First Steps Towards a Semi-Analytical Description of Beam-Beam Driven Instabilities in the FCC-ee

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The Circulant Matrix Model

The **circulant matrix model (CMM)** is based on the decomposition of the longitudinal phase space distribution

$$\Psi(R), \quad R = \sqrt{(\frac{s}{\sigma_s})^2 + (\frac{\delta}{\sigma_\delta})^2}, \qquad (1)$$

σ_s and σ_δ being the RMS bunch length and momentum deviation.

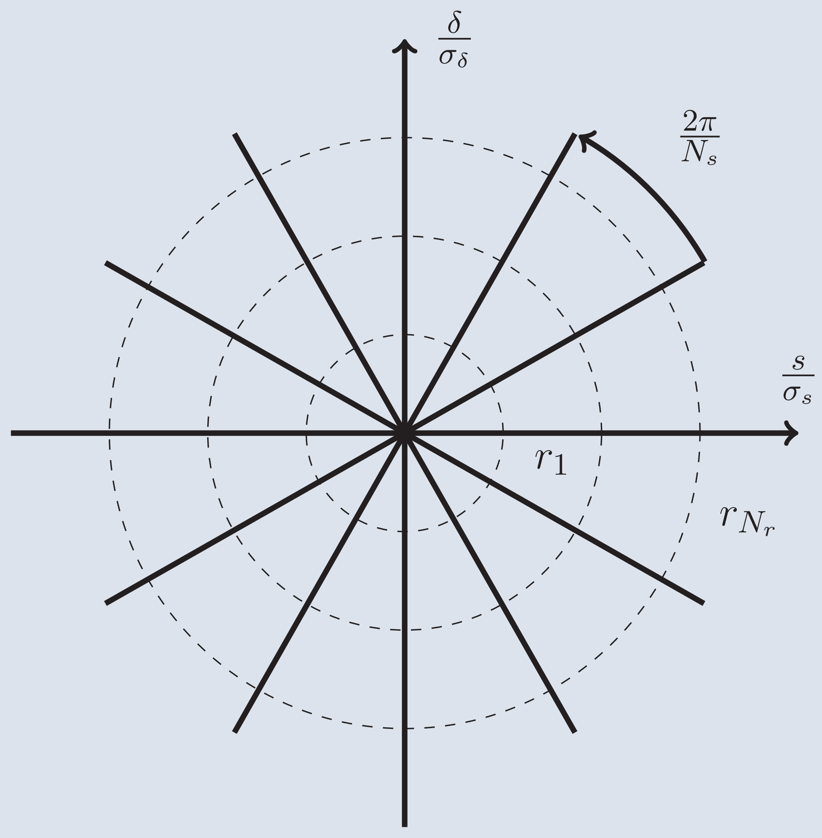

Figure 1: Discretisation of the longitudinal phase space in the CMM (source: [1]).

Beam-Beam Kick Matrix

The coupling between two elements of the distribution by the Beam Beam force, with two transverse planes, is given by:

$$C_{\mathrm{BB},0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\partial \Delta x'_{coh}}{\partial x} & 1 & -\frac{\partial \Delta x'_{coh}}{\partial y} & 0 & \frac{\partial \Delta x'_{coh}}{\partial x} & 0 & \frac{\partial \Delta x'_{coh}}{\partial y} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\partial \Delta y'_{coh}}{\partial x} & 0 & -\frac{\partial \Delta y'_{coh}}{\partial y} & 1 & \frac{\partial \Delta y'_{coh}}{\partial x} & 0 & \frac{\partial \Delta y'_{coh}}{\partial y} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\partial \Delta x'_{coh}}{\partial x} & 0 & \frac{\partial \Delta x'_{coh}}{\partial y} & 0 & -\frac{\partial \Delta x'_{coh}}{\partial x} & 1 & -\frac{\partial \Delta x'_{coh}}{\partial y} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{\partial \Delta y'_{coh}}{\partial x} & 0 & \frac{\partial \Delta y'_{coh}}{\partial y} & 0 & -\frac{\partial \Delta y'_{coh}}{\partial x} & 0 & -\frac{\partial \Delta y'_{coh}}{\partial y} & 1 \end{pmatrix}$$

where all derivatives are evaluated at the closed orbit, (x_0, y_0) and are given by the 2 dimensional extension:

$$\Delta x'_{coh}(x,y) \approx \boldsymbol{\Delta x'_{coh}} + \frac{\partial \boldsymbol{\Delta x'_{coh}}}{\partial x}\Delta x + \frac{\partial \boldsymbol{\Delta x'_{coh}}}{\partial y}\Delta y, \quad (2)$$

$$\Delta y'_{coh}(x,y) \approx \boldsymbol{\Delta y'_{coh}} + \frac{\partial \boldsymbol{\Delta y'_{coh}}}{\partial x}\Delta x + \frac{\partial \boldsymbol{\Delta y'_{coh}}}{\partial y}\Delta y, \quad (3)$$

where the bold quantities are being evaluated at (x_0, y_0).

The **beam-beam kick matrix** was developed for **VEPP** and **LHC** with round beams and crossing angle, but it needs to be extended to flat beams.

Mode Coupling Instabilities

The coherent beam-beam force modifies the frequency of head-tail modes and can result in mode coupling instabilities (source: [2]).

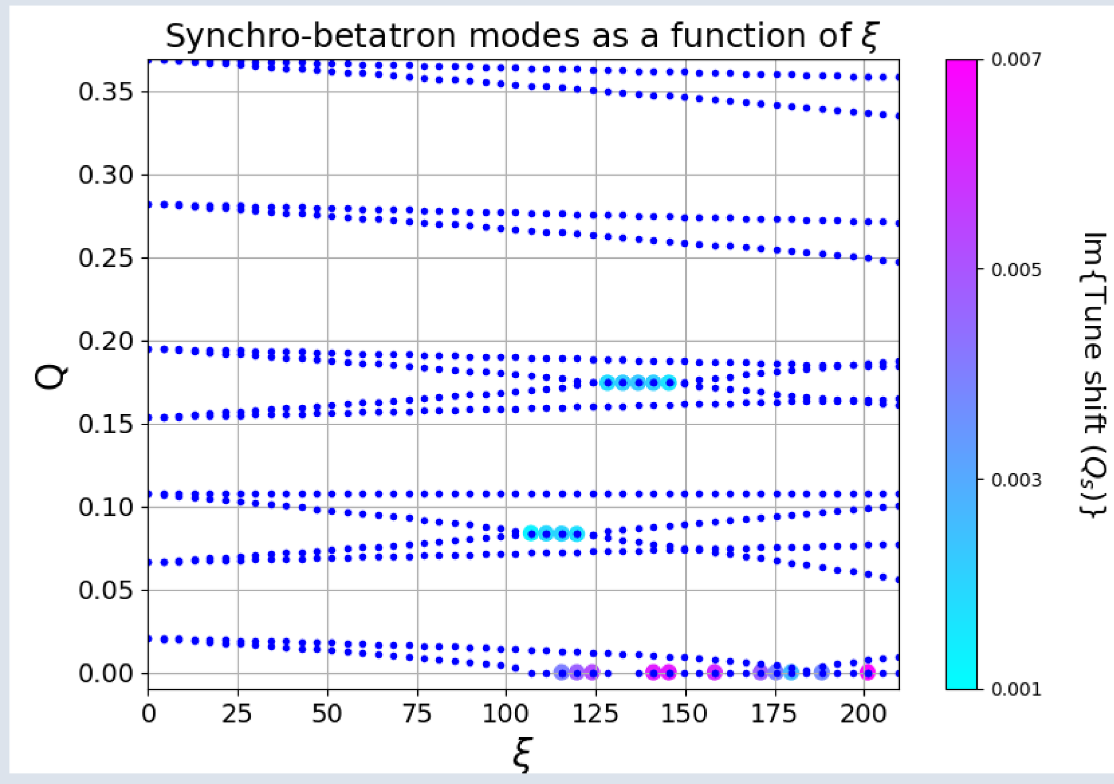

Figure 2: $\sigma_s = 2.54$[mm], $\beta_x^* = \beta_y^* = 1$[m] and $\epsilon_N (= \epsilon_x) = 1.46$[nm]. The colour-bar corresponds to the imaginary part of the eigenvalues.

The linearized **beam parameter** for the case of the round beam is given by: $\xi = \frac{N r_e}{4\pi\epsilon_N}$, where N is the bunch population, r_e is the classical radius and ϵ_N is the normalised emittance.

Round Beam vs Flat Beam Case

The total beam-beam kick called, **coherent kick**, is obtained by integration of the single particle kicks over the beam distribution $\Psi(x,y)$ (7). For a **Bi-Gaussian** distribution the explicit form for the **incoherent kick**, i.e. the force experienced by a single particle, is derived from the well known Basseti-Erskine formula [3] †

$$\Delta^{\mathbf{F}} x'(x,y) = \frac{N r_e}{\gamma_r}\sqrt{\frac{2\pi}{\sigma_x^2 - \sigma_y^2}} \mathrm{Im}\left\{ w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right) w\left(\frac{\frac{\sigma_y}{\sigma_x}x + i\frac{\sigma_x}{\sigma_y}y}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right\}, \qquad (4)$$

where w is the Fadeeva complex error function: $w(z) = \exp(-z^2)\left[1 + \frac{2i}{\sqrt{\pi}}\int_0^z \exp\zeta^2 d\zeta\right]$. The kick felt by a test particle assuming a **round Gaussian** distribution is:

$$\Delta^{\mathbf{R}} x'(x,y) = -\frac{2 r_0 N}{\gamma_r}\frac{x}{r^2}\left(1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right), \qquad (5)$$

$$\Delta^{\mathbf{R}} x'_{coh}(x,y) = -\frac{2 r_0 N}{\gamma_r}\frac{x}{r^2}\left(1 - \exp\left(-\frac{r^2}{4\sigma^2}\right)\right), \; r = \sqrt{x^2+y^2}, \quad (6)$$

$$\Delta^{\mathbf{F}} x'_{coh}(x,y) = \int_{-\infty}^{\infty} \Delta x'(X,Y)\Psi(X-x, Y-y)dXdY. \qquad (7)$$

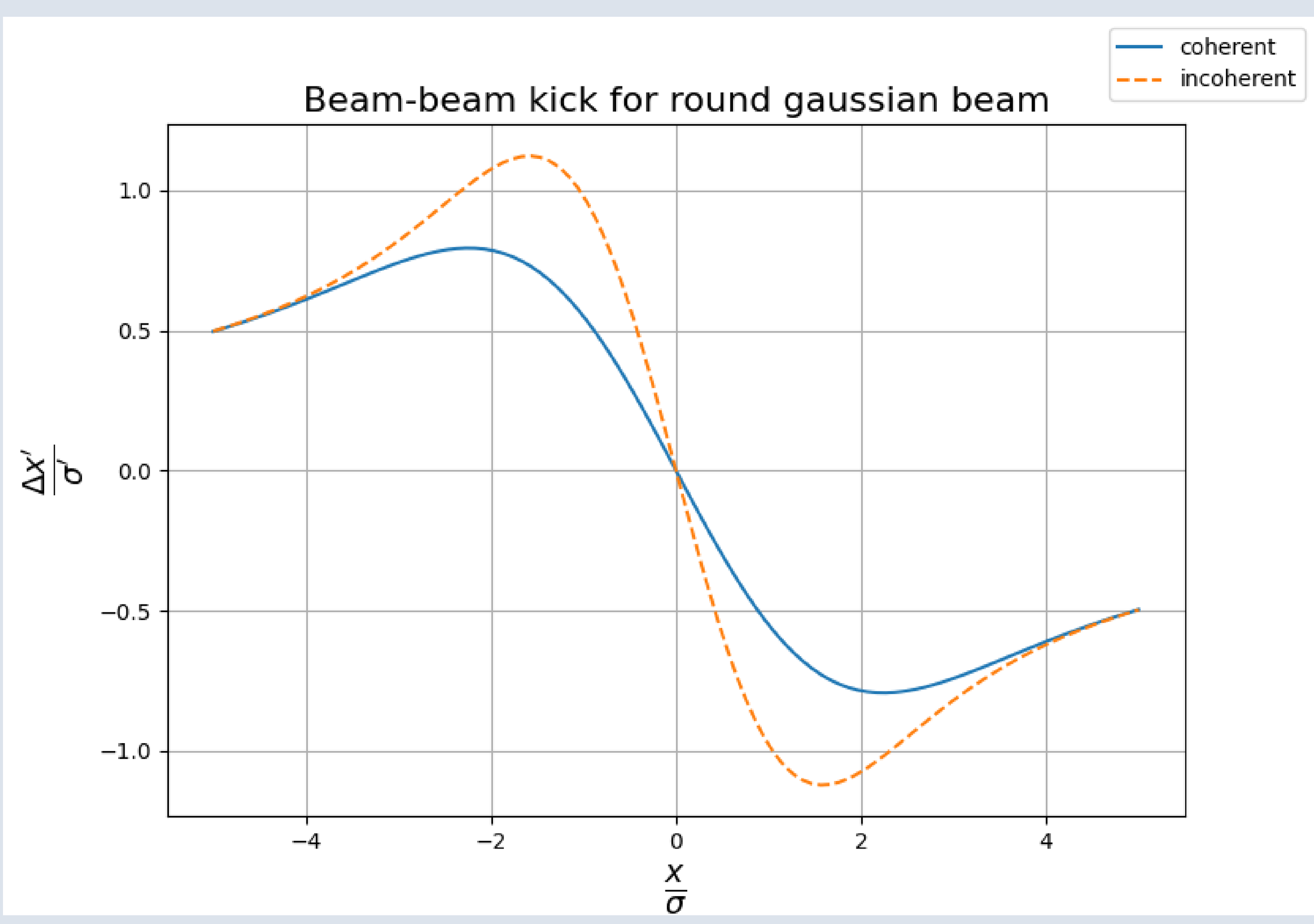

Figure 3: Comparison between the incoherent and coherent beam–beam kick for **round** Gaussian beam distributions.

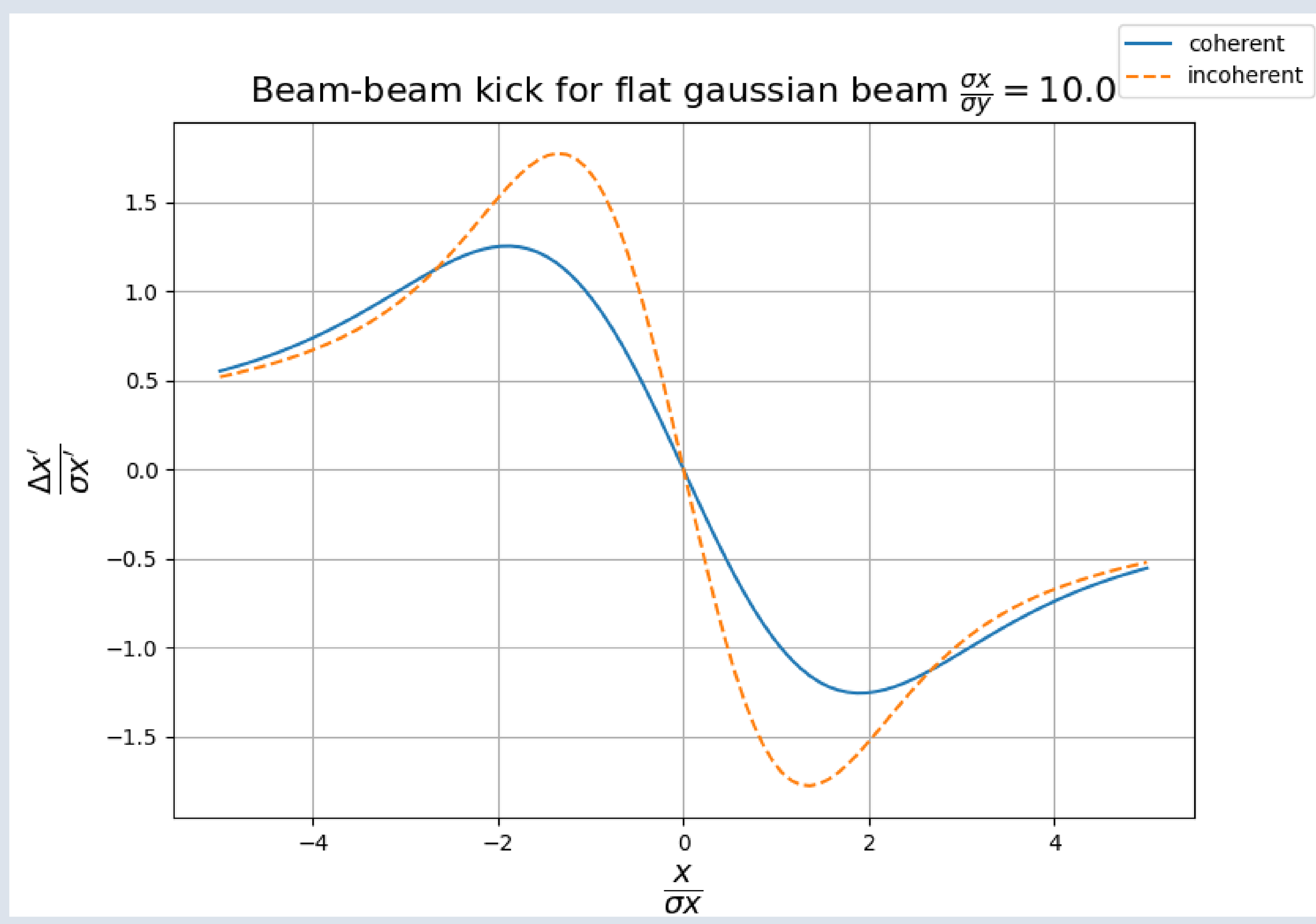

Figure 4: Comparison between the incoherent and coherent beam–beam kick for **flat** Gaussian beam distributions.

† The superscripts $\mathbf{R}$ and $\mathbf{F}$ correspond to the **round** and **flat** cases respectively.

Summary

Completed:

- ☑ Integrated the coherent force numerically.

Next steps:

- ☐ Attempt for an analytical solution.
- ☐ Implement it in the **BimBim** code.
- ☐ Study mode coupling instabilities.

References

[1] X. Buffat, "Transverse beams stability studies at the Large Hadron Collider," Presented 30 Jan 2015, Jan. 2015. [Online]. Available: `https://cds.cern.ch/record/1987672`.

[2] S. White, X. Buffat, N. Mounet, and T. Pieloni, "Transverse mode coupling instability of colliding beams," *Phys. Rev. ST Accel. Beams*, vol. 17, p. 041 002, 4 Apr. 2014. DOI: `10.1103/PhysRevSTAB.17.041002`. [Online]. Available: `https://link.aps.org/doi/10.1103/PhysRevSTAB.17.041002`.

[3] M. Bassetti and G. A. Erskine, "Closed expression for the electrical field of a two-dimensional Gaussian charge," CERN, Geneva, Tech. Rep., 1980. [Online]. Available: `https://cds.cern.ch/record/122227`.