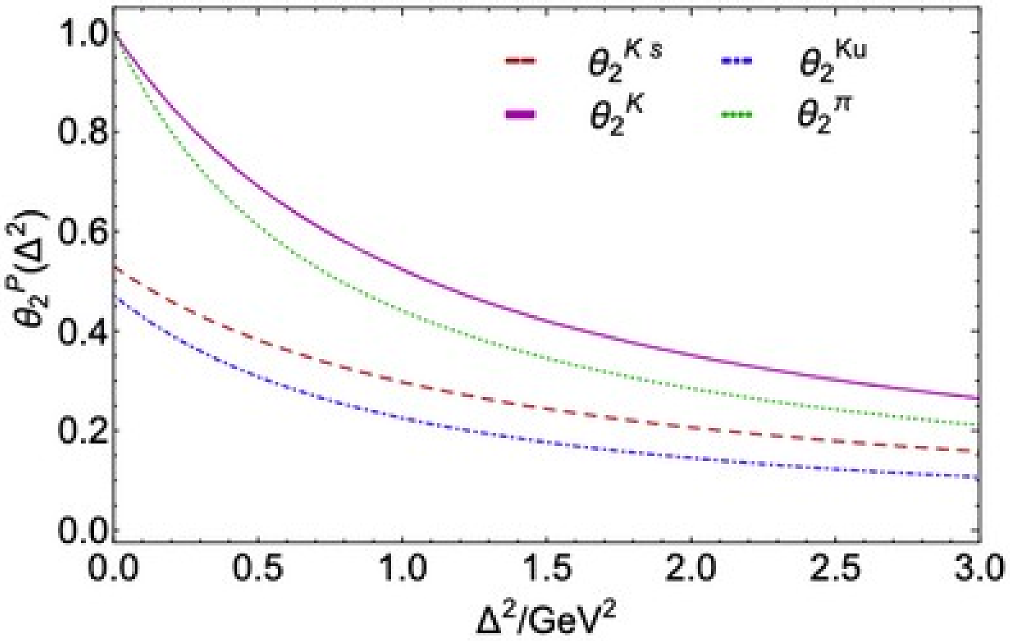

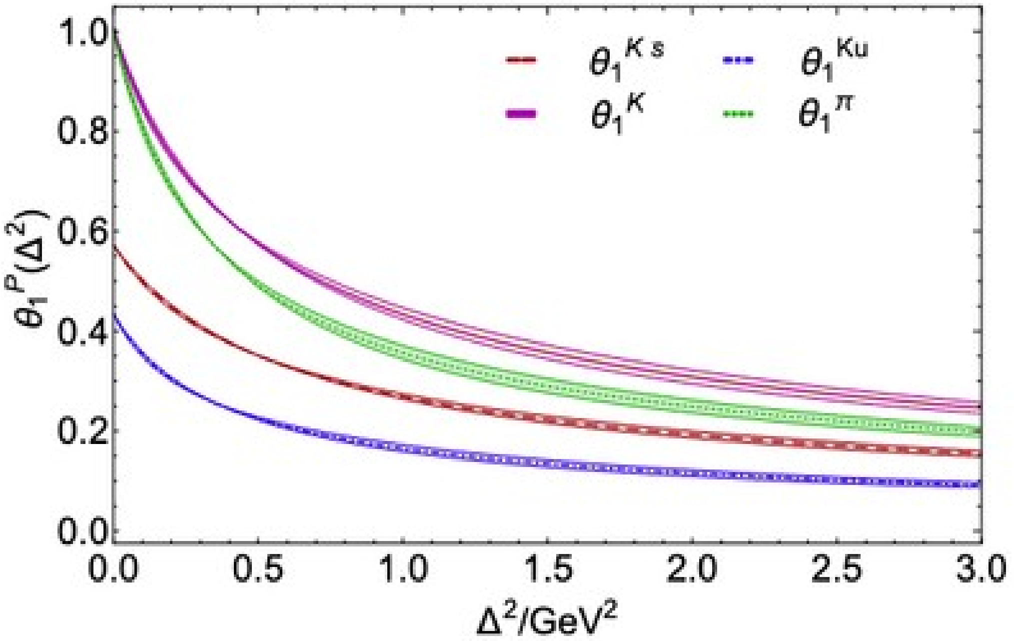

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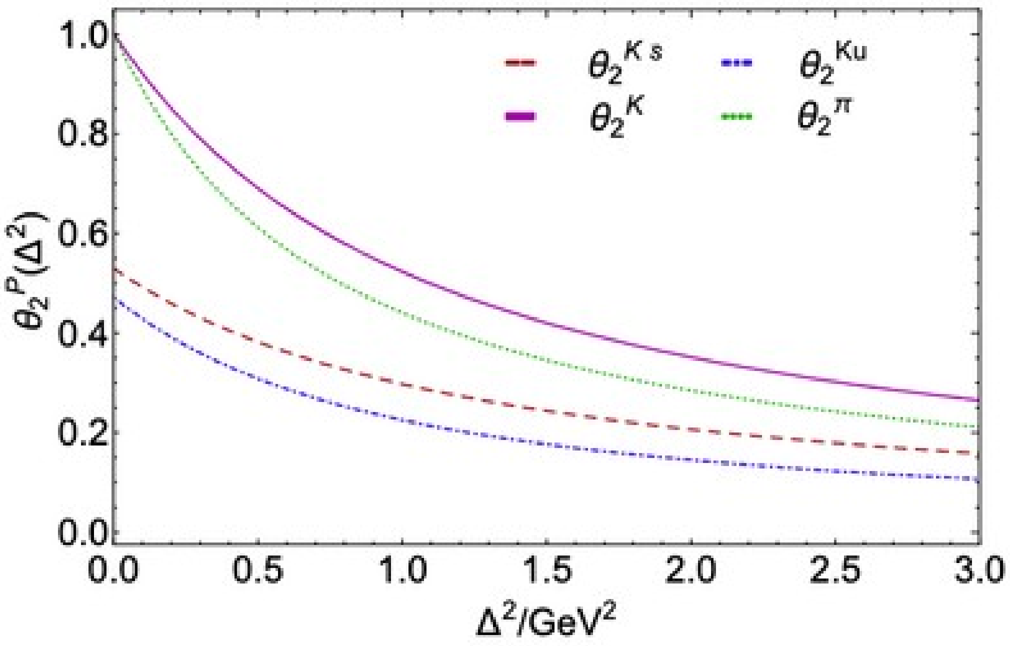
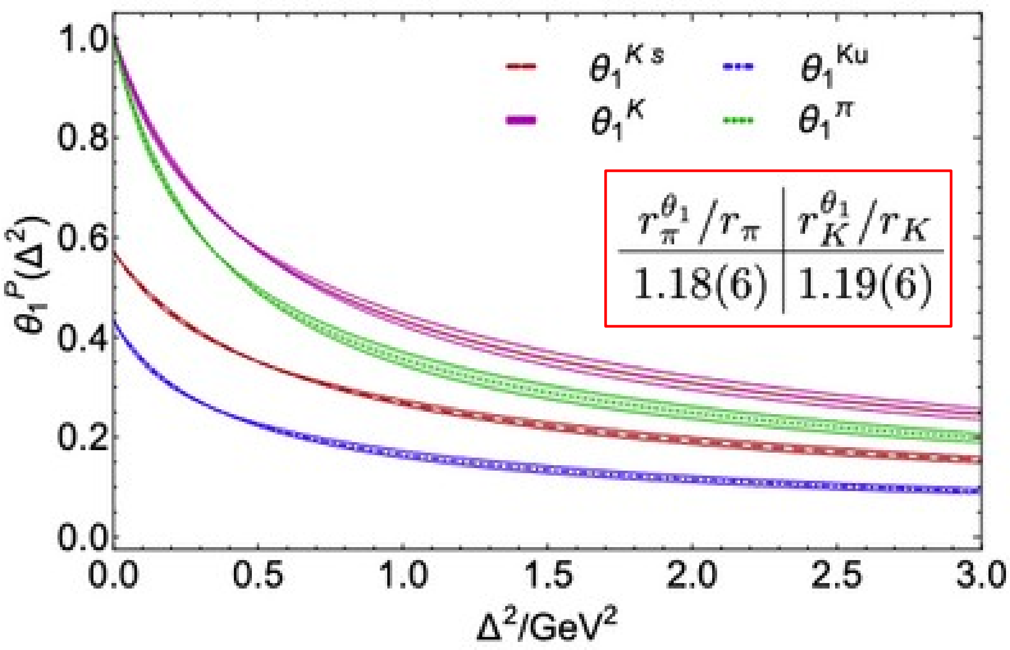


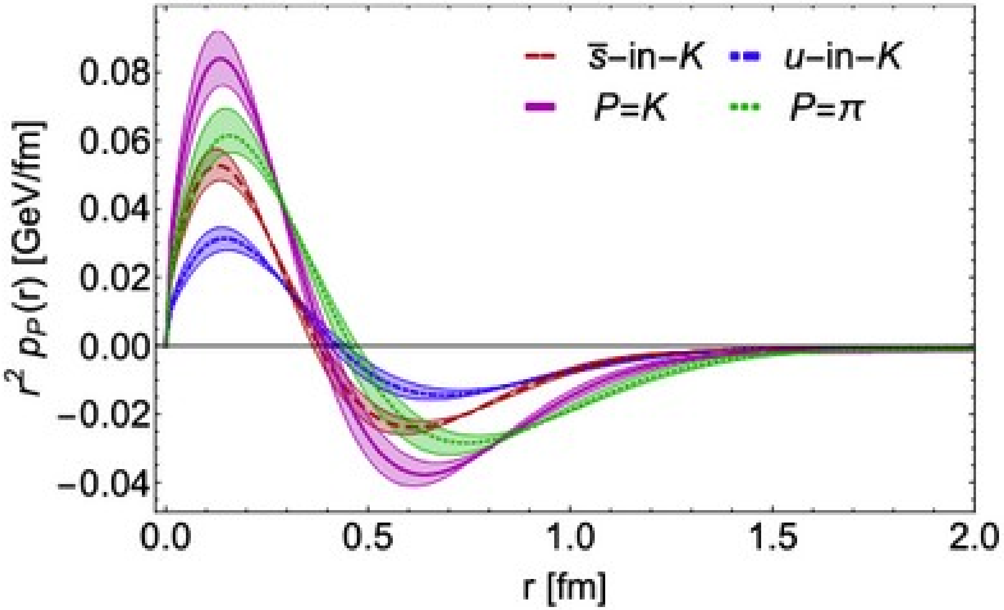


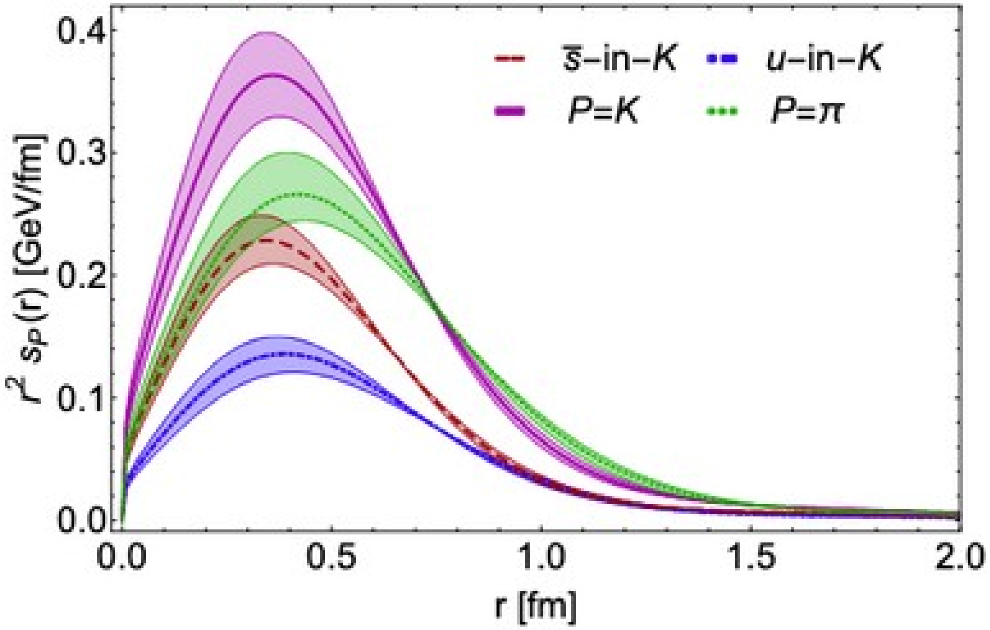
$r_\pi^{\theta_1}/r_\pi$	$r_K^{\theta_1}/r_K$
1.18(6)	1.19(6)

Meson gravitational Form Factors

Thus, the normal pressure can be sketched via the Fourier transform of the GFF:

$$p_\pi(r) = \frac{1}{6\pi^2 r}\int_0^\infty d\Delta \frac{\Delta}{2E(\Delta)} \sin(\Delta r)\left[\Delta^2\theta_1\left(\Delta^2\right)\right]$$





And a shear pressure as:

$$s_\pi(r) = \frac{3}{16\pi^2}\int_0^\infty d\Delta \frac{\Delta}{2E(\Delta)} \Delta j_2(\Delta r)\left[\Delta^2\theta_1\left(\Delta^2\right)\right]$$

Meson gravitational Form Factors

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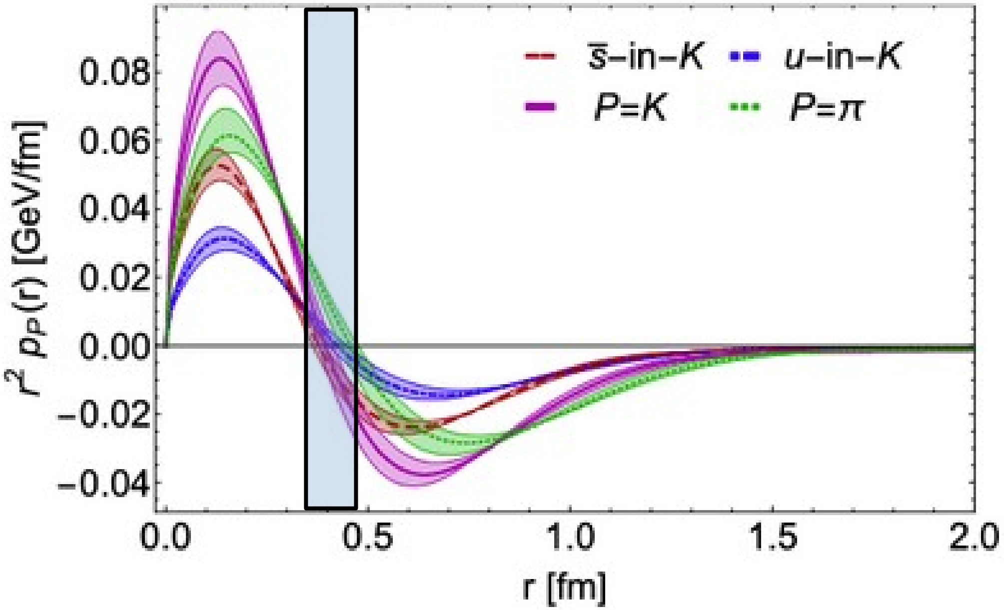
which displays a zero crossing lying around 0.5 fm for both pion and kaon, indicating where forces switch from “repulsive” (positive pressure) from “confining” (negative).



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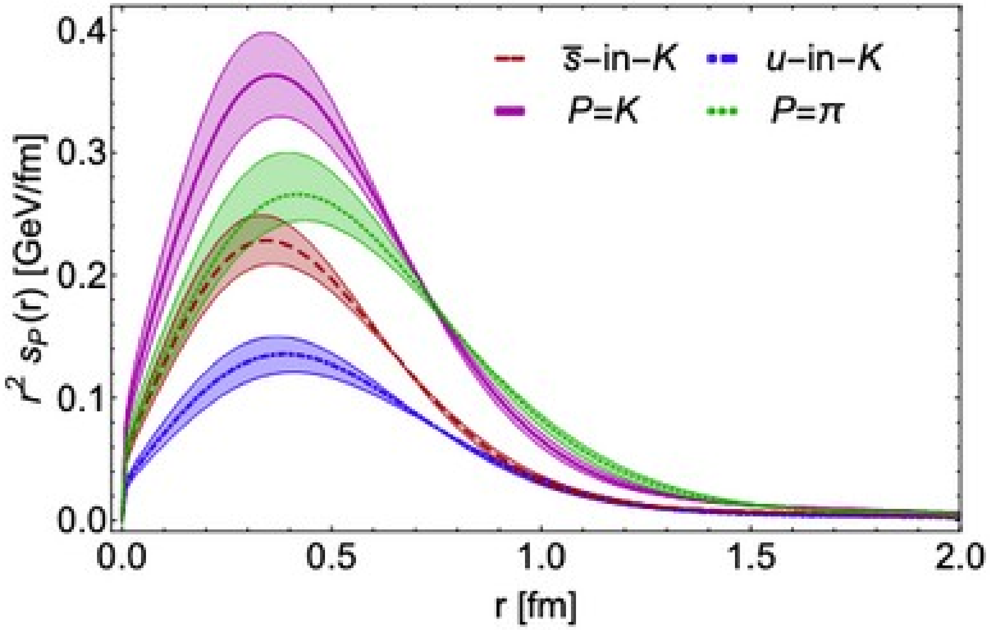

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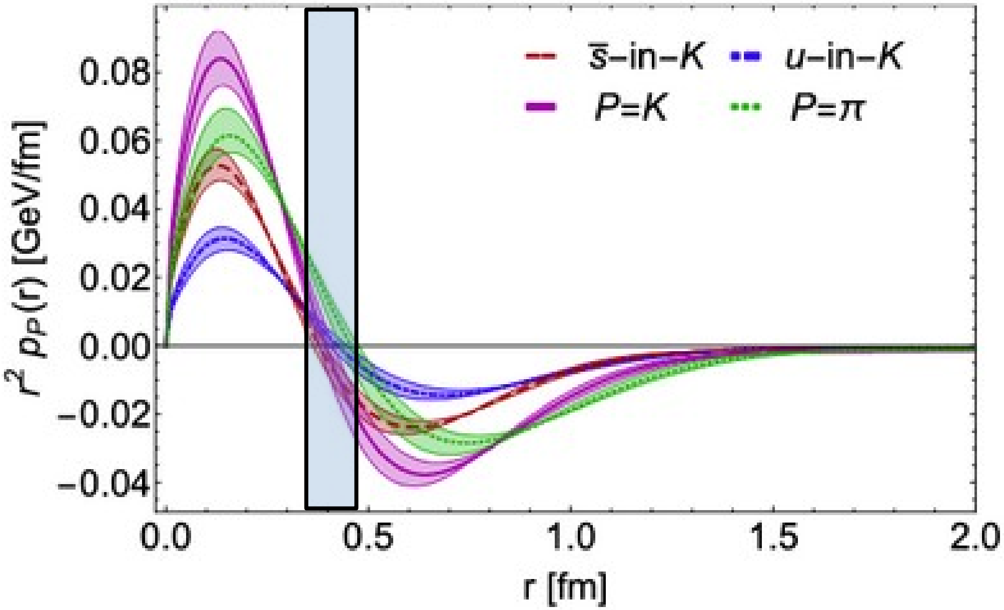

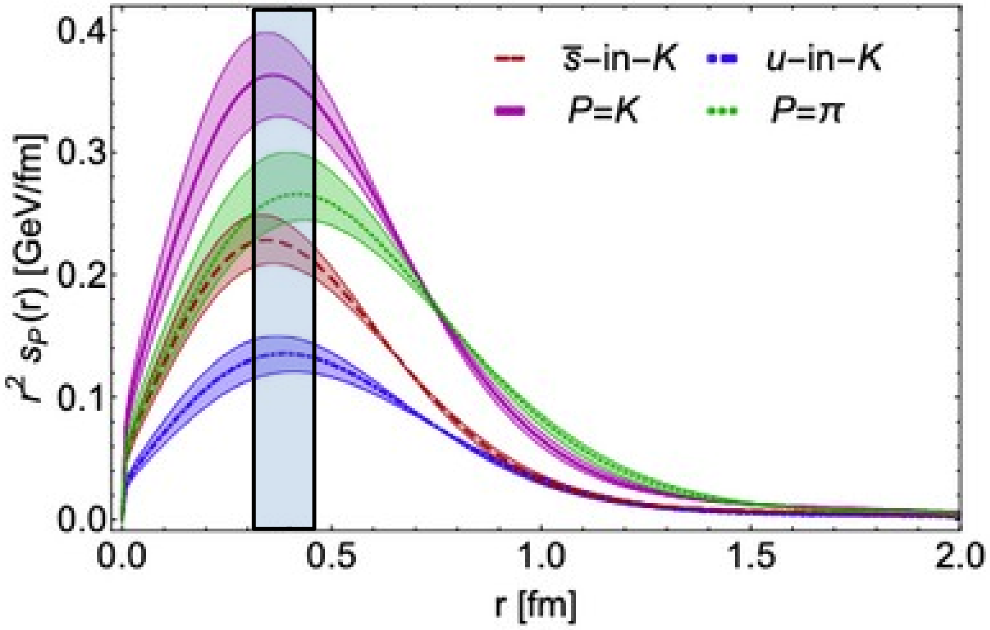

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that peaks up roughly where the normal pressure takes its zero, indicating that “repulsive” and “confining” forces maximally interfere with each other.

QCD evolution

DGLAP leading-order evolution of forward and non-skewed GPDs:

$$\left\{\zeta^2\frac{d}{d\zeta^2}\int_0^1 dy\,\delta(y-x) - \frac{\alpha(\zeta^2)}{4\pi}\int_x^1 \frac{dy}{y}\begin{pmatrix} P_{qq}^{\mathrm{NS}}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\mathrm{S}}\left(\frac{\mathbf{x}}{\mathbf{y}}\right)\end{pmatrix}\right\}\begin{pmatrix} H_\pi^{\mathrm{NS},+}(y,t;\zeta) \\ \mathbf{H}_\pi^{\mathrm{S}}(y,t;\zeta)\end{pmatrix} = 0$$

$$\mathbf{P}^{\mathrm{S}}\left(\frac{x}{y}\right) = \begin{pmatrix} P_{qq}^{\mathrm{S}}\left(\frac{x}{y}\right) & 2n_f P_{qg}^{\mathrm{S}}\left(\frac{x}{y}\right) \\ P_{gq}^{\mathrm{S}}\left(\frac{x}{y}\right) & P_{gg}^{\mathrm{S}}\left(\frac{x}{y}\right)\end{pmatrix}$$

$$\mathbf{H}_\pi^{\mathrm{S}}(y,t;\zeta) = \begin{pmatrix} H_\pi^{\mathrm{S},+}(y,t;\zeta) \\ \frac{1}{x}H_\pi^{g}(y,t;\zeta)\end{pmatrix}$$

QCD evolution

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$$\mathbf{H}_\pi^{\mathrm{S}}(y,t;\zeta) = \begin{pmatrix} H_\pi^{\mathrm{S},+}(y,t;\zeta) \\ \frac{1}{x}H_\pi^{g}(y,t;\zeta)\end{pmatrix}$$

Approach: a charge is defined such that the leading-order evolution kernel gives all-orders evolution.

QCD evolution

DGLAP ~~leading-order~~ evolution of forward and non-skewed GPDs:

Let’s illustrate with pion PDFs (forward limit)

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle_q^\zeta = -\frac{\alpha(\zeta^2)}{4\pi}\gamma_{qq}^n\langle x^n\rangle_q^\zeta$$

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle_S^\zeta = -\frac{\alpha(\zeta^2)}{4\pi}\left\{\gamma_{qq}^n\langle x^n\rangle_S^\zeta + 2n_f\gamma_{qg}^n\langle x^n\rangle_g^\zeta\right\}$$

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle_g^\zeta = -\frac{\alpha(\zeta^2)}{4\pi}\left\{\gamma_{gq}^n\langle x^n\rangle_S^\zeta + \gamma_{gg}^n\langle x^n\rangle_g^\zeta\right\}$$

Anomalous dimensions from splitting functions:

$$\gamma_{ij}^n = -\int_0^1 dz z^n P_{i\leftarrow j}(z)$$

Approach: a charge is defined such that the leading-order evolution kernel gives all-orders evolution.

QCD evolution

DGLAP ~~leading-order~~ evolution of forward and non-skewed GPDs:

Let’s illustrate with pion PDFs (forward limit)

$$\boxed{\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_q^\zeta = -\frac{\alpha(\zeta^2)}{4\pi} \gamma_{qq}^n \langle x^n \rangle_q^\zeta}$$

Non-singlet (valence-quark) sector

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_S^\zeta = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{qq}^n \langle x^n \rangle_S^\zeta + 2n_f \gamma_{qg}^n \langle x^n \rangle_g^\zeta \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_g^\zeta = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{gq}^n \langle x^n \rangle_S^\zeta + \gamma_{gg}^n \langle x^n \rangle_g^\zeta \right\}$$

Anomalous dimensions from splitting functions:

$$\gamma_{ij}^n = -\int_0^1 dz z^n P_{i\leftarrow j}(z)$$

Approach: a charge is defined such that the leading-order evolution kernel gives all-orders evolution.

ζ : “experimental” scale

$$\langle x^n \rangle_u^\zeta = \langle x^n \rangle_u^{\zeta_H} \left(\langle 2x \rangle_u^\zeta \right)^{9\gamma_0^n/32}$$

ζ_H : hadron scale

QCD evolution

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Let’s illustrate with pion PDFs (forward limit)

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_q^\zeta = -\frac{\alpha(\zeta^2)}{4\pi} \gamma_{qq}^n \langle x^n \rangle_q^\zeta$$

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Singlet (sea and glue) sector

Anomalous dimensions from splitting functions:

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Approach: a charge is defined such that the leading-order evolution kernel gives all-orders evolution.

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$$\langle x^n \rangle_u^\zeta = \langle x^n \rangle_u^{\zeta_H} \left(\langle 2x \rangle_u^\zeta \right)^{9\gamma_0^n/32}$$

$$\begin{pmatrix} \langle x^n \rangle_S^\zeta \\ \langle x^n \rangle_g^\zeta \end{pmatrix} = W_n \begin{pmatrix} [\langle 2x \rangle_u^\zeta]^{\lambda_+^n/\gamma_0^1} & 0 \\ 0 & [\langle 2x \rangle_u^\zeta]^{\lambda_-^n/\gamma_0^1} \end{pmatrix} W_n^{-1} \begin{pmatrix} \langle 2x^n \rangle_u^{\zeta_H} \\ 0 \end{pmatrix}$$

ζ_H : hadron scale

$$\begin{pmatrix} \gamma_{uu}^n & 2n_f \gamma_{ug}^n \\ \gamma_{gu}^n & \gamma_{gg} \end{pmatrix} = W_n \begin{pmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{pmatrix} W_n^{-1}$$

Modal matrix for the diagonalisation of the anomalous dimension array

QCD evolution

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Let's illustrate with pion PDFs (forward limit)

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All the information from the charge is encoded in the valence-quark momentum fraction at the experimental scale

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$$\begin{pmatrix} \gamma_{uu}^n & 2n_f \gamma_{ug}^n \\ \gamma_{gu}^n & \gamma_{gg} \end{pmatrix} = W_n \begin{pmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{pmatrix} W_n^{-1}$$

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$$\begin{pmatrix}\langle x^n\rangle_S^\zeta \\ \langle x^n\rangle_g^\zeta\end{pmatrix} = W_n \begin{pmatrix}\boxed{[\langle 2x\rangle_u^\zeta]}^{\lambda_+^n/\gamma_0^1} & 0 \\ 0 & \boxed{[\langle 2x\rangle_u^\zeta]}^{\lambda_-^n/\gamma_0^1}\end{pmatrix} W_n^{-1}\begin{pmatrix}\boxed{\langle 2x^n\rangle_u^{\zeta_H}} \\ 0\end{pmatrix}$$

The inputs are the valence-quark Mellin moments from CSF and GPD modeling at the hadron scale.

ζ_H : hadron scale

$$\begin{pmatrix}\gamma_{uu}^n & 2n_f\gamma_{ug}^n \\ \gamma_{gu}^n & \gamma_{gg}\end{pmatrix} = W_n\begin{pmatrix}\lambda_+^n & 0 \\ 0 & \lambda_-^n\end{pmatrix}W_n^{-1}$$

Modal matrix for the diagonalisation of the anomalous dimension array

QCD evolution

DGLAP ~~leading-order~~ evolution of forward and non-skewed GPDs:

Let’s illustrate with pion PDFs (forward limit)

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_q^\zeta = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle_q^\zeta$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_S^\zeta = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle_S^\zeta + 2n_f \gamma^n_{qg} \langle x^n \rangle_g^\zeta \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_g^\zeta = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gq} \langle x^n \rangle_S^\zeta + \gamma^n_{gg} \langle x^n \rangle_g^\zeta \right\}$$

Anomalous dimensions from splitting functions:

$$\gamma^n_{ij} = -\int_0^1 dz z^n P_{i\leftarrow j}(z)$$

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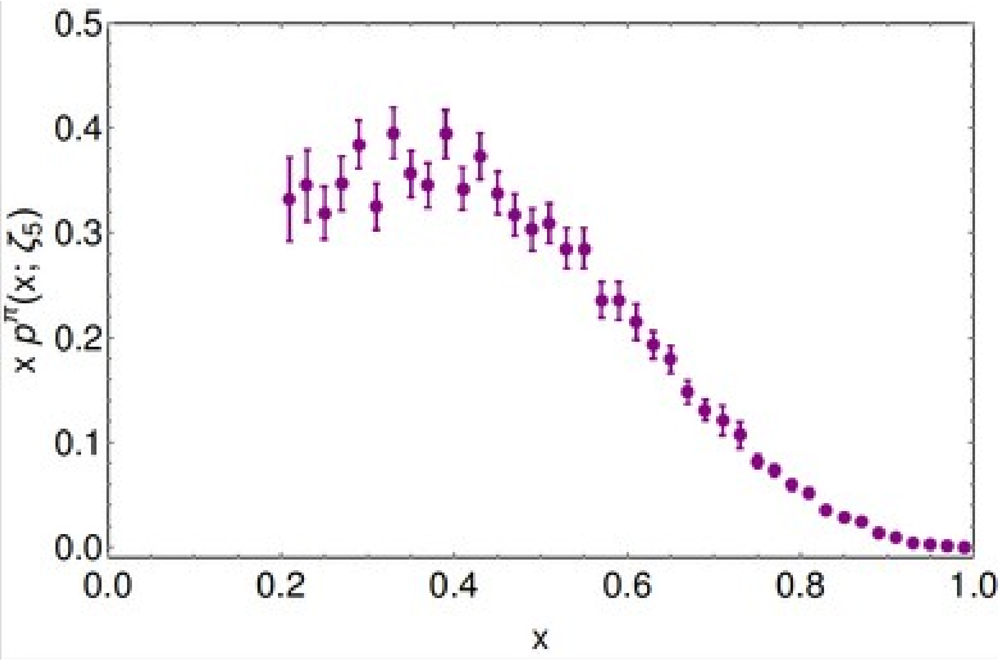
Data from Aicher et al. reanalysis of E615 exp., [PRL 105(2010)2502003]



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QCD evolution

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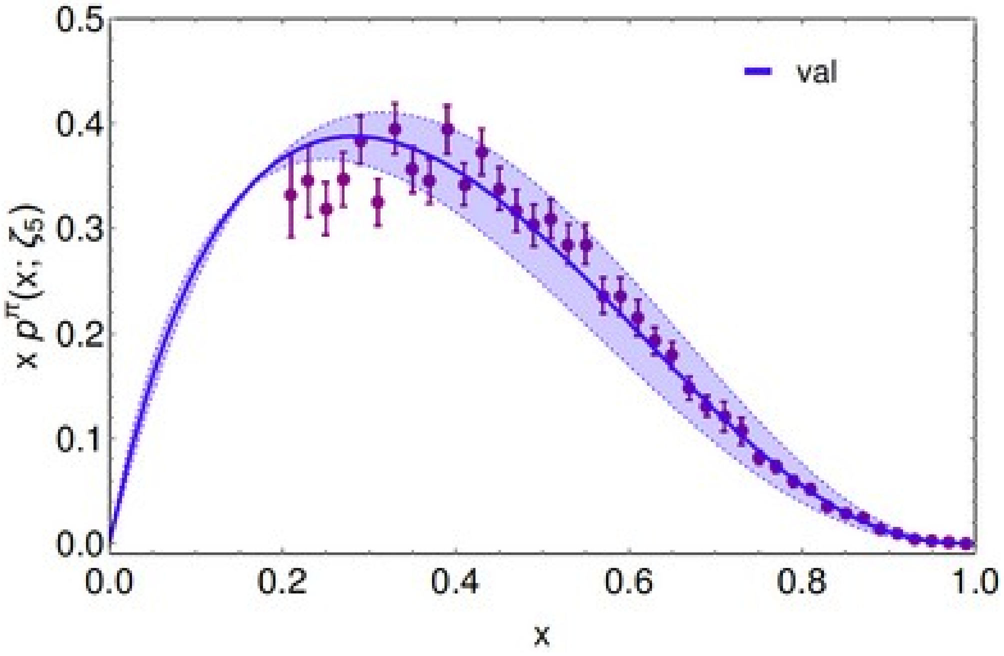
Data from Aicher et al. reanalysis of E615 exp., [PRL 105(2010)2502003]

Evolution results with $\langle 2x\rangle_u^\zeta = 0.42(4)$ from leading-logarithm, NLO pQD fit to Drell-Yan pion data [$\zeta = 5.2$ GeV]



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QCD evolution

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Anomalous dimensions from splitting functions:

$$\gamma_{ij}^n = -\int_0^1 dz z^n P_{i\leftarrow j}(z)$$

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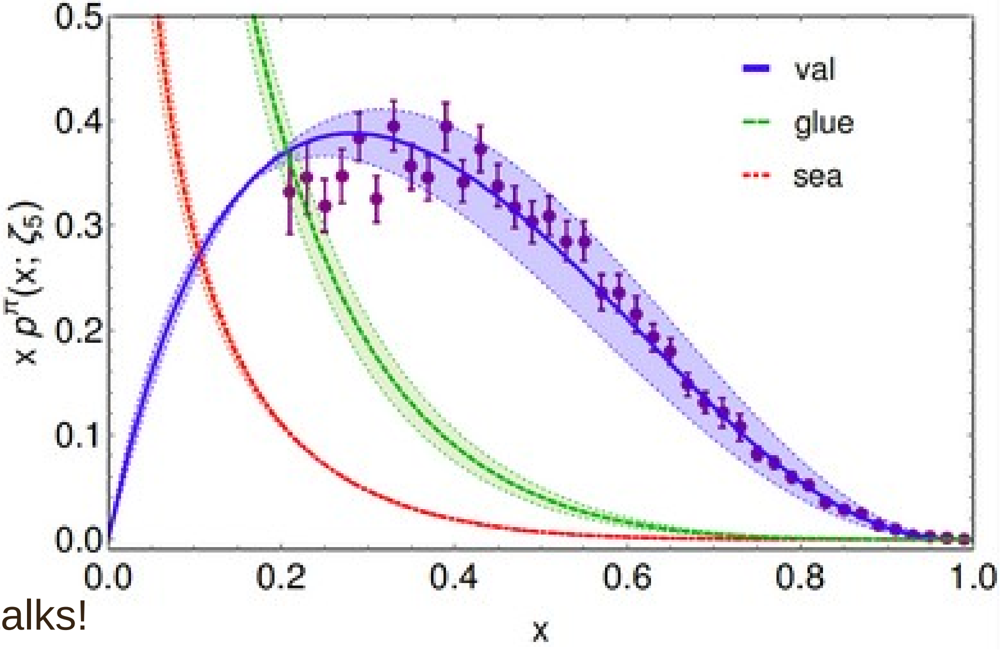
Sea and glue pion PDFs, correspondingly obtained.



ζ : “experimental” scale

ζ_H : hadron scale

c.f. K. Raya’s and L. Chang’s talks!

QCD evolution

DGLAP ~~leading-order~~ evolution of forward and non-skewed GPDs:

Let’s illustrate with pion PDFs (forward limit)

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^\zeta_q = -\frac{\alpha(\zeta^2)}{4\pi}\gamma^n_{qq}\langle x^n\rangle^\zeta_q$$

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^\zeta_S = -\frac{\alpha(\zeta^2)}{4\pi}\left\{\gamma^n_{qq}\langle x^n\rangle^\zeta_S + 2n_f\gamma^n\langle x^n\rangle^\zeta\right\}$$

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^\zeta$$

Anomalous dimensions from splitting functions:

$$\int^1 dz z^n P_{i\leftarrow j}(z)$$

Reminding K. Raya’s talk!

PI Effective charge also predicts:

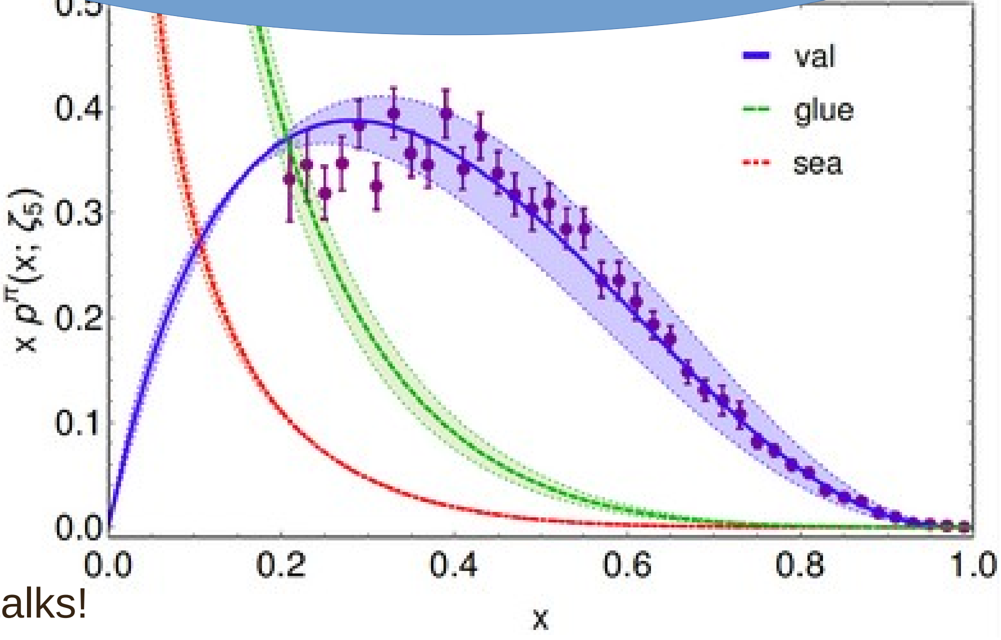
$$\langle 2x\rangle^\zeta_u = \exp\left[\frac{\gamma^1_{qq}}{2\pi}\int_{\zeta_H}^{\zeta}\frac{dy}{y}\alpha(y)\right] = 0.40(4)$$

Approach: a charge is all-orders evolution.

Data from Aicher et al. reanalysis of E615 exp., [PRL 105(2010)2502003]

Evolution results with $\langle 2x\rangle^\zeta_u = 0.42(4)$ from leading-logarithm, NLO pQD fit to Drell-Yan pion data [$\zeta = 5.2$ GeV]

Sea and glue pion PDFs, correspondingly obtained.



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QCD evolution

DGLAP ~~leading-order~~ evolution of forward and non-skewed GPDs:

Let’s illustrate with pion PDFs (forward limit)

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle_q^\zeta = -\frac{\alpha(\zeta^2)}{4\pi}\gamma_{qq}^n\langle x^n\rangle_q^\zeta$$

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$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle_g^\zeta = -\frac{\alpha(\zeta^2)}{4\pi}\left\{\gamma_{gq}^n\langle x^n\rangle_S^\zeta + \gamma_{gg}^n\langle x^n\rangle_g^\zeta\right\}$$

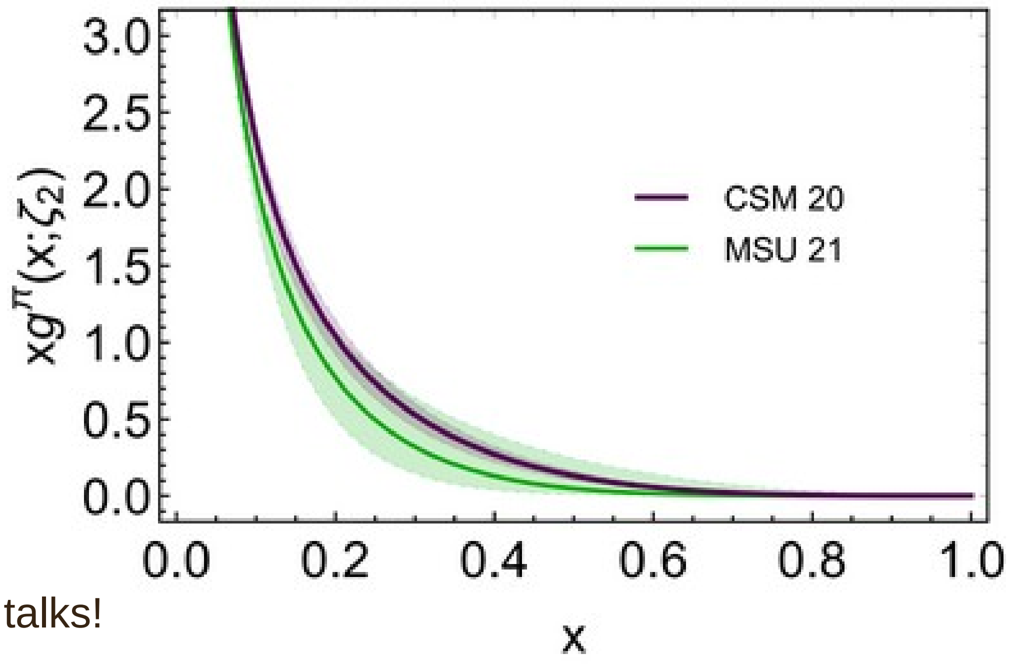
Anomalous dimensions from splitting functions:

$$\gamma_{ij}^n = -\int_0^1 dz z^n P_{i\leftarrow j}(z)$$

Approach: a charge is defined such that the leading-order evolution kernel gives all-orders evolution.

Focus on glue DF and compare with recent lattice MSU results: [Z. Fan and H-W. Lin, arXiv:2104.06372]

Evolution results with $\langle 2x\rangle_u^\zeta = 0.50(5)$ from leading-logarithm, NLO pQD fit to Drell-Yan pion data [$\zeta = 2.0$ GeV]



ζ : “experimental” scale

ζ_H : hadron scale

c.f. K. Raya’s and L. Chang’s talks!

QCD evolution

DGLAP ~~leading-order~~ evolution of forward and non-skewed GPDs:

Let's illustrate with pion PDFs (forward limit)

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_q^\zeta = -\frac{\alpha(\zeta^2)}{4\pi} \gamma_{qq}^n \langle x^n \rangle_q^\zeta$$

$$\boxed{\begin{aligned} \zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_S^\zeta &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{qq}^n \langle x^n \rangle_S^\zeta + 2n_f \gamma_{qg}^n \langle x^n \rangle_g^\zeta \right\} \\ \zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_g^\zeta &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{gq}^n \langle x^n \rangle_S^\zeta + \gamma_{gg}^n \langle x^n \rangle_g^\zeta \right\} \end{aligned}}$$

Singlet (sea and glue) sector

Anomalous dimensions from splitting functions:

$$\gamma_{ij}^n = -\int_0^1 dz z^n P_{i \leftarrow j}(z)$$

The results can be readily extended to non-skewed GPDs.

i.e., particularizing for the first moment:

$$2\theta_2^{\pi\text{val}}(\Delta^2;\zeta) + \theta_2^{\pi\text{sea}}(\Delta^2;\zeta) = 2\theta_2^{\pi\text{val}}(\Delta^2;\zeta_{\mathcal{H}}) \left[\tfrac{3}{7} + \tfrac{4}{7} (\langle 2x \rangle_u^\zeta)^{\frac{7}{4}} \right]$$

$$\theta_2^{\pi\text{g}}(\Delta^2;\zeta) = \tfrac{4}{7} 2\theta_2^{\pi\text{val}}(\Delta^2;\zeta_{\mathcal{H}}) \left[1 - (\langle 2x \rangle_u^\zeta)^{\frac{7}{4}} \right]$$

ζ : "experimental" scale

ζ_H : hadron scale

$$\lambda_+^1 = \frac{56}{9};\ \lambda_-^1 = 0, W_1 = \begin{pmatrix} 1 & \frac{3}{4} \\ -1 & 1 \end{pmatrix}.$$

QCD evolution

DGLAP ~~leading-order~~ evolution of forward and non-skewed GPDs:

Let’s illustrate with pion PDFs (forward limit)

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_q^\zeta = - \frac{\alpha(\zeta^2)}{4\pi} \gamma_{qq}^n \langle x^n \rangle_q^\zeta$$

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$+$

$$\boxed{2\theta_2^{\pi \mathrm{val}}(\Delta^2;\zeta) + \theta_2^{\pi \mathrm{sea}}(\Delta^2;\zeta) + \theta_2^{\pi \mathrm{g}}(\Delta^2;\zeta) = 2\theta_2^{\pi \mathrm{val}}(\Delta^2;\zeta_{\mathcal{H}}) = \theta_2^{\pi}(\Delta^2)}$$

ζ_H : hadron scale

$$\lambda_+^1 = \tfrac{56}{9};\ \lambda_-^1 = 0,\ W_1 = \begin{pmatrix} 1 & \frac{3}{4} \\ -1 & 1 \end{pmatrix}.$$

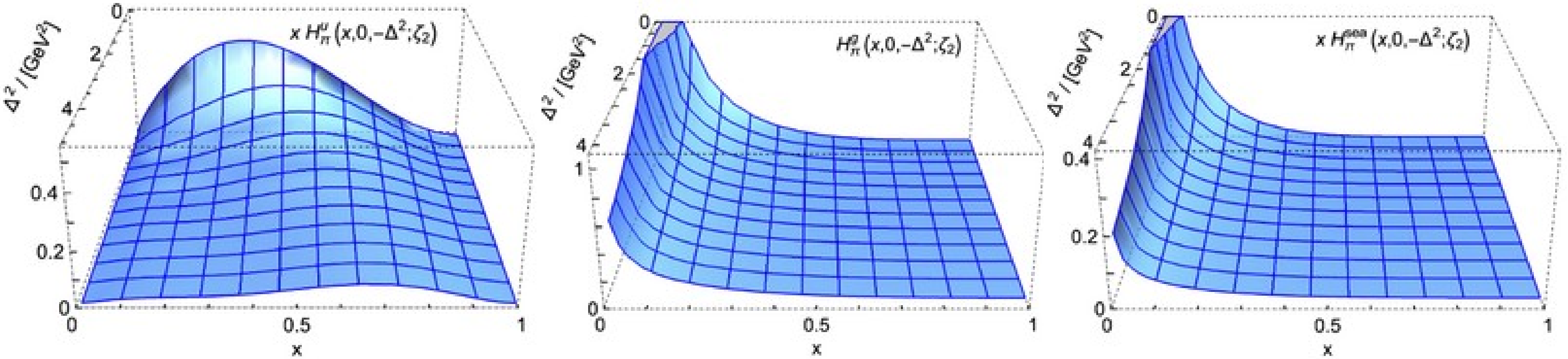
Momentum conservation implies scale invariance of mass-squared distributions (as should be!).

QCD evolution

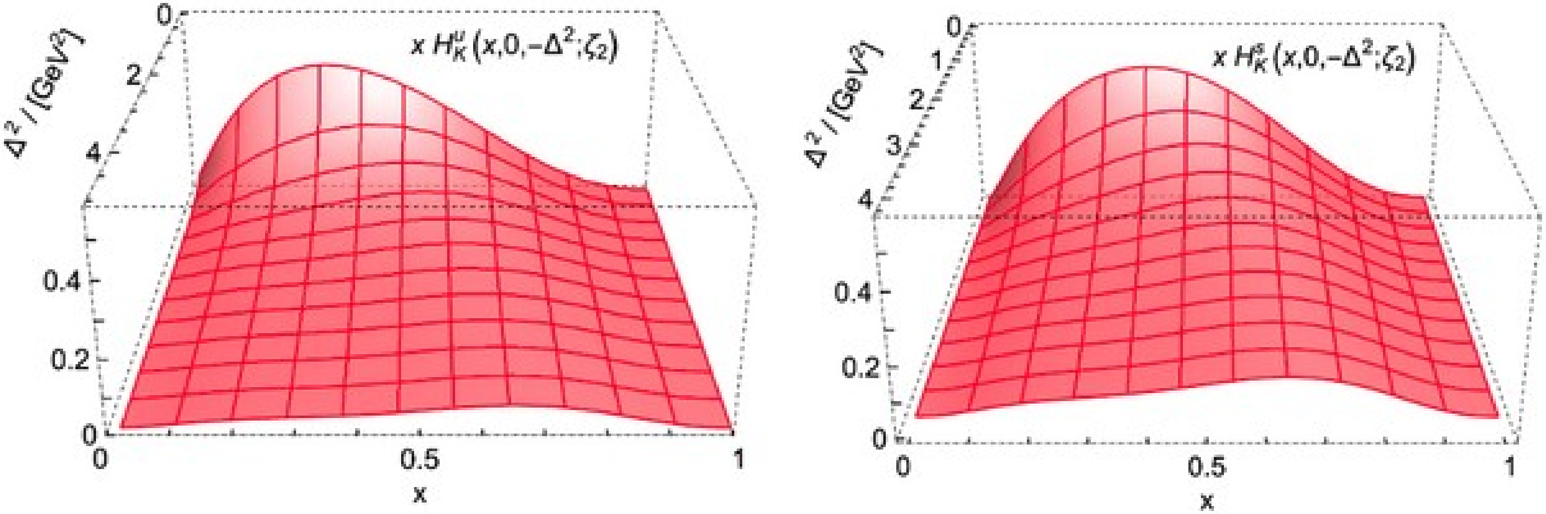
DGLAP ~~leading-order~~ evolution of forward and non-skewed GPDs.

Reconstruction of DFs from their evolved Mellin moments:

Pion



Kaon



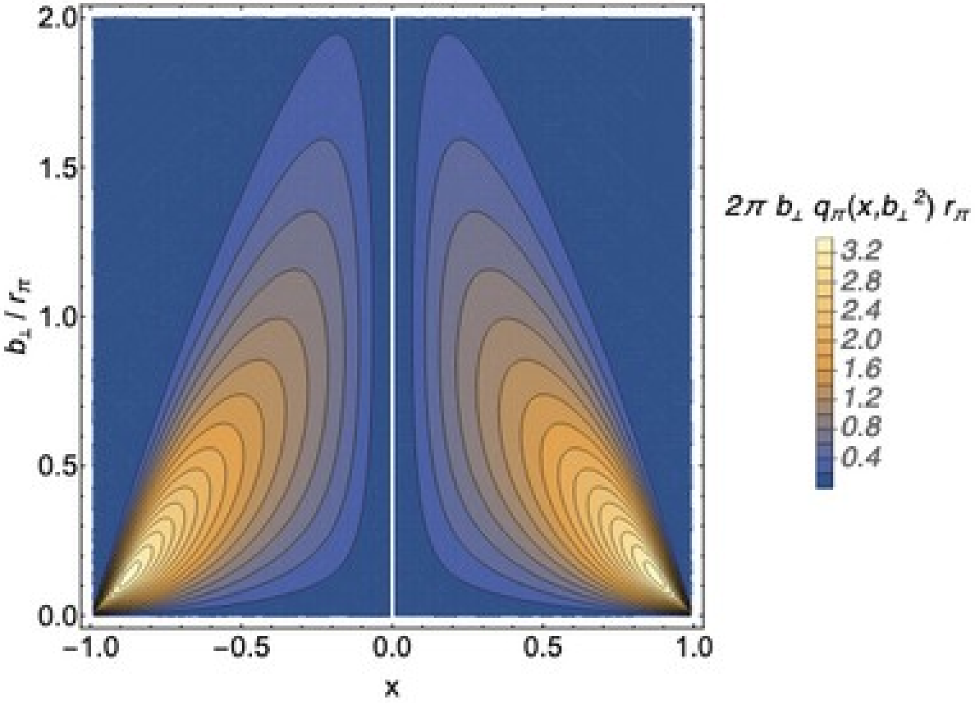
QCD evolution

Pion IPS GPD:

$$u^M(x, b_\perp^2; \zeta_H) = \int_0^\infty \frac{d\Delta_\perp}{2\pi} \Delta_\perp J_0(b_\perp \Delta_\perp)\ H_M^u(x,\xi,t;\zeta_H)|_{\xi=0}$$

The probability of finding the pion’s u-quark (x>0) or d-antiquark (x<0) at a distance $b_\perp$ away from the CoTM peaks up at a small but non-zero value and at |x| near 1.

$$\boxed{(|x|, b_\perp/r_\pi)=(0.91,0.065)}$$



QCD evolution

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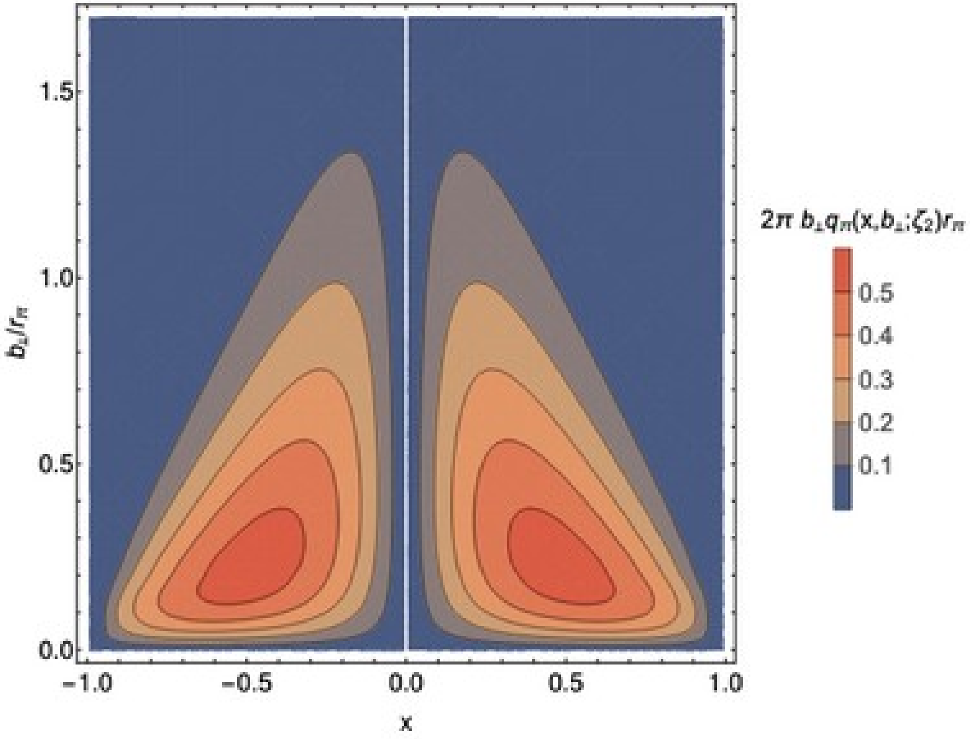
$$\boxed{(|x|, b_\perp/r_\pi)=(0.53, 0.21)}$$

The peaks clearly broaden, the maximum clearly decreases and drifts towards lower values of the momentum fraction, implying that the dressed quasi-particles share the momentum with the “interacting cloud”, losing identity!



QCD evolution

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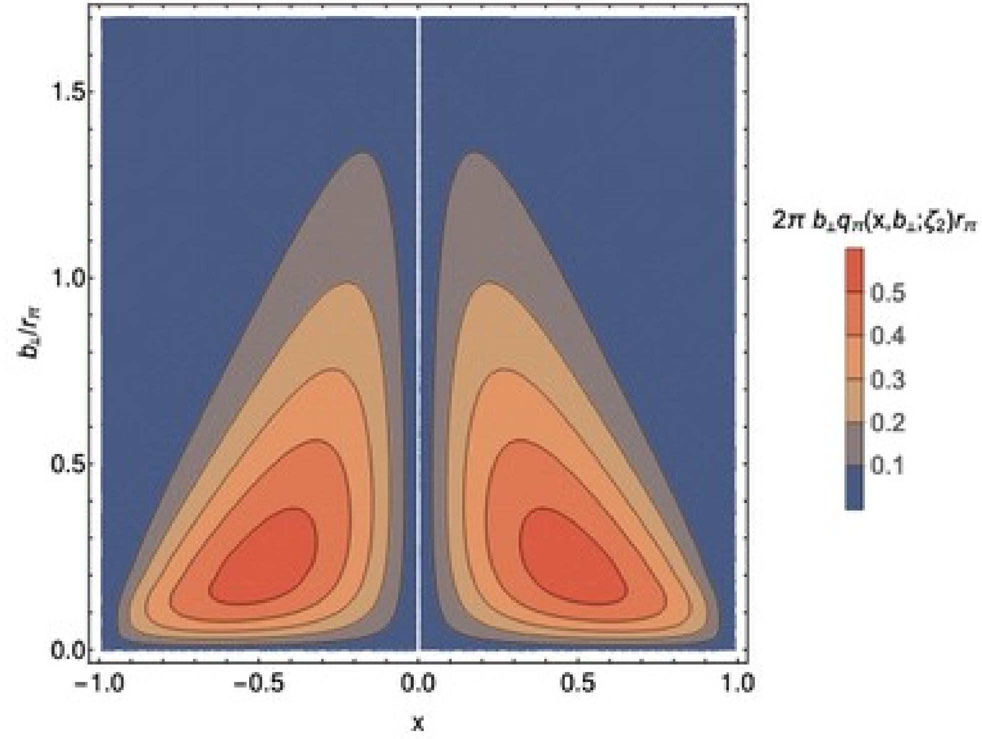
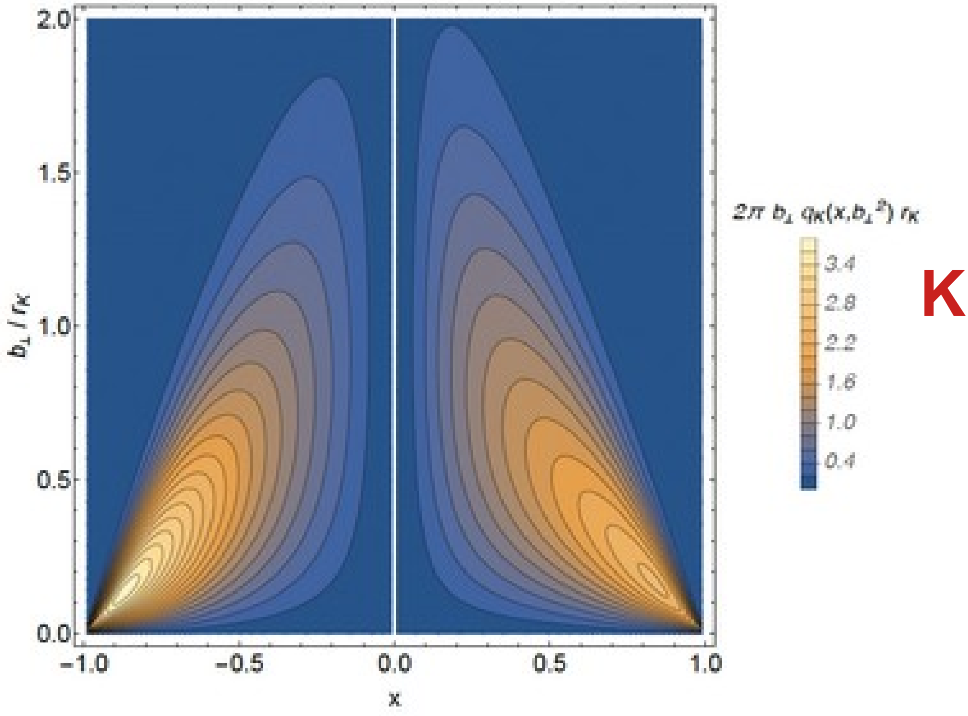


Kaon IPS GPD:

$$u^K(x,b_\perp^2;\zeta_H) = \int_0^\infty \frac{d\Delta_\perp}{2\pi}\Delta_\perp J_0(b_\perp\Delta_\perp)\; H_K^u(x,\xi,t;\zeta_H)|_{\xi=0}$$



QCD evolution

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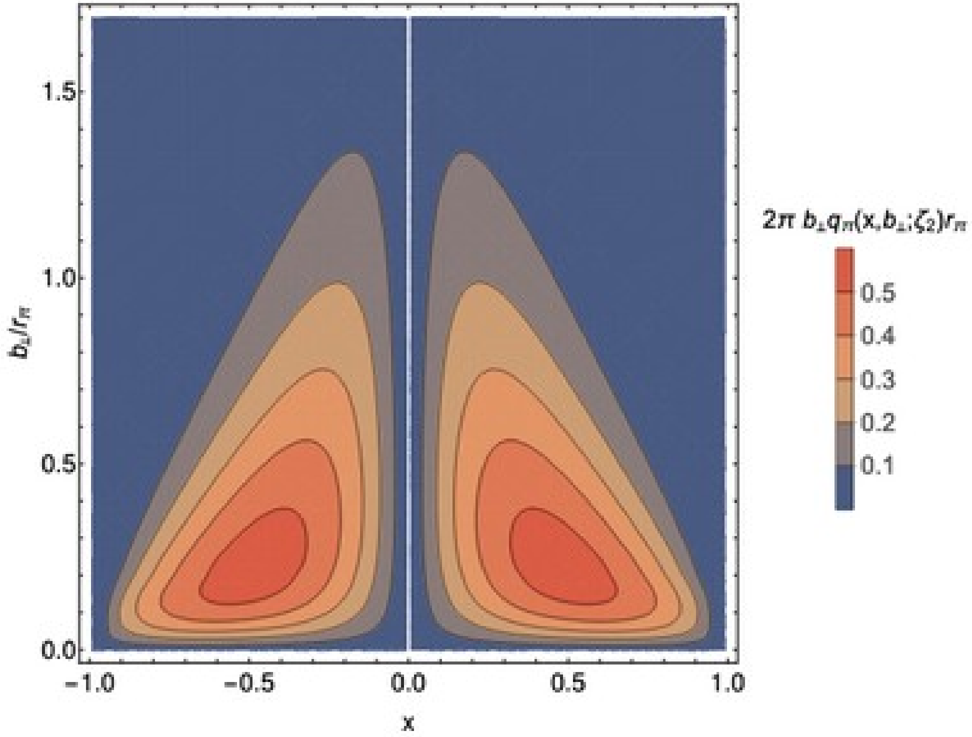
$$u^M(x, b_\perp^2; \zeta_H) = \int_0^\infty \frac{d\Delta_\perp}{2\pi} \Delta_\perp J_0(b_\perp \Delta_\perp)\; H_M^u(x,\xi,t;\zeta_H)|_{\xi=0}$$

The probability of finding the pion’s u-quark (x>0) or d-antiquark (x<0) at a distance $b_\perp$ away from the CoTM peaks up at a small but non-zero value and at |x| near 1.

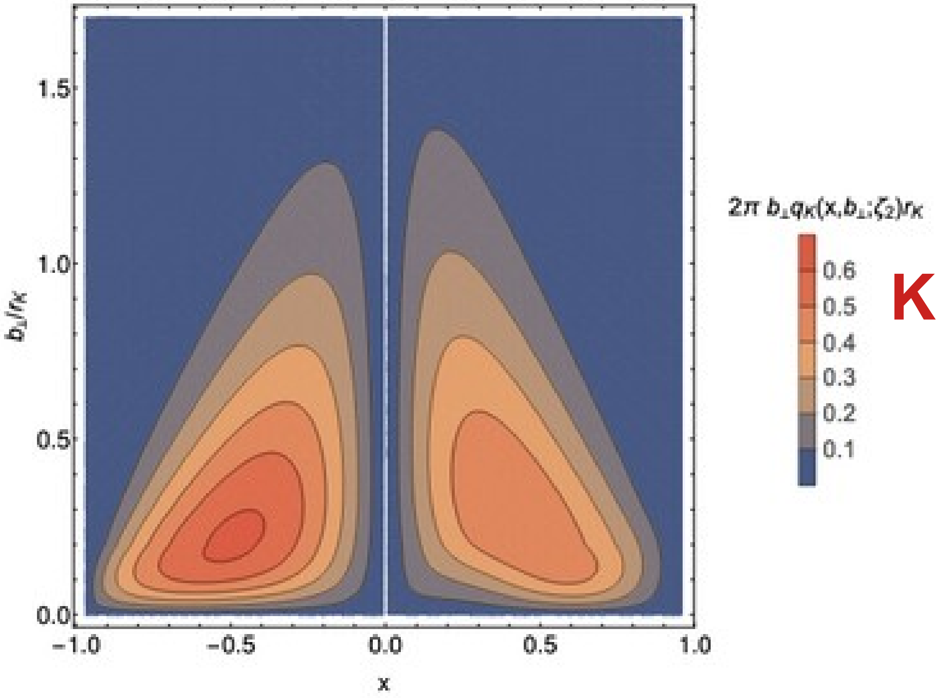
$$\boxed{(|x|, b_\perp/r_\pi) = (0.53, 0.21)}$$

The peaks clearly broaden, the maximum clearly decreases and drifts towards lower values of the momentum fraction, implying that the dressed quasi-particles share the momentum with the “interacting cloud”, losing identity!



Kaon IPS GPD:

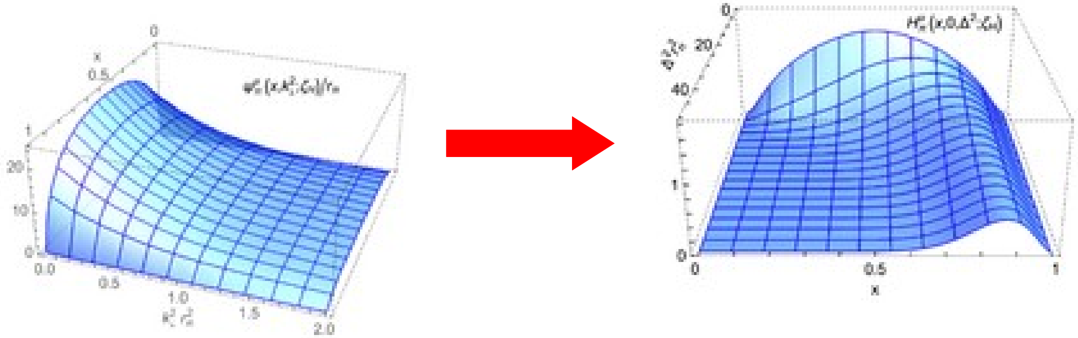
$$u^K(x, b_\perp^2; \zeta_H) = \int_0^\infty \frac{d\Delta_\perp}{2\pi} \Delta_\perp J_0(b_\perp \Delta_\perp)\; H_K^u(x,\xi,t;\zeta_H)|_{\xi=0}$$



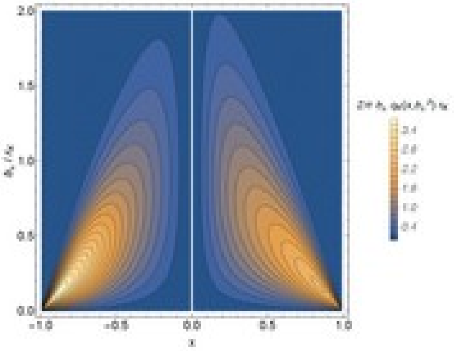
Epilogue

Pion and kaon Bethe-Salpeter wave functions have been modeled, with the help of either factorization approximation or PTIR representation, on the ground of a realistic DSE estimate of PDAs and used to obtain LFWFs.

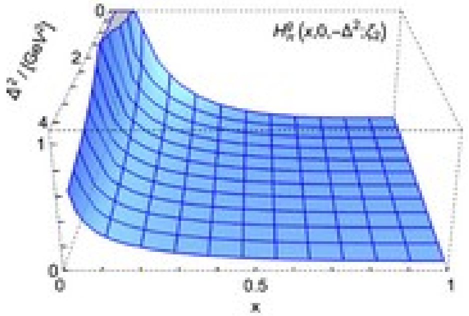
Pion and kaon GPDs are then estimated, within the DGLAP kinematic domain, from the overlap representation of their LFWFs



Electric FFs, IPS GPDs and GFFs have been also obtained and displayed, thus featuring the spatial distribution of the partons inside the mesons.



Mass and pressure distributions have been also derived and, from them, mass-squared radius that can be compared to electric charge one.



All-orders evolution have been then applied and either with an empirical input or invoking the PI charge, we accounted for reanalysed E615 data and delivered sea-quark and glue Dfs.

Prospect: owing to mass-dependent corrections for the evolution kernels, evaluate flavor-separated sea-quark DFs and contents.