Form Factors of the Nucleon Axial and Pseudoscalar Currents

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Emergence of Hadron Mass through AMBER@CERN - VI

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Quantum Chromodynamics in its non-perturbative regime

Emergence

Low-level rules producing high-level phenomena with enormous apparent complexity

Start from the QCD Lagrangian:

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}G_a^{\mu\nu}G^a_{\mu\nu} + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \partial^\mu \bar{c}^a \partial_\mu c^a + g\, f^{abc}(\partial^\mu \bar{c}^a) A^b_\mu c^c \,.$$

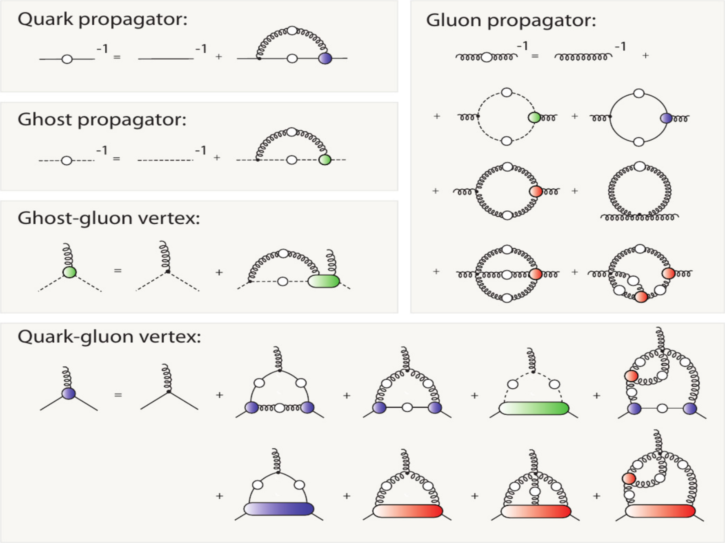
Lattice-regularized QCD, Continuum Schwinger-function methods, ...

And obtain:

- Dynamical generation of fundamental mass scale in pure Yang-Mills (gluon mass).
- Quark constituent masses and dynamical chiral symmetry breaking.
- Bound state formation: mesons, baryons, glueballs, hybrids, multiquark systems...
- Signals of confinement.

These (emergent) phenomena is not apparent in the QCD Lagrangian; however, they characterized the nonperturbative regime of QCD where hadrons live

Emergent phenomena *could be associated with dramatic, dynamically driven changes in the analytic structure of QCD's Schwinger functions, which are solutions of the DSEs*



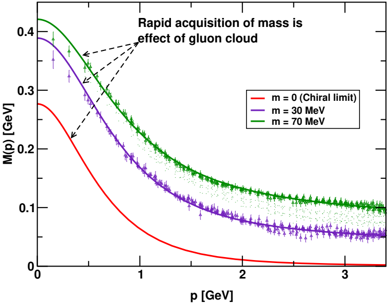
Non-perturbative QCD:
Dynamical generation of quark mass

☞ Dressed-quark propagator in Landau gauge:

$$S^{-1}(p) = Z_2(i\gamma\cdot p + m) + \Sigma(p) = \left(\frac{Z(p^2)}{i\gamma\cdot p + \mathrm{M}(\mathrm{p}^2)}\right)^{-1}$$

- Mass generated from the interaction of quarks with the gluon-medium.
- Light quarks acquire a HUGE constituent mass.
- Responsible of the 98% of proton's mass, the large splitting between parity partners, . . .

M.S. Bhagwat *et al.*, Phys.Rev. C68 (2003) 015203.



☞ Goldberger-Treiman relation at the quark level:

Quark propagator: $S^{-1}(p) = i\gamma\cdot p\,A(p^2) + B(p^2)\,,$

Pion's BS-amplitude: $\Gamma_\pi(p,P) \propto \gamma^5 E_\pi(p;P)\,.$

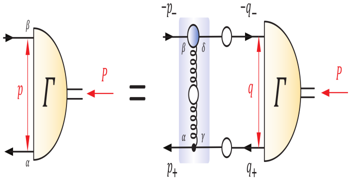
$$\mathrm{f}_\pi \mathrm{E}_\pi(\mathrm{p};0) = \mathrm{B}(\mathrm{p}^2)$$

Properties of the massless pion are a direct measure of the dressed-quark mass function

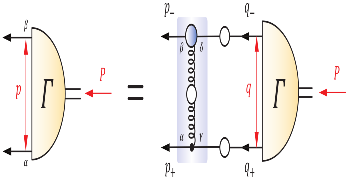
Cleanest expression of the mechanism that is responsible for almost all the visible mass in the universe

Diquark correlations

Any interaction able to create Goldstone modes as bound-states of light dressed-quark and -antiquark will generate strong $\bar{3}_c$ correlations between any two dressed quarks.



Meson BSE



Diquark BSE

☞ Owing to properties of charge-conjugation, a diquark with spin-parity J^P may be viewed as a partner to the analogous J^{-P} meson:

$$\Gamma_{q\bar{q}}(p;P) = - \int \frac{d^4q}{(2\pi)^4}\, g^2 D_{\mu\nu}(p-q)\frac{\lambda^a}{2}\gamma_\mu\, S(q+P)\Gamma_{q\bar{q}}(q;P)S(q)\,\frac{\lambda^a}{2}\,\gamma_\nu$$

$$\Gamma_{qq}(p;P)C^\dagger = -\frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\, g^2 D_{\mu\nu}(p-q)\frac{\lambda^a}{2}\gamma_\mu\, S(q+P)\Gamma_{qq}(q;P)C^\dagger S(q)\,\frac{\lambda^a}{2}\,\gamma_\nu$$

☞ Whilst no pole-mass exists, the following mass-scales express the strength and range of the correlation:

$$m_{[ud]_{0^+}} = 0.7-0.8\,\text{GeV}, \quad m_{\{uu\}_{1^+}} = 0.9-1.1\,\text{GeV}, \quad m_{\{dd\}_{1^+}} = m_{\{ud\}_{1^+}} = m_{\{uu\}_{1^+}}$$

☞ Diquark correlations are soft, they possess an electromagnetic size:

$$r_{[ud]_{0^+}} \gtrsim r_\pi, \qquad r_{\{uu\}_{1^+}} \gtrsim r_\rho, \qquad r_{\{uu\}_{1^+}} > r_{[ud]_{0^+}}$$

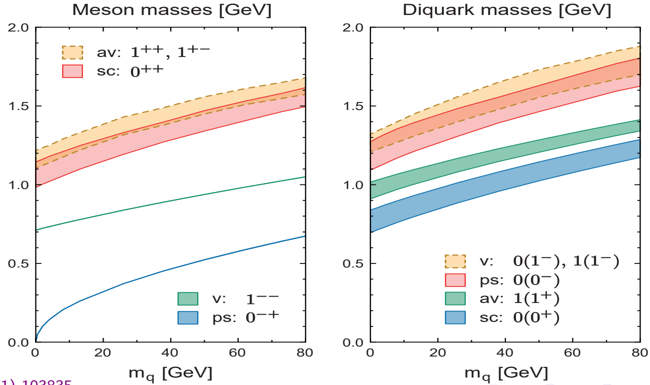
Diquark species

Octet and decuplet baryons

	[nn]	{nn}	[ns]	{ns}	{ss}
N	●	●			
Δ		●			
Λ	●		●	●	
Σ		●	●	●	
Ξ			●	●	●
Ω					●

Other baryons as parity partners

- ☞ $[I = 0, J^P = 0^+]$: Isoscalar-scalar.
- ☞ $[I = 1, J^P = 1^+]$: Isovector-pseudovector.
- ☞ $[I = 0, J^P = 0^-]$: Isoscalar-pseudoscalar.
- ☞ $[I = 0, J^P = 1^-]$: Isoscalar-vector.
- ☞ $[I = 1, J^P = 1^-]$: Isovector-vector.



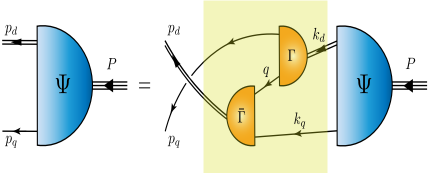
M. Yu. Barabanov *et al.*,

Prog. Part. Nucl. Phys. 116 (2021) 103835

The quark+diquark structure of a baryon

☞ A baryon can be viewed as a **Borromean bound-state**, the binding within which has two contributions:

- Formation of tight diquark correlations.
- Quark exchange depicted in the shaded area.



☞ The exchange ensures that diquark correlations within the baryon are fully dynamical: no quark holds a special place.

☞ The rearrangement of the quarks guarantees that the baryon's wave function complies with Pauli statistics.

☞ The number of states in the spectrum of baryons obtained is similar to that found in the three-constituent quark model, just as it is in today's LQCD calculations.

☞ Modern diquarks are different from the old static, point-like diquarks which featured in early attempts to explain the so-called missing resonance problem.

☞ Modern diquarks enforce certain distinct interaction patterns for the singly- and doubly-represented valence-quarks within the baryon.

S.-S. Xu *et al.*, Phys. Rev. D92 (2015) 114034; Y. Lu *et al.*, Phys. Rev. C96 (2017) 015208; C. Chen *et al.*, Phys. Rev. D100 (2019) 054009; P.-L. Yin *et al.*, Phys. Rev. D100 (2019) 034008.

Consequence of solving Poincaré-covariant bound-state equations

non-relativistic

relativistic

Mesons: $P=(-1)^{L+1}$

S	L	J^{PC}
0	0	0^{-+}
1	0	1^{--}
0	1	1^{+-}
1	1	0^{++}

$\cancel{P=(-1)^{L+1}}$

Bethe,Salpeter, Llewelyn-Smith 1950ies

$$\begin{aligned}\Gamma_\pi(P,p)=\gamma_5\,[&F_1(P,p) &&\text{s-wave}\\ &+F_2(P,p)\,i\slashed{P}\\ &+F_3(P,p)\,pP\,i\slashed{p} &&\text{p-wave}\\ &+F_4(P,p)\,[\slashed{p}\,,\slashed{P}]]\end{aligned}$$

Baryons: $P=(-1)^{L}$

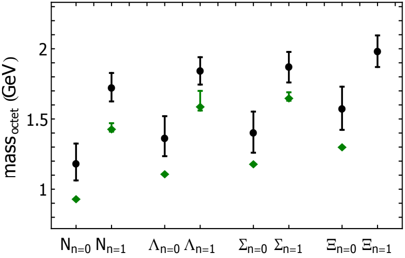
S	L	J^P
$1/2$	0	$1/2^+$
$3/2$	2	

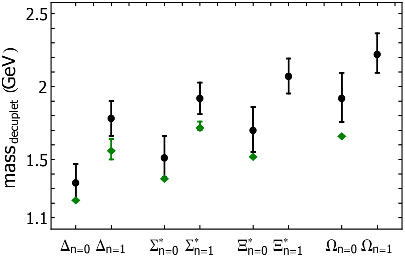
$\cancel{P=(-1)^{L}}$

J^P	total	s-wave	p-wave	d-wave	f-wave
$1/2^+$	64	8	36	20	
$3/2^+$	128	4	36	60	28

Masses of the octet and decuplet

- ☞ The computed masses are uniformly larger than the corresponding empirical values.
- ☞ The quark-diquark kernel omits all resonant contributions associated with meson-baryon final state interactions, which typically generate a measurable reduction.
- ☞ The Faddeev equations analyzed to produce the results should be understood as producing the dressed-quark core of the bound state, not the completely dressed and hence observable object.

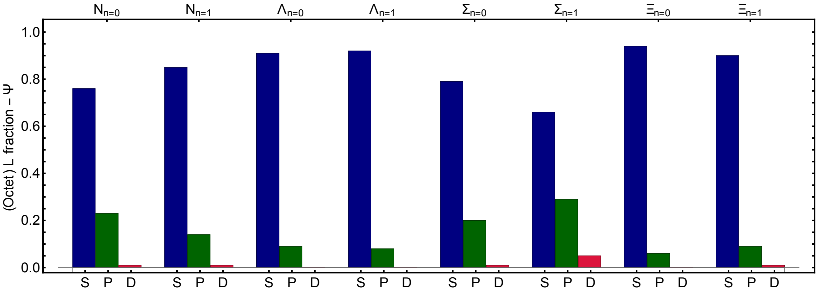


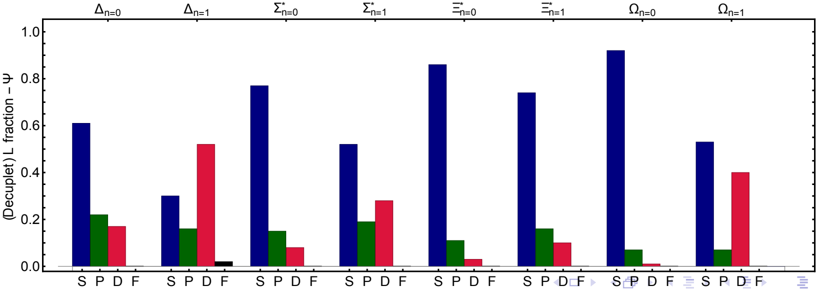


C. Chen *et al.*, Phys. Rev. D100 (2019) 054009.

Angular momenta of the octet and decuplet

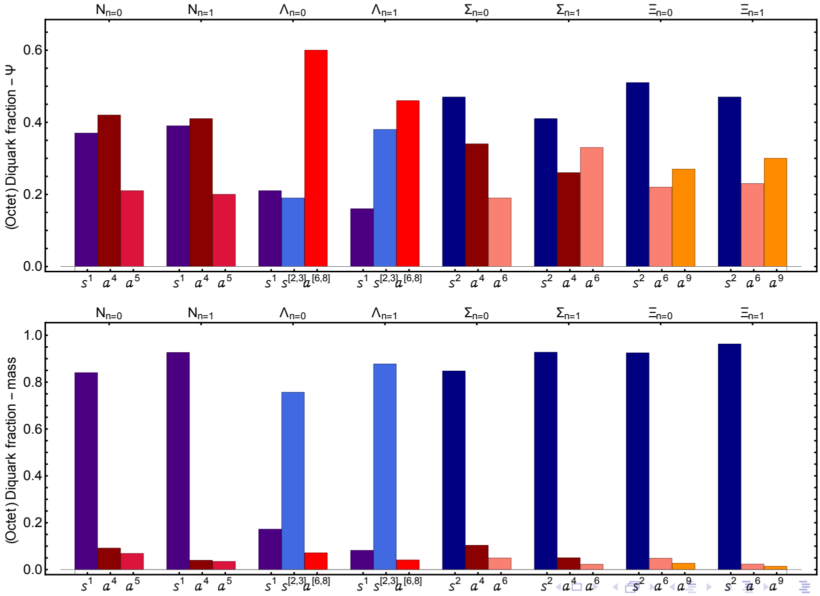
The P- and D-wave components play a measurable role in octet and decuplet baryons





Diquark content of the octet and decuplet

Only axial-vector diquarks are present in the decuplet baryons. For the octet case...



Understanding the structure of the nucleon

☞ The proton and neutron, collectively known as nucleons, lie at the heart of any nucleus and their properties are determined by the complex nature of the strong interaction.

☞ A central aim of ongoing experimental and theoretical efforts is the understanding of the structure of the nucleon because it is a composite object.

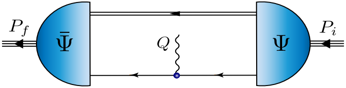
☞ The electron scattering off nucleons, mostly protons, is a well developed experimental technique and it has delivered, for instance, precise measurements of the EM FFs

☞ While the EM FFs are relatively well determined, less constrained are the axial and induced-pseudoscalar form factors derived from the isovector axial-vector current.

☞ These nucleon axial form factors are important quantities for the understanding of weak interactions, neutrino-nucleus scattering and parity violation experiments.

Computing form factors in the quark-diquark picture

Contain important information about the structure and the properties of hadrons.



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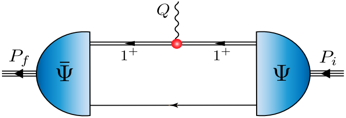

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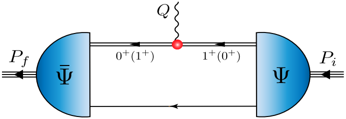

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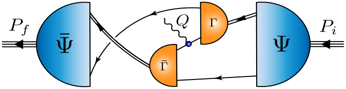

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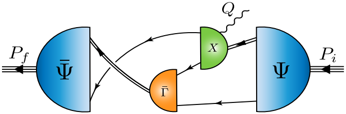

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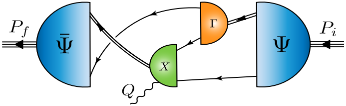


(6)

C. Chen *et al.*, Phys. Lett. B815 (2020) 136150

C. Chen *et al.*, arXiv:hep-ph/2103.02054.

Different probes correspond to different form factors

☞ The nucleon electromagnetic current: $F_1(Q^2)$ and $F_2(Q^2)$.

$$J_\mu^{em}(K,Q) = \bar{u}(P_f)\left[\gamma_\mu F_1(Q^2) + \frac{1}{2m_N}\sigma_{\mu\nu}Q_\nu F_2(Q^2)\right]u(P_i)$$

- A large number of experimental measurements, with high precision and up to large momentum transfer.

☞ The nucleon axial-vector current: $G_A(Q^2)$ and $G_P(Q^2)$.

$$J_{5\mu}^j(K,Q) = \bar{u}(P_f)\frac{\tau^j}{2}\gamma_5\left[\gamma_\mu G_A(Q^2) + i\frac{Q_\mu}{2m_N}G_P(Q^2)\right]u(P_i)$$

- The measurements are much more difficult, since they are related to weak processes. Axial form factor: Experimental data of G_A are rather sparse and with large uncertainties; only 4 empirical results exist for G_P.

☞ The nucleon pseudoscalar current: $G_5(Q^2)$.

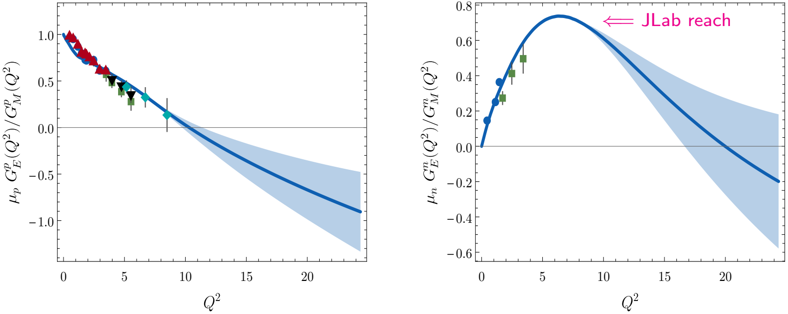
$$J_5^j(K,Q) = \bar{u}(P_f)\frac{\tau^j}{2}\gamma_5 G_5(Q^2)u(P_i)$$

- PCAC at the form factor level: $G_A(Q^2) - \frac{Q^2}{4m_N^2}G_P(Q^2) = \frac{m_q}{m_N}G_5(Q^2)$

The nucleon electromagnetic current: $F_1(Q^2)$ and $F_2(Q^2)$

☞ The Sachs electric and magnetic form factors are:

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4m_N^2} F_2(Q^2)\,, \qquad G_M(Q^2) = F_1(Q^2) + F_2(Q^2)\,.$$

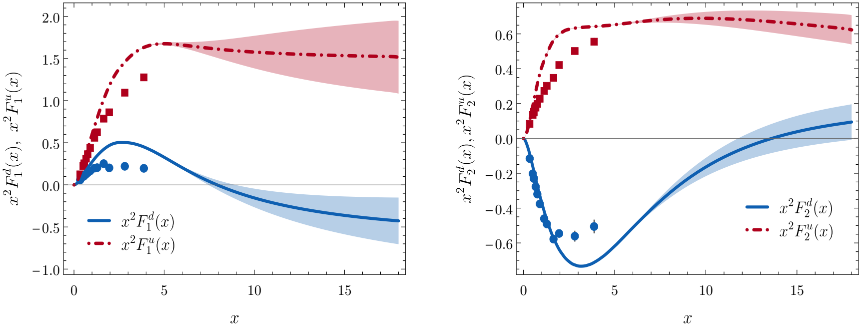


C.-F. Cui *et al.*, Phys. Rev. D102 (2020) 014043

☞ Observations:

- There is no evidence for scaling in Dirac and Pauli form factors, and thus in the electromagnetic Sach form factors.
- Our analysis predicts a zero for the proton's electromagnetic ratio at $Q^2 = 10.3^{+1.1}_{-0.7}\,\text{GeV}^2$.
- The neutron's electromagnetic ratio has a peak at $Q^2 \approx 6\,\text{GeV}^2$ and then crossed zero for $Q^2 = 20.1^{+10.6}_{-3.5}\,\text{GeV}^2$.

The nucleon electromagnetic current: $F_1(Q^2)$ and $F_2(Q^2)$



C.-F. Cui *et al.*, Phys. Rev. D102 (2020) 014043

☞ Observations:

- F_1^d is smaller than F_1^u, even allowing for the difference in normalisation, and decreases more quickly as x increases.
- The location of the zero in F_1^d is a measure of the relative probability of finding pseudovector and scalar diquarks in the proton.
- The u- and d-quark Pauli form factors are roughly equal in magnitude on $x \lesssim 5$; *i.e.* F_2^d is suppressed with respect F_2^u but only at large momentum transfer.
- There are contributions playing an important role in F_2, like the anomalous magnetic moment of dressed-quarks or meson-baryon final-state interactions.

Building block 1: Weak-boson–dressed-quark interaction

The current-quark vertices

- **The axial-vector Ward-Takahashi identity:**

$$Q_\mu \Gamma^j_{5\mu}(k_+,k_-) + 2im_q\Gamma^j_5(k_+,k_-) = S^{-1}(k_+)i\gamma_5\frac{\tau^j}{2} + \frac{\tau^j}{2}i\gamma_5 S^{-1}(k_-)$$

- **The Bethe-Salpeter Amplitude of the pion:**

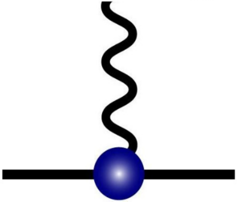
$$\Gamma^j_\pi(k,Q) = \tau^j\gamma_5\left[\boxed{iE_\pi(k,Q)}\right]$$

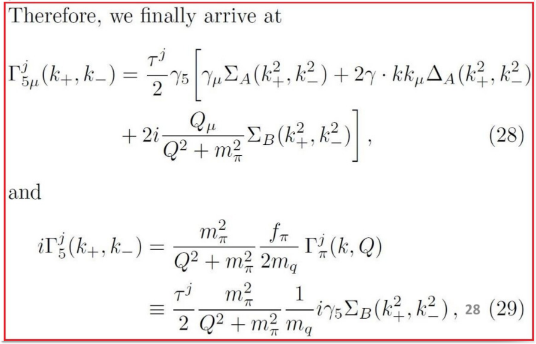
- **One *Ansatz*:** $E_\pi(k,Q) = \dfrac{1}{2f_\pi}\left(B(k_+^2)+B(k_-^2)\right)$

$$S^{-1}(k) = i\gamma\cdot kA(k^2) + B(k^2)$$

in the chiral limit:

$$E_\pi(k,0) = \frac{B(k^2)}{f_\pi}$$





Therefore, we finally arrive at

$$\Gamma^j_{5\mu}(k_+,k_-) = \frac{\tau^j}{2}\gamma_5\Big[\gamma_\mu\Sigma_A(k_+^2,k_-^2) + 2\gamma\cdot kk_\mu\Delta_A(k_+^2,k_-^2) + 2i\frac{Q_\mu}{Q^2+m_\pi^2}\Sigma_B(k_+^2,k_-^2)\Big], \qquad (28)$$

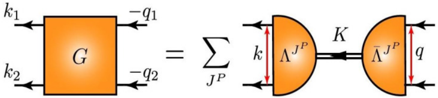
and

$$i\Gamma^j_5(k_+,k_-) = \frac{m_\pi^2}{Q^2+m_\pi^2}\frac{f_\pi}{2m_q}\Gamma^j_\pi(k,Q) \equiv \frac{\tau^j}{2}\frac{m_\pi^2}{Q^2+m_\pi^2}\frac{1}{m_q}i\gamma_5\Sigma_B(k_+^2,k_-^2), \text{ 28 } \qquad (29)$$

Building block 2: Seagull terms

The seagull terms

- **The diquark *Ansatz* for the 4-point Green's function of the quark-quark correlations:**

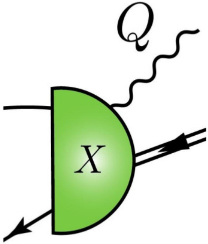


✓ Martin Oettel, Mike Pichowsky, Lorenz von Smekal, Eur.Phys.J. A8 (2000) 251-281

- **The equaltime commutators of the axial current operator:**

$$[\mathcal{A}^j_{5\mu=4}(x), \psi(y)]_{x_4=y_4} = \frac{\tau^j}{2}\gamma_5\psi(x)\delta^{(4)}(x-y)$$

$$[\mathcal{A}^j_{5\mu=4}(x), \bar{\psi}(y)]_{x_4=y_4} = \bar{\psi}(x)\gamma_5\frac{\tau^j}{2}\delta^{(4)}(x-y)$$



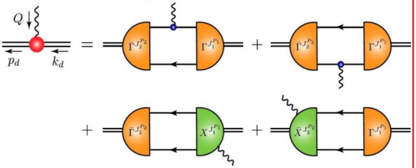
$$\chi^{j,J^P}_{5\mu,[\mathrm{sg}]}(k,Q) = -\frac{Q_\mu}{Q^2+m_\pi^2}\left[\frac{\tau^j}{2}i\gamma_5\Gamma^{J^P}(k-Q/2)+ \right.$$
$$\left. \Gamma^{J^P}(k+Q/2)(i\gamma_5\frac{\tau^j}{2})^{\mathrm{T}}\right], \qquad (57)$$

and

$$i\chi^{j,J^P}_{5,[\mathrm{sg}]}(k,Q) = -\frac{1}{2m_q}\frac{m_\pi^2}{Q^2+m_\pi^2}\left[\frac{\tau^j}{2}i\gamma_5\Gamma^{J^P}(k-Q/2)+ \right.$$
$$\left. \Gamma^{J^P}(k+Q/2)(i\gamma_5\frac{\tau^j}{2})^{\mathrm{T}}\right]. \qquad 29 \quad (58)$$

Building block 3: Weak-boson–dressed-diquark interaction

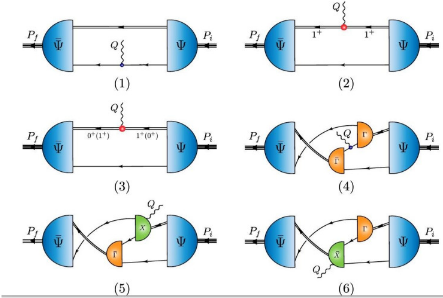
- **The current-diquark vertices**



- **AXWTIs:**

$$Q_\mu\,\Gamma^{aa}_{5\mu,\alpha\beta}(p_d,k_d) + 2im_q\,\Gamma^{aa}_{5,\alpha\beta}(p_d,k_d) = 0$$

$$Q_\mu\,\Gamma^{sa}_{5\mu,\beta}(p_d,k_d) + 2im_q\,\Gamma^{sa}_{5,\beta}(p_d,k_d) = 0$$



i) The $\{qq\}_{1^+}$–pseudoscalar-current vertex

$$\Gamma^{aa}_{5,\alpha\beta}(p_d,k_d) = \\ = \frac{1}{2m_q}\frac{m_\pi^2}{Q^2+m_\pi^2}\left(\kappa^{aa}_{\rm ps}\frac{M_q^E}{m_N}\epsilon_{\alpha\beta\gamma\delta}(p_d+k_d)_\gamma Q_\delta\right)d(\tau^{aa})\,, \tag{61}$$

ii) The $\{qq\}_{1^+}$–axial-current vertex

$$\Gamma^{aa}_{5\mu,\alpha\beta}(p_d,k_d) = \left(\frac{\kappa^{aa}_{\rm ax}}{2}\epsilon_{\mu\alpha\beta\nu}(p_d+k_d)_\nu + \right. \\ \left. + \frac{Q_\mu}{Q^2+m_\pi^2}\left(\kappa^{aa}_{\rm ps}\frac{M_q^E}{m_N}\epsilon_{\alpha\beta\gamma\delta}(p_d+k_d)_\gamma Q_\delta\right)\right)d(\tau^{aa})\,, \tag{62}$$

iii) The pseudoscalar-current induced $0^+ \leftarrow 1^+$ transition vertex

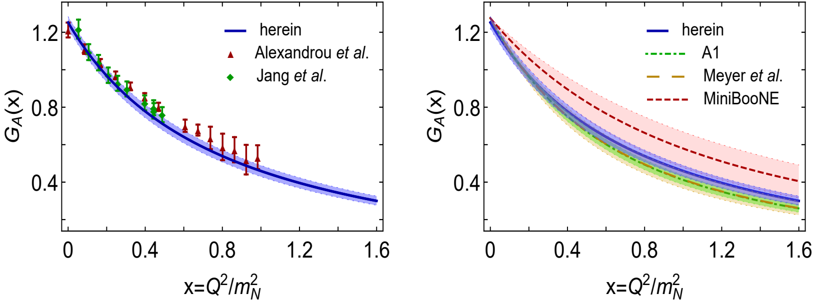
$$\Gamma^{sa}_{5,\beta}(p_d,k_d) = \\ = \frac{1}{2m_q}\frac{m_\pi^2}{Q^2+m_\pi^2}\left(-2i\kappa^{sa}_{\rm ps}M_q^E Q_\beta\right)d(\tau^{sa})\,, \tag{63}$$

iv) The axial-current induced $0^+ \leftarrow 1^+$ transition vertex

$$\Gamma^{sa}_{5\mu,\beta}(p_d,k_d) = \left(im_N\kappa^{sa}_{\rm ax}\delta_{\mu\beta} + \right. \\ \left. + \frac{Q_\mu}{Q^2+m_\pi^2}\left(-2i\kappa^{sa}_{\rm ps}M_q^E Q_\beta\right)\right)d(\tau^{sa})\,.30 \tag{64}$$

The nucleon axial-vector current: $G_A(Q^2)$

> *G_A should provide a sound foundation for analyses of the (anti-)neutrino–nucleus cross-sections that are relevant to modern accelerator neutrino experiments*



☞ Observations:

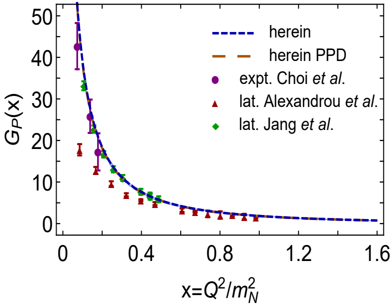
- The result for G_A can reliably be represented by a dipole form factor characterised by an axial charge $g_A = G_A(0) = 1.25(3)$ and a mass-scale $M_A = 1.23(3)m_N$.
- The axial mean-square radius is calculated to be

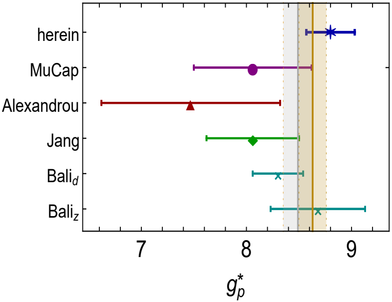
$$\langle r_A^2 \rangle^{\frac{1}{2}} = -6 \frac{1}{G_A(0)} \left. \frac{dG(Q^2)}{dQ^2} \right|_{Q^2=0} = 3.25(4)/m_N$$

- The ratio $g_A^d / g_A^u = -0.16(2)$ expresses a marked suppression of the d-quark component and owes to the presence of strong diquark correlations.

The nucleon axial-vector current: $G_P(Q^2)$

G_P can be extracted from longitudinal cross-section in pion electro-production or single spin asymmetries in $\nu(\bar{\nu})$ charged-current quasielastic scattering on free nucleons.





☞ Observations:

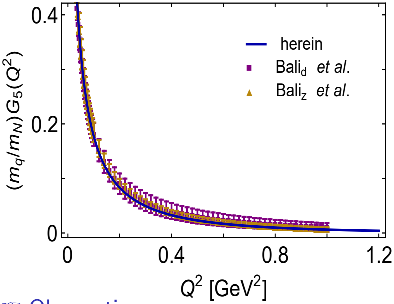
- The pion pole dominance *ansatz*, $G_P \approx 4m_N^2 G_A/(Q^2+m_\pi^2)$, is found to provide a reliable estimate of the directly computed result of the induced pseudoscalar form factor, G_P.
- Muon capture experiments $(\mu + p \to \nu_\mu + n)$ determine the induced pseudoscalar charge

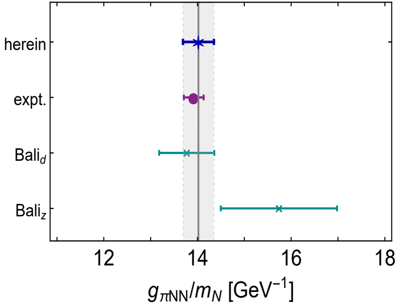
$$g_p^* = \frac{m_\mu}{2m_N} G_p(Q^2 = 0.88 m_\mu^2)\,,$$

where m_μ is the muon mass. We obtain $g_p^* = 8.80(23)$.

The nucleon pseudoscalar current: $G_5(Q^2)$

The pseudoscalar form factor G_5, or equivalently the pion-nucleon form factor, $G_{\pi NN}$, cannot be directly measured except at the pion mass point $Q^2 = -m_\pi^2$, i.e. $g_{\pi NN}$.





☞ Observations:

- At the pion mass pole, the residue of G_5 is the pion-nucleon coupling constant $g_{\pi NN}$. Thus one can define the pion-nucleon form factor $G_{\pi NN}$:

$$G_5(Q^2) \equiv \frac{m_\pi^2}{Q^2 + m_\pi^2} \frac{f_\pi}{m_q} G_{\pi NN}(Q^2)$$

- One cannot claim a reliable calculation of the axial and pseudoscalar form factors if they do not satisfy the Goldberger-Treiman relation at the form factor level:

$$\frac{f_\pi}{m_N} G_{\pi NN}(0) = 1.25(3) = G_A(0)$$

Partially conserved axial current (PCAC)

The PCAC relation at the nucleon level reads

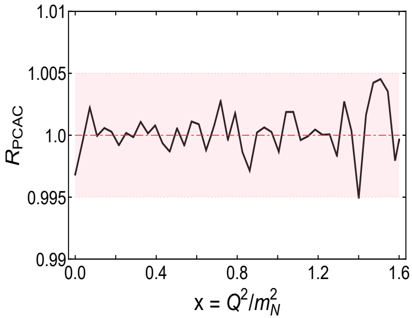
$$Q_\mu J^j_{5\mu}(K,Q) + 2im_q J^j_5(K,Q) = 0\,,$$

and this entails

$$G_A(Q^2) - \frac{Q^2}{4m_N^2} G_P(Q^2) = \frac{m_q}{m_N} G_5(Q^2)\,.$$

It should be pointed out that PCAC is an operator relation and thus any realistic results for G_A, G_P and G_5 should precisely satisfy the PCAC relation.

$$R_{\text{PCAC}} \equiv \frac{4m_N^2 G_A}{Q^2 G_P + 4m_q m_N G_5}$$



☞ Observations:

- The computed R_{PCAC} is practically indistinguishable from unity for the whole range of momentum-transfer depicted.
- We have demonstrated that our theoretical approach is consistent with fundamental symmetry requirements in an analytic way, but also numerically.
- This result does not rely on any fine-tuned set of parameters; instead, it is automatically satisfied owing to our careful construction of the currents.

Contributions from various diagrams to $G_A(0)$, $G_P(0)$ and $G_5(0)$

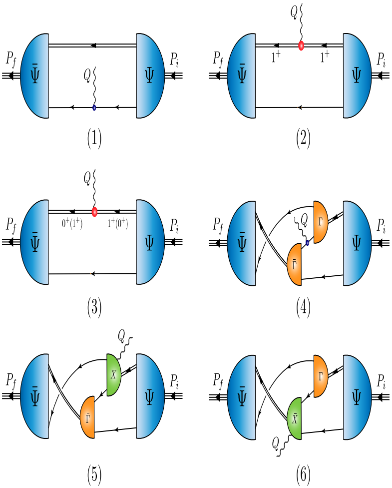


Table. Separation of $G_A(0)$, $G_P(0)$ and $G_5(0)$ into contributions from various diagrams, listed as a fraction of the total $Q^2 = 0$ value. Diagram (1): $\langle J\rangle_{\rm q}^{S}$ – weak-boson strikes dressed-quark with scalar diquark spectator; and $\langle J\rangle_{\rm q}^{A}$ – weak-boson strikes dressed-quark with axial-vector diquark spectator. Diagram (2): $\langle J\rangle_{\rm qq}^{AA}$ – weak-boson interacts strikes axial-vector diquark with dressed-quark spectator. Diagram (3): $\langle J\rangle_{\rm dq}^{SA+AS}$ – weak-boson mediates transition between scalar and axial-vector diquarks, with dressed-quark spectator. Diagram (4): $\langle J\rangle_{\rm ex}$ – weak-boson strikes dressed-quark "in-flight" between one diquark correlation and another. Diagrams (5) and (6): $\langle J\rangle_{\rm sg}$ – weak-boson couples inside the diquark correlation amplitude. The listed uncertainty in these results reflects the impact of $\pm 5\%$ variations in the diquark masses.

	$\langle J\rangle_{\rm q}^{S}$	$\langle J\rangle_{\rm q}^{A}$	$\langle J\rangle_{\rm qq}^{AA}$	$\langle J\rangle_{\rm qq}^{SA+AS}$	$\langle J\rangle_{\rm ex}$	$\langle J\rangle_{\rm sg}$
$G_A(0)$	$0.71_{4_\mp}$	$0.064_{2_\pm}$	$0.025_{5_\pm}$	$0.13_{0_\mp}$	$0.072_{32_\pm}$	0
$G_P(0)$	$0.74_{4_\mp}$	$0.070_{5_\pm}$	$0.025_{5_\pm}$	$0.13_{0_\mp}$	$0.22_{4_\pm}$	$-0.19_{1_\mp}$
$G_5(0)$	$0.74_{4_\mp}$	$0.069_{5_\pm}$	$0.025_{5_\pm}$	$0.13_{0_\mp}$	$0.22_{4_\pm}$	$-0.19_{1_\mp}$

Conclusions

- A dipole form factor defined by an axial charge $g_A = G_A(0) = 1.25(3)$ and a mass-scale $M_A = 1.23(3) m_N$ can accurately describe the Q^2-behavior of G_A.
- The pion pole dominance approach delivers a reliable estimate of the the Q^2-behavior of G_P; we find a pseudoscalar charge $g_p^* = 8.80(23)$.
- The pion-nucleon coupling constant, $g_{\pi NN}/m_N = 14.02(33)/\text{GeV}$ is consistent with a Roy–Steiner-equation analysis of pion-nucleon scattering.
- A suppression of the *d*-quark component relative to that of the *u*-quark in the ratio $g_A^d/g_A^u = -0.16(2)$ highlights the presence of strong diquark correlations.
- Our nucleon's axial, induced-pseudoscalar and explicit pseudoscalar form factors analytically satisfy the PCAC relation and verified this numerically.