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Simplifying calculations with chirality flow

PARTIKELDAGARNA 2021 - MALIN SJÖDAHL

BASED ON HEP-PH:2003.05877 (EPJC) AND HEP-PH:2011.10075 (EPJC)

IN COLLABORATION WITH JOAKIM ALNEFJORD, ANDREW LIFSON AND CHRISTIAN REUSCHLE

Introduction

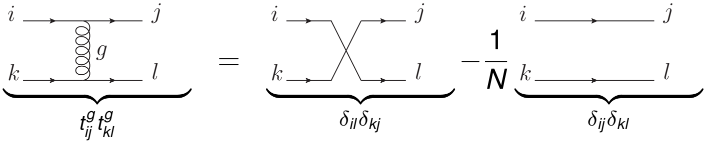
Motivation

- Question: Calculations in QCD color space, su(N), $N = 3$, can be elegantly simplified using a flow picture for color, can we do the same for the Lorentz structure $\sim \underbrace{su(2)}_{\text{left}}, \underbrace{su(2)}_{\text{right}}$?
- For color one can formulate color-flow Feynman rules, can we similarly formulate some chirality-flow Feynman rules?
- Answer: YES!
- Feynman rules can be rewritten in terms of chirality flows and this beautifully simplifies calculations!

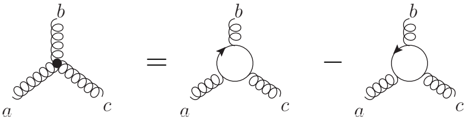
Builds on hep-ph:2003.05877 and hep-ph:2011.10075, both published in EPJC

In QCD we translate color structures to flows of color

- SU(N) Fierz identity: remove adjoint indices ($T_R = 1$)



- Remove gluon vertices similarly

$if^{abc} =$ 

- In the end every amplitude is a linear combination of products of δs

Idea:

Try to remove Lorentz indices in analogy with removing gluon indices, with the goal of recasting all Feynman rules to chirality-flow rules

Can we do something similar for spacetime?

- At the (complexified) algebra level, the Lorentz group consists of two copies of su(2), $so(3,1) \cong su(2) \oplus su(2)$
- The Dirac spinor structure transforms under the direct sum representation $\underbrace{(\frac{1}{2},0)}_{\text{left}} \oplus \underbrace{(0,\frac{1}{2})}_{\text{right}}$ in the chiral/Weyl basis

$$\begin{pmatrix} u_L \\ u_R \end{pmatrix} \to \begin{pmatrix} e^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}+\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} & 0 \\ 0 & e^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}-\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix}$$

i.e. actually two copies of **SL(2,$\mathbb{C}$)**, generated by the complexified su(2) algebra

Spinor-helicity: its building blocks

- Consider massless particles: chirality $\sim$ helicity
- Spinors

$$u^{+}(p) = v^{-}(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \qquad u^{-}(p) = v^{+}(p) = \begin{pmatrix} |p] \\ 0 \end{pmatrix}$$

$$\bar{u}^{+}(p) = \bar{v}^{-}(p) = \left([p| \, , \quad 0 \right) \qquad \bar{u}^{-}(p) = \bar{v}^{+}(p) = \left(0 \, , \quad \langle p| \right)$$

- Polarization vectors

$$\epsilon_L^{\mu}(p,r) \to \frac{|r\rangle [p|}{\langle rp \rangle} \text{ or } \frac{|p]\langle r|}{\langle rp \rangle}, \qquad \epsilon_R^{\mu}(p,r) \to \frac{|r]\langle p|}{[pr]} \text{ or } \frac{|p\rangle [r|}{[pr]}$$

where ϵ_L is for incoming negative helicity or outgoing positive helicity and ϵ_R is for incoming positive helicity or outgoing negative helicity

- Amplitudes built up from Lorentz invariant inner products
- Lorentz inner products formed using **the only SL(2,$\mathbb{C}$) invariant object** $\epsilon^{\alpha\beta}$, $\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12}$

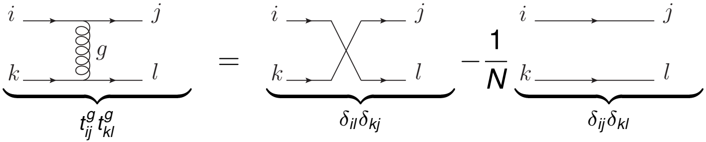
$$\underbrace{\epsilon^{\alpha\beta}|i\rangle_\beta}_{\equiv\langle i|^\alpha} |j\rangle_\alpha = \langle i|^\alpha |j\rangle_\alpha = \langle ij\rangle, \quad \underbrace{\epsilon_{\dot\alpha\dot\beta}|i]^{\dot\beta}}_{\equiv[i|_{\dot\alpha}} |j]^{\dot\alpha} = [i|_{\dot\alpha}|j]^{\dot\alpha} = [ij] \,,$$

- $\implies$ Amplitudes are built up of contractions of form $\langle ij\rangle, [ij] \sim \sqrt{s_{ij}}$
- If we manage to create a flow picture, the “flow” must contract dotted and undotted indices separately

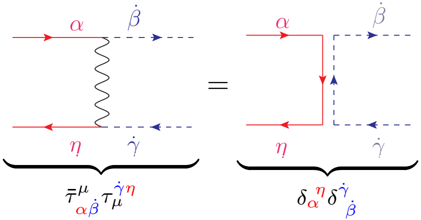
Massless chirality flow

Towards chirality flow: Photon exchange

- Recall color, a single $SU(N)$: generators $t^a \to \delta$'s



- For Lorentz structure $\gamma^\mu = \sqrt{2}\begin{pmatrix} 0 & \tau^\mu \\ \bar{\tau}^\mu & 0 \end{pmatrix}$ in vertices, split in two terms and use spinor Fierz in flow form is (always read indices along arrow):



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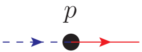
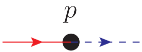
Towards chirality flow: Fermion propagators

- We split $\not{p}_{4d} \equiv p_\mu \gamma^\mu = p_\mu \sqrt{2} \begin{pmatrix} 0 & \tau^\mu \\ \bar{\tau}^\mu & 0 \end{pmatrix}$ into two terms

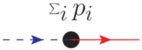
$$\not{p} \equiv \sqrt{2} p^\mu \tau_\mu^{\dot{\alpha}\beta} =$$ 

$$\bar{\not{p}} \equiv \sqrt{2} p_\mu \bar{\tau}^\mu_{\alpha\dot{\beta}} =$$ 

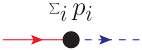
- Momentum dot defined to represent slashed momenta
- For massless momenta we have

$$\sqrt{2} p^\mu \tau_\mu \equiv \not{p} = |p]\langle p| \;, \quad \sqrt{2} p^\mu \bar{\tau}_\mu \equiv \bar{\not{p}} = |p\rangle [p|$$

- In a propagator, we have $p^\mu = \sum p_i^\mu \;, \; p_i^2 = 0$

$$\not{p} =$$ 

$$= \sum_i |i]^{\dot{\alpha}} \langle i|^{\beta} \quad \text{for} \quad p_i^2 = 0$$

$$\bar{\not{p}} =$$ 

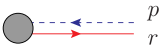
$$= \sum_i |i\rangle_\alpha [i|_{\dot{\beta}} \quad \text{for} \quad p_i^2 = 0$$

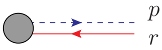
Towards chirality flow: external gauge bosons

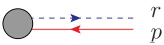
- In the spinor-helicity formalism

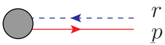
$$\epsilon_L^\mu(p,r) \to \frac{|r\rangle[p|}{\langle rp\rangle} \text{ or } \frac{|p]\langle r|}{\langle rp\rangle}, \qquad \epsilon_R^\mu(p,r) \to \frac{|r]\langle p|}{[pr]} \text{ or } \frac{|p\rangle[r|}{[pr]}$$

- $\implies$ easy to translate to chirality flow

$\epsilon_L^\mu(p,r) \to \frac{1}{\langle rp\rangle}$ 

, or $\epsilon_L^\mu(p,r) \to \frac{1}{\langle rp\rangle}$ 

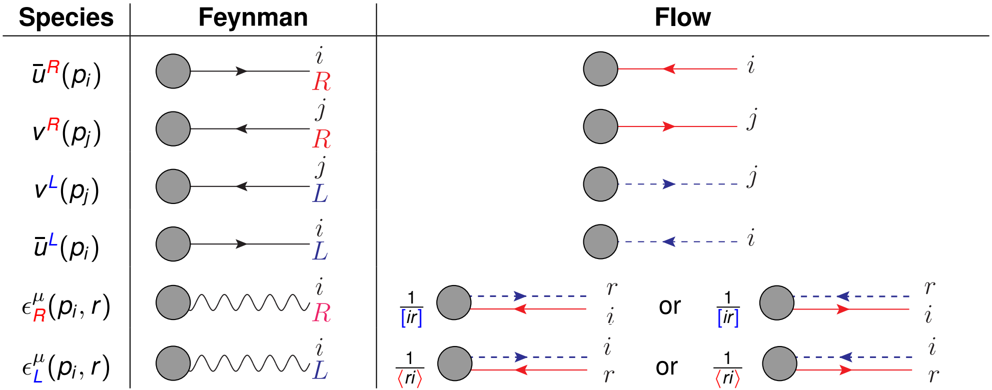
$\epsilon_R^\mu(p,r) \to \frac{1}{[pr]}$ 

, or $\epsilon_R^\mu(p,r) \to \frac{1}{[pr]}$ 

- In Feynman diagram choose arrow direction which gives aligned arrows

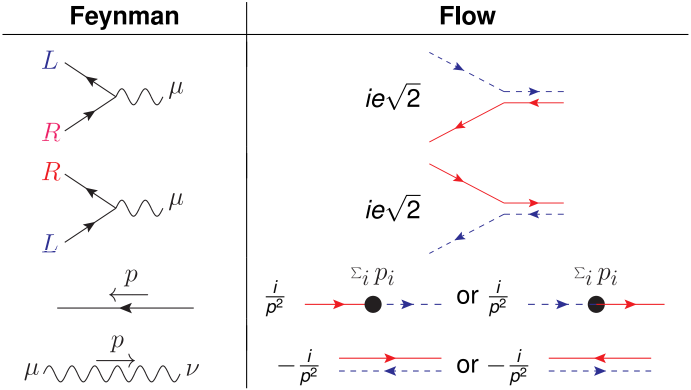
After careful consideration we conclude that this flow picture works always works

The QED flow rules: outgoing particles



Use left and right chiral spinors $\underbrace{su(2)}_{\text{dotted}}$, $\underbrace{su(2)}_{\text{undotted}}$ and R/L for outgoing $-/+$ helicity

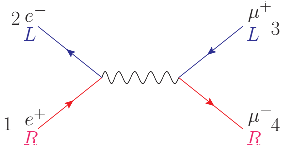
The QED flow rules: vertices and propagators



Stitch together such that arrow direction match

Simplest QED example, all particles outgoing

- Regular spinor-helicity = easy



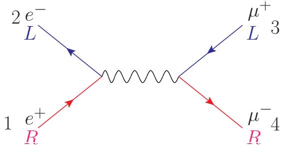
$$= \frac{2ie^2}{s_{e^+e^-}}\left([2|_{\dot\alpha}\tau_\mu^{\dot\alpha\beta}|1\rangle_\beta\right)\left(\langle 4|^\alpha\bar\tau^\mu_{\alpha\dot\beta}|3]^{\dot\beta}\right)$$

$$= \frac{2ie^2}{s_{e^+e^-}}[2|_{\dot\alpha}|3]^{\dot\alpha}\langle 4|^\beta|1\rangle_\beta \equiv \frac{2ie^2}{s_{e^+e^-}}[2\,3]\langle 4\,1\rangle$$

- Chirality flow = super easy and intuitive

Simplest QED example, all particles outgoing

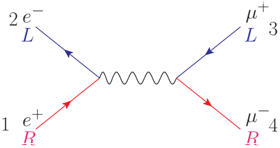
- Regular spinor-helicity = easy



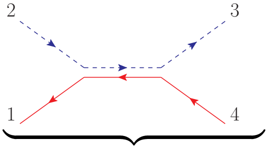
$$= \frac{2ie^2}{s_{e^+e^-}} \left([2|_{\dot{\alpha}} \tau_\mu^{\dot{\alpha}\beta} |1\rangle_\beta\right)\left(\langle 4|^{\alpha} \bar{\tau}^{\mu}_{\alpha\dot{\beta}} |3]^{\dot{\beta}}\right)$$

$$= \frac{2ie^2}{s_{e^+e^-}} [2|_{\dot{\alpha}} |3]^{\dot{\alpha}} \langle 4|^{\beta} |1\rangle_\beta \equiv \frac{2ie^2}{s_{e^+e^-}} [2\,3]\langle 4\,1\rangle$$

- Chirality flow = super easy and intuitive

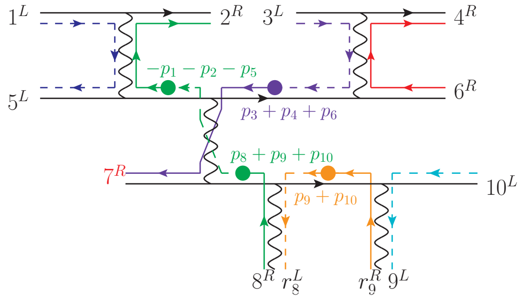


$$= \frac{2ie^2}{s_{e^+e^-}}$$



$[2\,3]\langle 4\,1\rangle$

A complicated QED example



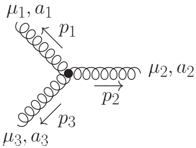
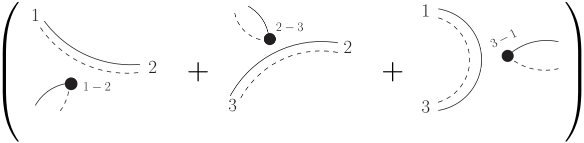
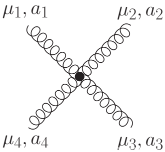
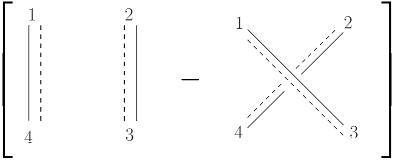
- Here all particles crossed to outgoing
- Pick any consistent arrow direction
- Compare to standard QFT: 12 γ^μ matrices

$$= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{1\,2}\; s_{3\,4}\; s_{8\,9\,10}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{1\,2\,5}\; s_{3\,4\,6}\; s_{8\,9\,10}\; s_{9\,10}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8 r_8]\langle r_9 9\rangle}}_{\text{polarization vectors}} \quad [15]\langle 64\rangle[10\;9]$$

$$\times \left(\langle r_9 9\rangle[9 r_8] + \langle r_9 10\rangle[10 r_8]\right)\left(\underbrace{[33]}_{0}\langle 37\rangle + [34]\langle 47\rangle + [36]\langle 67\rangle\right)$$

$$\times \left(-\langle 89\rangle[91]\langle 12\rangle - \langle 89\rangle[95]\langle 52\rangle - \langle 8\,10\rangle[10\;1]\langle 12\rangle - \langle 8\,10\rangle[10\;5]\langle 52\rangle\right)$$

The non-abelian massless QCD flow vertices

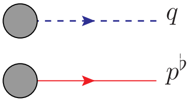
Feynman	Flow
	$-\frac{g_s f^{abc}}{2}$ 
	$ig_s^2 \sum_{Z(2,3,4)} f^{a_1 a_2 b} f^{b a_4 a_3}$ 

Arrow directions only consistently set within full diagram

Double line $\equiv g_{\mu\nu}$, momentum dot $\equiv p_\mu$

Massive chirality flow

Massive spinor helicity basics

- Massive spinors and polarization vectors written in terms of massless Weyl spinors of momentum $p^\flat$, q, $u^+(p) = \Bigg(-\frac{m}{[qp^\flat]}$ 

$\Bigg)$, etc.

- Decompose massive momentum p as sum of massless ones
$p^\mu = p^{\flat,\mu} + \alpha q^\mu \;, \quad (p^\flat)^2 = q^2 = 0 \;, \quad \alpha = \frac{p^2}{2p\cdot q}$
$\not{p} \equiv \sqrt{2} p^\mu \tau_\mu = |p^\flat]\langle p^\flat| + \alpha |q]\langle q|$

- q is arbitrary but physical, as defines spin direction s^μ

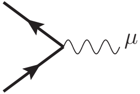
$$s^\mu = \frac{1}{m}(p^{\flat,\mu} - \alpha q^\mu) = \frac{1}{m}(p^\mu - 2\alpha q^\mu)$$

- $p^\mu = p_f^\mu + p_b^\mu$, $\alpha \to 1$, $p^\flat \to p_f = \frac{p^0+|\vec{p}|}{2}(1,\hat{p})$, $q \to p_b = \frac{p^0-|\vec{p}|}{2}(1,-\hat{p})$
Spin measured along $s^\mu = \frac{1}{m}(p_f^\mu - p_b^\mu) = \frac{1}{m}(|\vec{p}|, p^0\hat{p}) \equiv$ direction of motion!

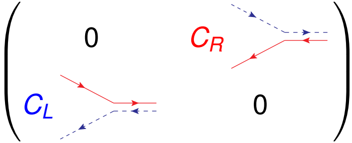
See e.g. hep-ph:0510157 for more details

Fermion vertices

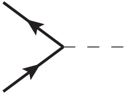
Fermion-vector vertex



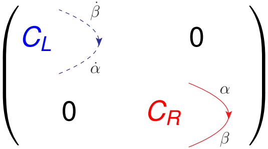
$= ie(P_L C_L + P_R C_R)\gamma^\mu = ie\sqrt{2}$



Fermion-scalar vertex



$= ie(P_L C_L + P_R C_R) = ie$

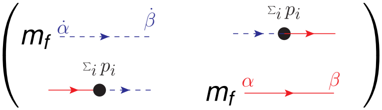


Left and right chiral couplings may differ, in particular $C_R = 0$ for $W^\pm$
$\Rightarrow$ **electroweak sector nice** in chirality flow

Fermion propagator

- Fermion propagator

$$\frac{i}{p^2-m_f^2}\begin{pmatrix} m_f\delta^{\dot{\alpha}}{}_{\dot{\beta}} & \sqrt{2}p^{\dot{\alpha}\beta} \\ \sqrt{2}\bar{p}_{\alpha\dot{\beta}} & m_f\delta_{\alpha}{}^{\beta}\end{pmatrix} = \frac{i}{p^2-m_f^2}$$



- Propagators and vertices don't always contribute factor $\tau/\bar{\tau}$
  $\Rightarrow$ may have even number of $\tau/\bar{\tau}$-matrices
- Have to update arrow swap procedure to include even number of $\tau/\bar{\tau}$

$$\langle i|\bar{\tau}^{\mu_1}\tau^{\mu_2}\ldots\bar{\tau}^{\mu_{2n+1}}|j] = [j|\tau^{\mu_{2n+1}}\bar{\tau}^{\mu_{2n}}\ldots\tau^{\mu_1}|i\rangle$$

$$\langle i|\bar{\tau}^{\mu_1}\tau^{\mu_2}\ldots\tau^{\mu_{2n}}|j\rangle = -\langle j|\bar{\tau}^{\mu_{2n}}\bar{\tau}^{\mu_{2n-1}}\ldots\tau^{\mu_1}|i\rangle$$

$$[i|\tau^{\mu_1}\bar{\tau}^{\mu_2}\ldots\bar{\tau}^{\mu_{2n}}|j] = -[j|\tau^{\mu_{2n}}\bar{\tau}^{\mu_{2n-1}}\ldots\bar{\tau}^{\mu_1}|i]$$

Arrow flips may induce minus signs! Care must be taken

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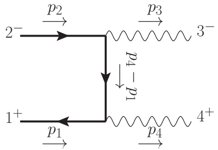
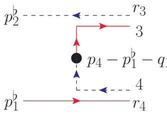
A massive example

Consider the diagram of $e_1^+ e_2^- \to \gamma_3^+ \gamma_4^-$ and include the mass m_e

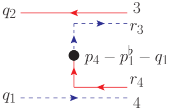


- Obtain 3 new terms
- Simplify with choices of q_1, q_2, r_3, r_4
- $e^{i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{\langle p_i^\flat q_i \rangle}$, $\quad e^{-i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^\flat]}$

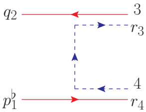
$$= \frac{-2ie^2}{((p_4 - p_1)^2 - m_e^2)[3r_3]\langle 4r_4\rangle} \Bigg\{$$



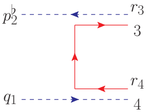
$$- \frac{m_e}{[q_1 p_1^\flat]} \frac{m_e}{\langle p_2^\flat q_2 \rangle}$$



$$+ m_e \left(\frac{m_e}{\langle p_2^\flat q_2 \rangle} \right.$$



$$- \frac{m_e}{[q_1 p_1^\flat]}$$



$$\left. \right) \Bigg\}$$

Towards an implementation

- Clearly much superior for pen and paper calculations
- What about implementation?
  - Each contribution free of matrix structure, only sum of products of spinor inner products
  - Natural to sum only over contributing helicities, most particles are effectively massless
  - Only some Feynman diagrams contribute to given helicity assignment, for fermions $(n_{\text{in,L}} + n_{\text{in,R}})! \to n_{\text{in,L}}! n_{\text{in,R}}!$
  - Assignment of the same reference momentum for massless gauge bosons removes many terms ($1/2$ of terms survive 3g vertex, $1/4$ survives 4g vertex for external gluons and random polarizations)
- Currently implementing in MadGraph (Zenny Wettersten's master thesis, in collaboration with Andrew Lifson, Oliver Mattelaer (Louvain La Neuve))

Application to resummation

- For color, resummation can be done in a color-flow basis
- Similarly here use chirality-flow we obtain a limited set of Lorentz structures
- Gain full control over analytic structure of resummation
- In collaboration with Simon Plätzer (Vienna and Graz)

... and beyond leading order

- Division in left and right chiral spinors is a four-dimensional construction
- ... but loops are calculated in $4 - 2\epsilon$ dimensions
- Need to consistently treat spinors in $4 - 2\epsilon$ dimension, solve Weyl eq. in $4 - 2\epsilon$ dimensions, treat spin sums and inner products...
- Worked out with Andrew Lifson and Simon Plätzer (Vienna and Graz)

Conclusion and outlook

Conclusion and outlook

- Splitting Lorentz structure into su(2), su(2), we have been able to recast all standard model Feynman rules to chirality-flow rules
- The chirality-flow formalism gives a transparent and intuitive way of understanding the Lorentz inner products appearing in amplitudes
  - Spinor-helicity formalism: 4×4 matrices $\gamma^\mu \rightarrow$ to 2×2 matrices σ^μ
  - Chirality-flow method: 2×2 matrices $\sigma^\mu \rightarrow$ scalars
- Shorter calculation of Feynman diagrams
  - Many processes within range of quick pen and paper calculations, often no intermediate steps
  - Final result transparent/intuitive
- Implementing in MadGraph
- Resummation paper close to ready, and loops on their way